Bogoliubov Theory for Bose Gases Interacting through Singular Potentials

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Abstract

This thesis deals with the theoretical investigation of spectral and dynamical properties of bosonic many-body quantum systems. Such systems are of particular theoretical and experimental interest since they may form a *Bose-Einstein condensate*: A macroscopic fraction of the particles occupies the same one-particle quantum state. This is interesting from a theoretical point of view, because the complex many-body quantum system can effectively be described by the behaviour of the one-particle condensate wavefunction. A particular effective theory is the so-called *Bogoliubov theory*. It was introduced by Bogoliubov in [17] and predicts the low-energy properties of the Bose gas under several physically motivated assumptions. This thesis deals with the mathematically rigorous derivation of Bogoliubov's predictions and related questions in physically interesting regimes.

Our first result is a rigorous derivation of complete Bose-Einstein condensation of low-energy states in the *Gross-Pitaevskii regime*. In this regime, the particles interact through a two-body potential of the form $\kappa N^2 V(N.)$, where N denotes the number of particles and $\kappa > 0$ is a coupling constant. Assuming $\kappa > 0$ to be sufficiently small, we show that low-energy states exhibit complete Bose-Einstein condensation with a uniform (in N) bound on the number of excitations, in the limit $N \to \infty$.

Strongly related to our first result, we consider next the dynamics of Bose-Einstein condensates in the Gross-Pitaevskii regime. Under a physically motivated assumption on the energy of the initial data, we show that the system exhibits complete Bose-Einstein condensation for all times if it exhibits condensation initially and that the dynamics can effectively be described by the time-dependent Gross-Pitaevskii equation. Compared to previously known results, we provide optimal rates of convergence.

Our third result is a rigorous approximation of the low-energy spectrum of the Bose gas in the Gross-Pitaevskii regime, up to errors that vanish in the limit $N \to \infty$. In particular, we verify the predictions of Bogoliubov theory for the ground state energy and the low-energy excitation spectrum of the system.

Our last result provides a norm approximation of the many-body Schrödinger evolution of a system of bosons interacting through a two-body potential scaling as $N^{3\beta-1}V(N^\beta.)$, for $\beta \in (0;1)$. Assuming that the system exhibits Bose-Einstein condensation initially, we approximate the many-body dynamics in the limit of large N by describing the condensate wavefunction by the solution of a cubic non-linear Schrödinger equation while describing the excitations of the system in terms of a unitary Fock space evolution, generated by a quadratic Fock space Hamiltonian.

This thesis is based on the articles [20], [13], [15] and [19].

Zusammenfassung

Diese Arbeit befasst sich mit der theoretischen Untersuchung von spektralen und dynamischen Eigenschaften von bosonischen Vielteilchen-Quantensystemen. Solche Systeme sind von besonderem theoretischen und experimentellen Interesse, da sie ein *Bose-Einstein Kondensat* bilden können: Ein makroskopischer Anteil der Teilchen befindet sich in ein und demselben Quantenzustand. Aus theoretischer Sicht ist dies interessant, da das komplexe Quantensystem effektiv durch die Einteilchen-Wellenfunktion des Kondensats beschrieben werden kann. Eine spezielle effektive Theorie ist die sogenannte *Bogoliubov Theorie*. Sie wurde von Bogoliubov in [17] eingeführt und trifft Vorhersagen über die nieder-energetischen Eigenschaften des Bose Gases unter Zuhilfenahme mehrerer physikalisch motivierter Annahmen. Diese Dissertation befasst sich mit der mathematisch rigorosen Herleitung von Bogoliubovs Vorhersagen und weiteren damit zusammenhängenden Fragen für physikalisch interessante Systeme.

Unser erstes Ergebnis ist eine rigorose Herleitung der Tatsache, dass nieder-energetische Zustände in dem *Gross-Pitaevskii Regime* ein Bose-Einstein Kondensat bilden. In diesem Regime wechselwirken die Teilchen durch ein Potential der Form $\kappa N^2 V(N.)$, wobei N die Teilchenanzahl des Systems und $\kappa > 0$ eine Kopplungskonstante bezeichnen. Unter der Annahme hinreichend kleiner $\kappa > 0$ zeigen wir, dass nieder-energetische Zustände ein Bose-Einstein Kondensat bilden mit gleichmässig (in N) beschränkter Anzahl angeregter Teilchen, im Grenzwert $N \to \infty$.

Eng zusammenhängend mit unserem ersten Ergebnis untersuchen wir dann die Dynamik von Bose-Einstein Kondensaten in dem Gross-Pitaevskii Regime. Unter einer physikalisch motivierten Annahme an die Energie der Anfangszustände zeigen wir, dass das System für alle Zeiten ein Bose-Einstein Kondensat bildet, falls es zu Beginn der Dynamik ein Kondensat bildet und dass die Dynamik effektiv durch die zeitabhängige Gross-Pitaevskii Gleichung beschrieben werden kann. Verglichen mit bisherigen Resultaten sind die Konvergenzraten unseres Ergebnisses optimal.

Unser drittes Ergebnis besteht in der rigorosen Approximation des nieder-energetischen Spektrums des Bose Gases in dem Gross-Pitaevskii Regime, bis auf Fehler, welche in dem Grenzwert $N \to \infty$ verschwinden. Insbesondere verifizieren wir Bogoliubovs Vorhersagen über die Grundzustandsenergie und das nieder-energetische Anregungsspektrum des Systems.

Unser letztes Ergebnis besteht in einer Norm Approximation der Vielteilchen Schrödinger Zeitentwicklung eines Systems von Bosonen, die durch ein Potential der Form $N^{3\beta-1}V(N^{\beta}.)$, für $\beta \in (0;1)$, miteinander wechselwirken. Unter der Annahme, dass das System zu Beginn der Dynamik ein Bose-Einstein Kondensat bildet, approximieren wir die Vielteilchen Zeitentwicklung im Grenzwert grosser N, indem wir die Kondensat-Wellenfunktion durch die Lösung einer kubischen, nicht-linearen Schrödinger-Gleichung beschreiben, während wir die Anregungen des Systems durch eine unitäre Fockraum Zeitentwicklung beschrieben, die von einem quadratischen Fockraum Hamilton Operator erzeugt wird.

Die vorliegende Arbeit basiert auf den Artikeln [20], [13], [15] und [19].

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Në fund, faleminderit zemra ime, te dua shumë, ti jeni thjesht e mrekullueshme dhe ti je dielli im.

Contents

Chapter 1

Introduction

This thesis deals with the theoretical investigation of properties of Bose gases interacting through very strong short-range interactions. A Bose gas is a quantum system composed of a large number of particles, such as atoms or molecules, obeying the so-called Bose-Einstein statistics. What makes Bose gases particularly interesting is the fact that, at extremely low-temperatures, they may undergo a phase transtion to form a *Bose-Einstein condensate*. In this state of matter, a macroscopic fraction of the particles occupies the same one-particle state. This phenomenon has not only been predicted theoretically by Bose and Einstein [18, 32, 33] in the early 20th century, but it has also been verified experimentally in 1995 by the Nobel laureates Cornell, Ketterle and Wieman [7, 29].

While a theoretical derivation of Bose-Einstein condensation for non-interacting gases can be found in many theoretical physics textbooks on condensed matter theory, a proof of condensation for Bose gases with realistic short-range interactions is much more difficult. More generally, the rigorous derivation of energetic and dynamical properties of interacting Bose Einstein condensates is highly non-trivial and there has been a lot of research devoted to it in the mathematical physics literature. Due to the large number of particles in typical experiments (the number may vary between 10^3 to 10^6 particles), a lot of attention has been devoted to the rigorous derivation of effective theories for large bosonic many-body quantum systems, such as Hartree theory, Gross-Pitaevskii theory or Bogoliubov theory. These theories provide effective descriptions of the complex many-body system, making explicit predictions for the energy and the dynamics of the system. This thesis deals with the mathematically rigorous justification of such theories for Bose gases interacting through very strong short-range potentials. In the following sections, we introduce basic notions of many-body quantum mechanics, the mathematical setting in which we work and we present our main results concerning the spectral and dynamical properties of large bosonic quantum many-body systems.

1.1 Bosonic Many-Body Systems in Quantum Mechanics

In the framework of quantum mechanics, a single particle is described by a vector $\psi \in \mathfrak{H}$ in a complex, separable Hilbert space $\mathfrak{H}$. A system of $N \in \mathbb{N}$ particles is described by a vector $\psi_N \in \mathfrak{H}_N = \mathfrak{H}^{\otimes N}$. Bosons are particles obeying Bose-Einstein statistics. A system of N identical bosons is described by a vector $\psi_N \in \mathfrak{H}_N$, invariant under permutations of its particles. More precisely (see for instance [93, Chapter II.4]), we denote by $\mathfrak{S}_N$ the group of permutations of N elements which acts on $\mathfrak{H}_N$ by requiring that

$$\sigma(\varphi_{j_1} \otimes \varphi_{j_2} \otimes \ldots \otimes \varphi_{j_N}) = \varphi_{\sigma(j_1)} \otimes \varphi_{\sigma(j_2)} \otimes \ldots \otimes \varphi_{\sigma(j_N)}$$

for all $\sigma \in \mathfrak{S}_N$, $j_1, j_2, \ldots, j_N \in \mathbb{N}$, where $\big\{ \bigotimes_{i=1}^N \varphi_{j_i} \in \mathfrak{H}_N : j_1, j_2, \ldots, j_N \in \mathbb{N} \big\}$ is a basis of $\mathfrak{H}_N$. It is then simple to verify that the symmetrization operator $S_N = (1/N!) \sum_{\sigma \in \mathfrak{S}_N} \sigma$ defines an orthogonal projection on $\mathfrak{H}_N$. Its image $S_N(\mathfrak{H}_N)$ describes the subspace of bosonic N-particle states which are symmtric with respect to permutations of its particles.

In this thesis, the one-particle space is given by $L^2(\Lambda)$ where either $\Lambda = \mathbb{R}^3$ or $\Lambda = \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, that is, we consider spinless particles either moving in the three-dimensional Euclidean space or being trapped in a box of volume one with periodic boundary conditions. As a consequence, the N-particle bosonic systems with which we are concerned are described by the Hilbert space $L^2_s(\Lambda^N) = S_N\big(L^2(\Lambda^N)\big)$.

The energy and the time-evolution of an N-particle system are determined by an Hamilton operator H_N which acts as a self-adjoint operator in $\mathfrak{H}_N$ and typically has the form

$$H_N = \sum_{i=1}^N \big(-\Delta_{x_i} + V_{\text{ext}}(x_i) \big) + \sum_{1 \le i < j \le N} v(x_i - x_j) \tag{1.1}$$

The operator $-\Delta_{x_i}$, $i = 1, \ldots, N$, denotes the Laplacian with respect to the variable $x_i \in \Lambda$. It measures the kinetic energy of the i-th particle. The multiplication operator V_{ext} denotes an external potential; in case of $\Lambda = \mathbb{R}^3$ it may trap the particles to a bounded domain. Finally, v is a real-valued, measurable function modeling the interactions among the particles and acting as a multiplication operator. We assume it to be radially symmetric and non-negative. In this thesis, we restrict our attention to systems described by Hamiltonians H_N of the form (1.1).

The conditions that we impose in the following sections on V_{ext} and v ensure that H_N is a densely defined, self-adjoint operator, $H_N : D(H_N) \to L^2_s(\Lambda^N)$, bounded from below. Under such conditions, the *ground state energy* E_N of H_N is defined by

$$E_N = \inf_{\substack{\psi_N \in D(H_N), \\ \|\psi_N\|_{L^2_s(\Lambda^N)} = 1}} \langle \psi_N, H_N \psi_N \rangle \tag{1.2}$$

As H_N is realized as a self-adjoint operator, also the dynamics of the system is well-defined. If ψ_N denotes the initial state of the system, its time-evolution is given by the solution $t \mapsto \psi_{N,t} = e^{-iH_N t}\psi_N$ of the Schrödinger equation

$$\begin{cases} i\partial_t \psi_{N,t} = H_N \psi_{N,t}, \\ (\psi_{N,t})_{|t=0} = \psi_N \end{cases} \tag{1.3}$$

Here, the operator-valued function $t \mapsto e^{-iH_N t}$ denotes the strongly continuous, one-parameter unitary group with infinitesimal generator H_N.

The ground state energy (1.2) describes the lowest possible energy the system can possess. If $\Lambda = \mathbb{T}^3$, we are not only interested in the ground state energy of the system, but also in estimating energy levels above E_N, the so-called *excitation spectrum* of H_N. In this case, the conditions on v ensure that H_N has purely discrete spectrum.

As mentioned in the introduction, this thesis deals with quantum systems with a very large number N of particles. As a consequence, an exact computation of the ground-state energy E_N, the excitation spectrum of H_N and the time-evolution $\psi_{N,t}$ are out of reach, at least for physically interesting systems with non-trivial interaction. Therefore, we are interested in rigorous approximations of (1.2) and (1.3) in the limit $N \to \infty$.

An important property of bosonic quantum systems is the fact that, at low temperature, they undergo a phase transition to form a Bose-Einstein condensate. This behaviour plays a crucial role, both to understand the low-energy spectrum and the time-evolution of the system. It was first predicted by Bose and Einstein in [18], [32, 33]. In this thesis, we are therefore also concerned with the proof of Bose-Einstein condensation for low-energy states at zero temperature. While it is rather clear what Bose-Einstein condensation means for a non-interacting Bose gas, a general definition for interacting Bose gases goes back to Penrose and Onsager in [84]. The definition makes use of the so-called one-particle reduced density $\gamma_N^{(1)} \in \mathcal{L}(L^2(\Lambda))$ of a many-body wavefunction $\psi_N \in L^2_s(\Lambda^N)$, defined as the non-negative trace class operator with integral kernel

$$\gamma_N^{(1)}(x;y) = \int_{\Lambda^{N-1}} dx_2 \dots dx_N \; \psi_N(x, x_2, \dots, x_N)\overline{\psi}_N(y, x_2, \dots, x_N) \qquad (x, y \in \Lambda) \tag{1.4}$$

We say that a sequence $(\psi_N)_{N\in\mathbb{N}}$, $\psi_N \in L^2_s(\Lambda^N)$, of many-body wavefunctions with associated sequence of one-particle reduced densities $(\gamma_N^{(1)})_{N\in\mathbb{N}}$ exhibits complete Bose-Einstein condensation in the one-particle wavefunction $\varphi \in L^2(\Lambda)$ with associated orthogonal rank-one projection $|\varphi\rangle\langle\varphi| \in \mathcal{L}(L^2(\Lambda))$ if

$$\lim_{N\to\infty} \operatorname{tr} \big|\gamma_N^{(1)} - |\varphi\rangle\langle\varphi|\big| = 0 \tag{1.5}$$

Intuitively, this means that all particles, up to a fraction vanishing as $N \to \infty$, are described by the condensate wavefunction φ.

1.2 Fock space and Excitations around a Bose-Einstein Condensate

Throughout this thesis, we work in a Fock space setting. In this section, we first introduce this setup and then explain how it may be used to describe excitations around a Bose-

Einstein condensate. Most of the objects we introduce are standard so that we only collect their definitions and some of their properties from the articles [20, Section 2], [13, Section 2], [15, Section 2] and [19, Sections 1 and 2].

First of all, the bosonic Fock space $\mathcal{F}$ is defined as the Hilbert space $\mathcal{F} = \bigoplus_{n\geq 0} L^2_s(\Lambda^n)$ equipped with the inner product

$$\langle \xi, \zeta \rangle_{\mathcal{F}} = \langle \xi, \zeta \rangle = \sum_{n\geq 0} \big\langle \xi^{(n)}, \zeta^{(n)} \big\rangle_{L^2_s(\Lambda^n)} \qquad (\xi, \zeta \in \mathcal{F})$$

The vacuum vector describing states of zero particles is denoted by $\Omega = \{1, 0, 0, \dots\} \in \mathcal{F}$.

In $\mathcal{F}$, one can create or annihilate a particle in the state $f \in L^2(\Lambda)$ using the creation operator $a^*(f)$ and the annihilation operator $a(f)$, respectively. They are defined by[1]

$$(a^*(f)\xi)^{(n)}(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \xi^{(n-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$$

$$(a(f)\xi)^{(n)}(x_1, \dots, x_n) = \sqrt{n+1} \int dx\ \bar{f}(x) \xi^{(n+1)}(x, x_1, \dots, x_n)$$

Notice that, given an n-particle state $\psi_n = (0, \dots, 0, \psi_n, 0, 0 \dots) \in \mathcal{F}$, $a^*(f)\psi_n = \sqrt{n+1} S_{n+1}(f \otimes \psi_n) \in \mathcal{F}$ is proportional to the symmetrization of the product $f \otimes \psi_n$. $a^*(f)$ and $a(f)$ are realized as closed and unbounded operators in $\mathcal{F}$, and they are defined such that $a^*(f)$ is the adjoint of $a(f)$ (see [94, Chapter X.7] for the details). In addition, they satisfy the canonical commutation relations

$$[a(f), a^*(g)] = \langle f, g \rangle_{L^2(\Lambda)} = \langle f, g \rangle_2, \qquad [a(f), a(g)] = [a^*(f), a^*(g)] = 0 \tag{1.6}$$

for all $f, g \in L^2(\Lambda)$.

When we consider translation invariant systems, that is $\Lambda = \mathbb{T}^3$, it is useful to work in momentum space. Indeed, in this case $\{x \mapsto \varphi_p(x) = e^{-ipx}, p \in 2\pi\mathbb{Z}^3\}$ is an orthonormal basis for $L^2(\Lambda)$. Hence, defining $\Lambda^* = 2\pi\mathbb{Z}^3$, every $f \in L^2(\Lambda)$ can be expanded in a Fourier series $f(.) = \sum_{p\in\Lambda^*} f_p e^{ip(.)}$ with the Fourier coefficients

$$\hat{f}_p = \int dx\ f(x) e^{-ip\cdot x} \qquad (p \in \Lambda^*)$$

For $\Lambda = \mathbb{T}^3$, it is moreover useful to define the operators

$$a^*_p = a^*(\varphi_p), \qquad a_p = a(\varphi_p) \qquad (p \in \Lambda^*) \tag{1.7}$$

creating and, respectively, annihilating a particle with momentum $p \in \Lambda^*$.

Sometimes, it is more convenient to carry out computations in position space. To this end, we will use the operator valued distributions $\check{a}^*_x, \check{a}_y$, $x, y \in \Lambda$, creating and, respectively, annihilating a particle at $x, y \in \Lambda$. They satisfy

$$a(f)^* = \int dx\ f(x)\, \check{a}^*_x, \quad a(f) = \int dy\ \bar{f}(y)\, \check{a}_y \tag{1.8}$$

[1] To simplify the notation, in the following all integrals are to be understood as taken over the whole domain under consideration, unless stated otherwise. For instance, $\int dxdy$ is to be read as $\int_{\Lambda\times\Lambda} dxdy$.

Formally, these creation and annihilation fields satisfy the (distributional) relations

$$[\check{a}_x, \check{a}_y^*] = \delta(x-y), \quad [\check{a}_x, \check{a}_y] = [\check{a}_x^*, \check{a}_y^*] = 0, \qquad (x, y \in \Lambda) \tag{1.9}$$

where δ denotes as usual the Dirac δ-distribution. If we consider systems in $\Lambda = \mathbb{R}^3$, we simply write a_x, a_y^* instead of $\check{a}_x, \check{a}_y^*$, since we do not use (1.7) in this case.

We emphasize that, in this thesis, we are mostly concerned with statements about expectations of quadratic forms that can be written conveniently as an integral of a suitable kernel multiplied by creation and annihilation fields. In this context, the fields $\check{a}_x, \check{a}_y^*$, are simply a device keeping track of the combinatorial factors due to the symmetry of bosonic many-body states which makes the computation of expectation values very efficient (see the corresponding remark in [65, Section 2]). For more details about their mathematically precise definition and their properties, we refer the reader to [94, Chapter X.7], in particular to [94, Theorem X.44].

Given a one particle operator B acting on (a dense subspace of) $L^2(\Lambda)$, we define its second quantization $d\Gamma(B)$ by setting $(d\Gamma(B)\xi)^{(n)} = \sum_{j=1}^n B_j \xi^{(n)}$, where $B_j = \mathbf{1} \otimes \cdots \otimes B \otimes \cdots \otimes \mathbf{1}$ acts as B on the j-th particle and as the identity on all other particles. If B has the integral kernel $B(x;y)$, we can use creation and annihilation fields to write

$$d\Gamma(B) = \int dxdy \; B(x;y) a_x^* a_y$$

Important operators are the number of particles operator $\mathcal{N} = d\Gamma(\mathbf{1})$ and the kinetic energy operator, given by $\mathcal{K} = d\Gamma(-\Delta)$. Equivalently, we have

$$\mathcal{N} = \int dx \; a_x^* a_x, \qquad \mathcal{K} = \int dx \; a_x^*(-\Delta_x) a_x = \int dx \; \nabla_x a_x^* \nabla_x a_x$$

The number of particles operator is useful to obtain upper bounds on creation and annihilation operators and second quantized, bounded operators. We have

$$\|a(f)\xi\| \leq \|f\|_2 \|\mathcal{N}^{1/2}\xi\|, \qquad \|a^*(f)\xi\| \leq \|f\|_2 \|(\mathcal{N}+1)^{1/2}\xi\| \tag{1.10}$$

for every $f \in L^2(\mathbb{R}^3)$. For a bounded operator $B \in \mathcal{L}(L^2(\Lambda))$, we also have

$$\pm\, d\Gamma(B) \leq \|B\|_{\text{op}}\, \mathcal{N}, \qquad \|d\Gamma(B)\xi\| \leq \|B\|_{\text{op}} \|\mathcal{N}\xi\| \tag{1.11}$$

where the second bound follows from the first by noting that $[d\Gamma(B), \mathcal{N}] = 0$.

We also need to bound operators that are quadratic in creation and annihilation operators, and that do not necessarily preserve the number of particles. Given $j \in L^2(\Lambda \times \Lambda)$, we let

$$A_{\sharp_1,\sharp_2}(j) = \int a^{\sharp_1}(j_x) a_x^{\sharp_2} \, dx = \int j^{\bar{\sharp}_1}(x;y) a_y^{\sharp_1} a_x^{\sharp_2} \, dxdy \tag{1.12}$$

where $j_x(y) = j(x;y)$, $\sharp_1, \sharp_2 \in \{\cdot, *\}$, $\bar{\sharp}_1 = \cdot$ if $\sharp_1 = *$ and $\bar{\sharp}_1 = *$ if $\sharp_1 = \cdot$, and where we write $a^\sharp = a$ if $\sharp = \cdot$, $a^\sharp = a^*$ if $\sharp = *$ and, similarly, $j^\sharp = j$ if $\sharp = \cdot$ and $j^\sharp = \bar{j}$ if

$\sharp = *$. If $\sharp_1 = \cdot$ and $\sharp_2 = *$ (i.e. if a creation operator lies on the right of an annihilation operator), in order to define[2] $A_{\sharp_1,\sharp_2}(j)$, we assume that the kernel $j \in L^2(\Lambda \times \Lambda)$ is such that $x \to j(x;x)$ is well-defined[3] and satisfies $x \to j(x;x) \in L^1(\Lambda)$. Operators of this form can be bounded as follows.

Lemma 1.2.1. *Let $j \in L^2(\Lambda \times \Lambda)$. Then for any $\xi \in \mathcal{F}$,*

$$\|A_{\sharp_1,\sharp_2}(j)\xi\| \leq \sqrt{2}\|(\mathcal{N}+1)\xi\| \begin{cases} \|j\|_2 + \int |j(x;x)|dx & \text{if } \sharp_1 = \cdot, \sharp_2 = * \\ \|j\|_2 & \text{otherwise} \end{cases}$$

When we work in $\Lambda = \mathbb{T}^3$, we only consider quadratic operators that are translation invariant and which, in momentum space, have the form

$$A_{\sharp_1,\sharp_2}(f) = \sum_{p \in \Lambda^*} f_p \, a^{\sharp_1}_{\alpha_1 p} a^{\sharp_2}_{\alpha_2 p} \tag{1.13}$$

Here, $f \in \ell^2(\Lambda^*)$, $\sharp_1, \sharp_2 \in \{\cdot, *\}$, and we use the notation $a^\sharp = a$, if $\sharp = \cdot$, and $a^\sharp = a^*$ if $\sharp = *$. Also, $\alpha_j \in \{\pm 1\}$ is chosen so that $\alpha_1 = 1$, if $\sharp_1 = *$, $\alpha_1 = -1$ if $\sharp_1 = \cdot$, $\alpha_2 = 1$ if $\sharp_2 = \cdot$ and $\alpha_2 = -1$ if $\sharp_2 = *$. In position space, these operators read

$$A_{\sharp_1,\sharp_2}(j) = \int dxdy \, \check{f}(x-y) \, \check{a}^{\sharp_1}_x \, \check{a}^{\sharp_2}_y$$

Analogously to Lemma 1.2.1, we have the following bounds.

Lemma 1.2.2. *Let $f \in \ell^2(\Lambda^*)$ and, if $\sharp_1 = \cdot$ and $\sharp_2 = *$ assume additionally that $f \in \ell^1(\Lambda^*)$. Then we have, for any $\xi \in \mathcal{F}$,*

$$\|A_{\sharp_1,\sharp_2}(f)\xi\| \leq \sqrt{2}\,\|(\mathcal{N}+1)\xi\| \begin{cases} \|f\|_2 + \|f\|_1 & \text{if } \sharp_1 = \cdot, \sharp_2 = * \\ \|f\|_2 & \text{otherwise} \end{cases}$$

Having introduced the Fock space $\mathcal{F}$ and standard creation and annihilation operators, let us now explain how a Fock space setting can be used to describe excitations around a Bose-Einstein condensate.

First, we fix a normalized function $\varphi \in L^2(\Lambda)$ which we think of as the condensate wavefunction. Following [64, Section 2.3], we then observe that every $\psi_N \in L^2_s(\Lambda^N)$ can be uniquely decomposed into the sum

$$\psi_N = \sum_{n=0}^{N} \psi_N^{(n)} \otimes_s \varphi^{\otimes(N-n)} \tag{1.14}$$

for a sequence $\psi_N^{(n)} \in L^2_{\perp\varphi}(\Lambda)^{\otimes_s n}$, $n = 0, \ldots, N$. Here, $L^2_{\perp\varphi}(\Lambda)^{\otimes_s n}$ denotes the symmetric tensor product of n copies of the orthogonal complement $L^2_{\perp\varphi}(\Lambda)$ of φ in $L^2(\Lambda)$. Given

[2] In this case, $A_{\sharp_1,\sharp_2}(j)$ is defined as $A_{\sharp_1,\sharp_2}(j) = A_{\sharp_2,\sharp_1}(j) + \int dx\, j(x;x)$.
[3] In this thesis, we in fact consider mostly continuous kernels $j \in C(\Lambda \times \Lambda)$.

$\psi_k \in L^2_s(\Lambda^k)$ and $\psi_l \in L^2_s(\Lambda^l)$, the symmetric tensor product $\psi_k \otimes_s \psi_l \in L^2_s(\Lambda^{k+l})$ is defined by

$$\psi_k \otimes_s \psi_\ell\, (x_1, ..., x_{k+l}) \\ = \frac{1}{\sqrt{k!\ell!(k+\ell)!}} \sum_{\sigma \in \mathfrak{S}_{k+\lambda}} \psi_k\big(x_{\sigma(1)}, ..., x_{\sigma(k)}\big)\psi_l\big(x_{\sigma(k+1)}, ..., x_{\sigma(k+l)}\big)$$

Using the decomposition (1.14), we can define a unitary map

$$U_N(\varphi) : L^2_s(\Lambda^N) \to \mathcal{F}^{\leq N}_{\perp\varphi} \quad \text{through} \quad U_N(\varphi)\psi_N = \{\psi^{(0)}_N, \psi^{(1)}_N, \dots, \psi^{(N)}_N\}. \tag{1.15}$$

Here $\mathcal{F}^{\leq N}_{\perp\varphi} = \bigoplus_{n=0}^N L^2_{\perp\varphi}(\Lambda)^{\otimes_s n}$ denotes the bosonic Fock space constructed over $L^2_{\perp\varphi}(\Lambda)$, truncated to sectors with at most N particles[4]. It naturally embeds into the full excitation Fock space $\mathcal{F}_{\perp\varphi}$, i.e. $\mathcal{F}^{\leq N}_{\perp\varphi} \hookrightarrow \mathcal{F}_{\perp\varphi} = \bigoplus_{n\geq 0} L^2_{\perp\varphi}(\Lambda)^{\otimes_s n}$. The image $U_N(\varphi)\psi_N$ describes particles in ψ_N that are not in the condensate. It allows us to focus on orthogonal excitations of the condensate. As we show in Chapter 2, see Remark 1) in Section 1.4, the criterion (1.5) for the sequence ψ_N to exhibit Bose-Einstein condensation is equivalent to

$$N^{-1}\langle U_N(\varphi)\psi_N, \mathcal{N}_{\perp\varphi}\, U_N(\varphi)\psi_N\rangle = 1 - \langle \varphi, \gamma^{(1)}_N \varphi\rangle \to 0 \tag{1.16}$$

as $N \to \infty$.

In most parts of this thesis (excluding Chapter 5), we work in the truncated Fock spaces $\mathcal{F}^{\leq N}$ and $\mathcal{F}^{\leq N}_{\perp\varphi}$. On these spaces, we define, for any $f \in L^2(\Lambda)$, modified creation and annihilation operators $b^*(f)$ and $b(f)$ by

$$b(f) = \sqrt{\frac{N-\mathcal{N}}{N}}\, a(f), \qquad \text{and} \qquad b^*(f) = a^*(f)\sqrt{\frac{N-\mathcal{N}}{N}} \tag{1.17}$$

Notice that $b(f), b^*(f)$ are bounded operators (since $\|\mathcal{N}_{|\mathcal{F}^{\leq N}}\|_{\text{op}} \leq N$) mapping $\mathcal{F}^{\leq N}$ to itself (and also $\mathcal{F}^{\leq N}_{\perp\varphi}$ to itself if $f \perp \varphi$). As becomes clear in the following chapters, these modified fields arise naturally from the application of the map $U_N(\varphi)$, defined in (1.14), since, for $f \perp \varphi$, we have

$$U_N(\varphi)a^*(f)a(\varphi)U^*_N(\varphi) = a^*(f)\sqrt{N - \mathcal{N}_{\perp\varphi}} = \sqrt{N}b^*(f) \tag{1.18}$$

Hence, on the level of $L^2_s(\Lambda^N)$, the operator $b^*(f)$ excites a particle from the condensate to the excited state f while $b(f)$ annihilates an excited particle, thereby creating a new particle in the condensate φ.

In analogy to the usual creation and annihilation operators, we introduce the distributional creation and annihilation fields

$$\check{b}_x = \sqrt{\frac{N-\mathcal{N}}{N}}\, \check{a}_x, \qquad \text{and} \qquad \check{b}^*_x = \check{a}^*_x\sqrt{\frac{N-\mathcal{N}}{N}}$$

[4]In the same way we define $\mathcal{F}^{\leq N} = \bigoplus_{n=0}^N L^2_s(\Lambda^n) \hookrightarrow \mathcal{F}$.

so that

$$b(f) = \int dx\ \bar{f}(x)\,\check{b}_x, \qquad \text{and} \qquad b^*(f) = \int dx\ f(x)\check{b}^*_x \tag{1.19}$$

These fields satisfy the modified commutation relations

$$\begin{aligned} [\check{b}_x, \check{b}^*_y] &= \left(1 - \frac{\mathcal{N}}{N}\right)\delta(x-y) - \frac{1}{N}\check{a}^*_y\check{a}_x \\ [\check{b}_x, \check{b}_y] &= [\check{b}^*_x, \check{b}^*_y] = 0 \end{aligned} \tag{1.20}$$

as well as

$$[\check{b}_x, \check{a}^*_y\check{a}_z] = \delta(x-y)b_z, \qquad [\check{b}^*_x, \check{a}^*_y\check{a}_z] = -\delta(x-z)b^*_y \tag{1.21}$$

In particular, these relations imply $[\check{b}_x, \mathcal{N}] = \check{b}_x$ and $[\check{b}^*_x, \mathcal{N}] = -\check{b}^*_x$. For translation invariant systems, i.e. $\Lambda = \mathbb{T}^3$, we define in addition

$$b_p = \sqrt{\frac{N - \mathcal{N}_{\perp\varphi_0}}{N}}\, a_p, \qquad b^*_p = a^*_p\sqrt{\frac{N - \mathcal{N}_{\perp\varphi_0}}{N}} \qquad (p \in \Lambda^*) \tag{1.22}$$

so that, in momentum space, the commutation relations (1.20) read

$$\begin{aligned} [b_p, b^*_q] &= \left(1 - \frac{\mathcal{N}_{\perp\varphi_0}}{N}\right)\delta_{p,q} - \frac{1}{N}a^*_q a_p \\ [b_p, b_q] &= [b^*_p, b^*_q] = 0 \end{aligned} \tag{1.23}$$

As an immediate consequence of (1.10), we obtain the following bounds on the b-fields.

Lemma 1.2.3. *Let $f \in L^2(\Lambda)$. For any $\xi \in \mathcal{F}^{\leq N}$, we have*

$$\begin{aligned} \|b(f)\xi\| &\leq \|f\|_2 \left\| \mathcal{N}^{1/2}\left(\frac{N - \mathcal{N} + 1}{N}\right)^{1/2}\xi \right\| \\ \|b^*(f)\xi\| &\leq \|f\|_2 \left\| (\mathcal{N}+1)^{1/2}\left(\frac{N - \mathcal{N}}{N}\right)^{1/2}\xi \right\| \end{aligned}$$

Since $\mathcal{N} \leq N$ on $\mathcal{F}^{\leq N}$, $b(f), b^*(f)$ are bounded by $\|b(f)\|, \|b^*(f)\| \leq (N+1)^{1/2}\|f\|_2$.

We also consider quadratic operators in the b fields. Given $j \in L^2(\Lambda \times \Lambda)$, we define, similarly to (1.13),

$$B_{\sharp_1,\sharp_2}(j) = \int \check{b}^{\sharp_1}(j_x)\check{b}^{\sharp_2}_x\, dx = \int j^{\bar{\sharp}_1}(x;y)\check{b}^{\sharp_1}_y\check{b}^{\sharp_2}_x\, dxdy \tag{1.24}$$

If $\sharp_1 = \cdot$ and $\sharp_2 = *$, we require that j is regular and that $x \to j(x;x)$ is integrable.

Lemma 1.2.4. *Let $j \in L^2(\Lambda \times \Lambda)$. Then*

$$\frac{\|B_{\sharp_1,\sharp_2}(j)\xi\|}{\left\|(\mathcal{N}+1)\left(\frac{N-\mathcal{N}+2}{N}\right)\xi\right\|} \leq \sqrt{2}\begin{cases} \|j\|_2 + \int |j(x;x)|dx & \text{if } \sharp_1 = \cdot, \sharp_2 = * \\ \|j\|_2 & \text{otherwise} \end{cases}$$

for all $\xi \in \mathcal{F}^{\leq N}$. Since $\mathcal{N} \leq N$ on $\mathcal{F}^{\leq N}$, the operator $B_{\sharp_1,\sharp_2}(j)$ is bounded, with

$$\|B_{\sharp_1,\sharp_2}(j)\| \leq \sqrt{2}N\begin{cases} \|j\|_2 + \int |j(x;x)|dx & \text{if } \sharp_1 = \cdot, \sharp_2 = * \\ \|j\|_2 & \text{otherwise} \end{cases}$$

Remark: For $\varphi \in L^2(\Lambda)$, let $q_\varphi = 1 - |\varphi\rangle\langle\varphi|$ be the orthogonal projection onto $L^2_{\perp\varphi}(\Lambda)$. If $j \in (q_{\varphi^{\bar\sharp_1}} \otimes q_{\varphi^{\bar\sharp_2}})(L^2(\Lambda\times\Lambda))$, we have $B_{\sharp_1,\sharp_2}(j) : \mathcal{F}^{\leq N}_{\perp\varphi} \to \mathcal{F}^{\leq N}_{\perp\varphi}$ (here we use the notation $\bar\sharp = *$ if $\sharp = \cdot$ and $\bar\sharp = \cdot$ if $\sharp = *$, and $\varphi^\sharp = \varphi$ if $\sharp = *$, $\varphi^\sharp = \bar\varphi$ if $\sharp = \cdot$).

When we consider $\Lambda = \mathbb{T}^3$, we restrict our attention to translation invariant operators which are quadratic in the modified fields. For $f \in \ell^2(\Lambda^*_+)$, we define

$$B_{\sharp_1,\sharp_2}(f) = \sum_{p\in\Lambda^*} f_p\, b^{\sharp_1}_{\alpha_1 p}\, b^{\sharp_2}_{\alpha_2 p}$$

with $\alpha_1 = 1$ if $\sharp_1 = *$, $\alpha_1 = -1$ if $\sharp_1 = \cdot$, $\alpha_2 = 1$ if $\sharp_2 = \cdot$ and $\alpha_2 = -1$ if $\sharp_2 = *$. By construction, $B_{\sharp_1,\sharp_2}(f) : \mathcal{F}^{\leq N}_{\perp\varphi_0} \to \mathcal{F}^{\leq N}_{\perp\varphi_0}$. In position space, these operators read

$$B_{\sharp_1,\sharp_2}(f) = \int \check{f}(x-y)\, \check{b}^{\sharp_1}_x \check{b}^{\sharp_2}_y \, dxdy$$

Similarly to Lemma 1.2.4, we close this section with the following lemma.

Lemma 1.2.5. *Let $f \in \ell^2(\Lambda^*)$. If $\sharp_1 = \cdot$ and $\sharp_2 = *$, we assume additionally that $f \in \ell^1(\Lambda^*)$. Then*

$$\frac{\|B_{\sharp_1,\sharp_2}(f)\xi\|}{\left\|(\mathcal{N}+1)\left(\frac{N-\mathcal{N}+2}{N}\right)\xi\right\|} \leq \sqrt{2}\begin{cases} \|f\|_2 + \|f\|_1 & \text{if } \sharp_1 = \cdot, \sharp_2 = * \\ \|f\|_2 & \text{otherwise} \end{cases}$$

for all $\xi \in \mathcal{F}^{\leq N}$. Since $\mathcal{N} \leq N$ on $\mathcal{F}^{\leq N}$, the operator $B_{\sharp_1,\sharp_2}(f)$ is bounded, with

$$\|B_{\sharp_1,\sharp_2}(f)\| \leq \sqrt{2}N\begin{cases} \|f\|_2 + \|f\|_1 & \text{if } \sharp_1 = \cdot, \sharp_2 = * \\ \|f\|_2 & \text{otherwise} \end{cases}$$

1.3 Bogoliubov Theory for Dilute Bose Gases

In this section we clarify what we mean by Bogoliubov theory and which of the mathematical questions it raises are addressed in this thesis. Our presentation of Bogoliubov theory follows [88] and [70, Appendix A], which is based on the article [65].

Bogoliubov theory, or Bogoliubov's method, is a microscopic quantum mechanical theory that was originally proposed by Bogoliubov in [17] in order to explain qualitative features of superfluid Helium. In modern physics textbooks (such as [88, Chapter 4])

Bogoliubov's method is used as an approximation scheme to calculate the ground state energy and the excitation energies of a dilute system of weakly interacting bosons with high accuracy.

To explain the ideas behind Bogoliubov's approximation, let us consider a system of N bosons moving in[5] $\Lambda_L = \mathbb{R}^3/(L\mathbb{Z}^3)$, i.e. in a box of volume L^3 with periodic boundary conditions. Switching to momentum space and using the formalism of second quantization, the Hamilton operator has the form

$$H_{N,L} = \sum_{p\in\Lambda_L^*} p^2 a_p^* a_p + \frac{1}{2L^3} \sum_{r,p,q\in\Lambda_L^*} \widehat{v}(r) a_{p+r}^* a_q^* a_p a_{q+r} \tag{1.25}$$

Bogoliubov was interested in the ground state and excitation energies of $H_{N,L}$ in the thermodynamic limit, where $N, L \to \infty$ with the density $\rho = N/L^3$ being fixed.

Bogoliubov's method is based on physically motivated assumptions that, starting from (1.25), lead to an operator whose spectrum can be computed explicitly. Motivated by the non-interacting Bose gas with $v = 0$, Bogoliubov's first assumption is that the system exhibits complete Bose-Einstein condensation in the one-particle state $\varphi_{0,L} = L^{-3/2} \in L^2(\Lambda_L)$. Condensation implies that, in low-energy states, the expectation of the operator $a_0^* a_0$ is close to N, with an error of lower order. Hence, as a first approximation, Bogoliubov replaced the operators a_0 and a_0^*, appearing in $H_{N,L}$, by the number $N^{1/2}$. This step is the so-called *c-number substitution.* The resulting operator $\widetilde{H}_{N,L}$ consists of a constant plus a sum of terms which are either quadratic, cubic or quartic in creation and annihilation operators. Since the cubic and quartic terms are of lower order in N than the constant and quadratic terms, the second step of Bogoliubov's approximation consists in dropping the cubic and quartic terms.

From these steps, $H_{N,L}$ can be approximated by the quadratic operator

$$\widetilde{Q}_{N,L} = \frac{N-1}{2}\rho\widehat{v}(0) + \sum_{0\neq p\in\Lambda_L^*} \big[p^2 + \rho\widehat{v}(p)\big] a_p^* a_p + \frac{1}{2}\sum_{0\neq p\in\Lambda_L^*} \rho\widehat{v}(p)\big[a_p^* a_{-p}^* + a_p a_{-p}\big]$$

which can be diagonlized explicitly by applying the unitary *Bogoliubov transformation*

$$T = \exp\left[\frac{1}{2}\sum_{0\neq p\in\Lambda_L^*} \nu_p\big(a_p^* a_{-p}^* - a_p a_{-p}\big)\right]$$

with $\nu(p) = \frac{1}{2}\tanh^{-1}\big(\rho\widehat{v}(p)/[p^2 + \rho\widehat{v}(p)]\big)$, $p \in \Lambda_L^*$. We obtain

$$\begin{aligned} T\widetilde{Q}_{N,L}T^* = {} & \frac{N-1}{2}\rho\widehat{v}(0) - \frac{1}{2}\sum_{0\neq p\in\Lambda_L^*}\left(\big[p^2 + \rho\widehat{v}(p)\big] - \sqrt{\big[p^2+\rho\widehat{v}(p)\big]^2 - \rho^2\widehat{v}(p)^2}\right) \\ & + \sum_{0\neq p\in\Lambda_L^*} \sqrt{p^4 + 2p^2\rho\widehat{v}(p)}\, a_p^* a_p \end{aligned} \tag{1.26}$$

[5] A complete orthonormal basis of $L^2(\Lambda_L)$ is given in this case by $\{x \mapsto L^{-3/2}e^{-ipx}, p \in \frac{2\pi}{L}\mathbb{Z}^3\}$.

At this point, Bogoliubov realized that, in the thermodynamic limit, the low-energy spectrum of the system only depends on v through the first and second Born approximations $\mathfrak{a}_0^{(1)}$ and $\mathfrak{a}_0^{(2)}$ of the scattering length $\mathfrak{a}_0 = \mathfrak{a}_0(v)$. The latter is a measure of the interaction's effective length scale on which it acts[6]. It is defined by the solution f of the zero-energy scattering equation

$$\left[-\Delta + \frac{1}{2}v(x)\right] f(x) = 0 \tag{1.27}$$

with the boundary condition[7] $f(x) \to 1$, as $|x| \to \infty$. For $|x| > r$, one has

$$f(x) = 1 - \frac{\mathfrak{a}_0}{|x|}$$

for a constant $\mathfrak{a}_0$ which is uniquely determined by the interaction v and which is called its scattering length. Equivalently, $\mathfrak{a}_0$ can be computed from

$$8\pi\mathfrak{a}_0 = \int dx\; v(x)f(x) \tag{1.28}$$

Writing $f = 1 - w$ and using the scattering equation (1.27), the last expression can be expanded into the *Born series* for $\mathfrak{a}_0$, given by

$$8\pi\mathfrak{a}_0 = \widehat{v}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k(2\pi)^{3k}} \int_{\mathbb{R}^{3k}} dp_1 \dots dp_k \, \frac{\widehat{v}(p_1)}{p_1^2} \left(\prod_{i=1}^{k-1} \frac{\widehat{v}(p_i - p_{i+1})}{p_{i+1}^2} \right) \widehat{v}(p_k) \tag{1.29}$$

where $\widehat{v}$ denotes the Fourier transform of the map v. From this series, we read off the first and second Born approximations $\mathfrak{a}_0^{(1)}$ and $\mathfrak{a}_0^{(2)}$ for $\mathfrak{a}_0$ as

$$\mathfrak{a}_0^{(1)} = \frac{\widehat{v}(0)}{8\pi}, \qquad \mathfrak{a}_0^{(2)} = \frac{\widehat{v}(0)}{8\pi} - \frac{\kappa^2}{16\pi(2\pi)^3} \int_{\mathbb{R}^3} dp \, \frac{\widehat{v}(p)^2}{p^2}$$

In the last step of his analysis, Bogoliubov argued that, for dilute systems with small density ρ, low-energy eigenvalues of $H_{N,L}$ should only depend on $\mathfrak{a}_0$; hence, he replaced $\mathfrak{a}_0^{(1)}$ and $\mathfrak{a}_0^{(2)}$ in the expressions for the low-energy spectrum by $\mathfrak{a}_0$. Arguing similarly for (1.26) (see [88, Chapter 4.2]), we approximate $T\widetilde{Q}_{N,L}T^*$ by the quadratic operator

$$Q_{N,L} = E_{N,L} + \sum_{0 \neq p \in \Lambda_L^*} \sqrt{p^4 + 16\pi\rho\mathfrak{a}_0 p^2}\; a_p^* a_p \tag{1.30}$$

[6]As an example, the box potential v_{box}, defined by $v_{\text{box}} = \infty$ for $|x| \leq \mathfrak{a}$ and vanishing otherwise, has scattering length $\mathfrak{a}_0(v_{\text{box}}) = \mathfrak{a}$, as mentioned in [70, Chapter 2].

[7]For a mathematically precise definition, we refer the reader to [70, Appendix C]: The scattering length $\mathfrak{a}_0$ is determined by first solving (1.27) in $H^1(B_R(0))$ for some fixed $R > r$, and noting that $\mathfrak{a}_0$ is independent of R which follows from [70, Appendix C, Remark 1]. In the notation of [70, Appendix C, Remark 1], the function $f : \mathbb{R}^3 \to \mathbb{R}$ solving (1.27) is given by $F_0(|\cdot|)$.

where the ground state energy $E_{N,L}$ is given by[8]

$$E_{N,L} = 4\pi\mathfrak{a}_0\rho(N-1) - \frac{1}{2}\sum_{0\neq p\in\Lambda_L^*}\left[p^2 + 8\pi\mathfrak{a}_0\rho - \sqrt{p^4 + 16\pi\rho\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0\rho)^2}{2p^2}\right] \quad (1.31)$$

(1.30) implies that low-energy eigenvalues of $H_{N,L} - E_{N,L}$ are given by finite sums of the form

$$\sum_{0\neq p\in\Lambda_L^*} n_p\sqrt{p^4 + 16\pi\rho\mathfrak{a}_0 p^2} \quad (1.32)$$

for integers $n_p \in \mathbb{N}$ such that $n_p \neq 0$ for only finitely many $p \in \Lambda_L^*$. Moreover, we remark that in the thermodynamic limit the ground state energy $E_{N,L}$ converges, up to leading order in $\rho\mathfrak{a}_0^3 \ll 1$, to the limit E_0 which is given by

$$E_0 = 4\pi\rho\mathfrak{a}_0 N\Big(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho\mathfrak{a}_0^3}\Big) \quad (1.33)$$

Equation (1.33) is the so called *Lee-Huang-Yang* formula (see [60]) for the ground state energy of the weakly interacting Bose gas. Finally, let us remark that the linearity of the dispersion relation $\sqrt{p^4 + 16\pi\rho\mathfrak{a}_0 p^2}$ for small values of $p \in \Lambda_L^*$, $|p| \ll 1$, verifies Landau's criterion of superfluidity (see, e.g. [88, Chapter 6.1]).

To date, there exist only few mathematically rigorous results related to parts of Bogoliubov theory. First of all, the mathematically rigorous justification of whether or not low-energy states of interacting many-body systems exhibit complete Bose-Einstein condensation in the thermodynamic limit is up to now, more than 70 years after Bogoliubov's paper [17], an open problem. A proof of condensation is currently pursued in a long-term project using renormalization group techniques by Balaban-Feldman-Knörrer-Trubowitz; we refer the reader to [8] for a review on recent progress. Concerning the ground state energy of three dimensional interacting Bose gases, the leading order term $4\pi\rho\mathfrak{a}_0 N$ of the Lee-Huang-Yang formula (1.33) could be verified as the correct upper bound by Dyson in [31] (for hard-sphere interactions). The validity of the leading order term as lower bound was verified only 40 years later by Lieb and Yngvason in [73]. Concerning the second order corrections to the ground state energy, Bogoliubov theory could be verified for the first time by Lieb-Solovej for bosonic jellium in [68] and for the two-component charged Bose gas in [69] as well as [97]. The Lee-Huang-Yang formula (1.33) was verified as correct upper bound for the ground state energy, up to leading order in the coupling constant of the interaction potential, by Erdős-Schlein-Yau in [38]. Their result was later improved in [99] by Yau-Yin whose upper bound agrees with the Lee-Huang-Yang formula. That the Lee-Huang-Yang formula describes correct upper and lower bounds for the ground state energy of systems of interacting Bose gases in a regime of weak coupling and high density was proved by Giuliani-Seiringer in [45]. In such a regime, the scattering length $\mathfrak{a}_0$ can indeed be replaced by its second order Born

[8] We point out that the leading order term $4\pi\mathfrak{a}_0\rho(N-1)$ is usually replaced by $4\pi\mathfrak{a}_0\rho N$ in the literature, see for instance equation (4.29) in [88, Chapter 4.2]. In the thermodynamic limit, the difference $4\pi\mathfrak{a}_0\rho$ is negligible compared to the leading order term (1.33) of the ground state energy.

approximation $\mathfrak{a}_0 \simeq \mathfrak{a}_0^{(1)} + \mathfrak{a}_0^{(2)}$, up to negligible errors. The results of Giuliani-Seiringer in [45] were recently improved by Brietzke-Solovej, see [21]. Finally, the rigorous verification of Bogoliubov's predictions for the excitation spectrum of the Hamiltonian $H_{N,L}$, in particular the dispersion relation $\sqrt{p^4 + 16\pi\rho\mathfrak{a}_0 p^2}$, is still an open problem.

In this thesis, we will not be interested in the thermodynamic limit. Instead, we will study systems in the so-called *Gross-Pitaevskii* regime, which is relevant to describe trapped Bose gases like the ones that are currently produced in labs. In the Gross-Pitaevskii regime, gases of N particles are trapped in a volume of order one and interact through a potential with scattering length of the order $\mathcal{O}(N^{-1})$. The Hamilton operator acts on (a dense subspace of) $L^2_s(\Lambda^N)$, for $\Lambda = \mathbb{R}^3$ or $\Lambda = \mathbb{T}^3$, and it has the form

$$H_N = \sum_{i=1}^{N} \big(-\Delta_{x_i} + V_{\text{ext}}(x_i)\big) + \sum_{1\leq i<j\leq N} N^2 V(N(x_i - x_j)) \tag{1.34}$$

By scaling, Bose gases in the Gross-Pitaevskii regime are mathematically equivalent to systems of N particles interacting through a fixed two-body potential in an extended volume of the order N^3. Compared with the thermodynamic limit discussed above, the Gross-Pitaevskii limit describes ultra-dilute gases, with the density $\rho = N^{-2}$ converging to zero as $N \to \infty$.

It is easy to translate Bogoliubov theory to Bose gases in the Gross-Pitaevskii regime. The goal of this thesis consists in providing rigorous mathematical justification for Bogoliubov Theory in the Gross-Pitaevskii regime. In particular, we will address the following questions.

The first ingredient in Bogoliubov theory is the assumption of complete Bose-Einstein condensation of low-energy states. If one tries to determine the excitation spectrum of the Hamiltonian H_N based on this assumption, it is clear that one first has to prove that the system indeed satisfies this hypothesis under physically reasonable assumptions. Moreover, in view of Bogoliubov's c-number substitution[9], which simply replaces an operator by a number, it is desireable to determine explicit error bounds on the rate of convergence in (1.5). Our first goal in this thesis consists therefore to prove condensation in the Gross-Pitaevskii regime. This question is discussed in Chapter 2 whose main result is presented in Section 1.4.

Naturally linked with the proof of condensation for low-energy states is the question of its dynamical stability. Indeed, in actual experiments the particles are first trapped by a confining potential and cooled down to extremely low temperatures. Condensation is measured by recording the particle's velocity distribution after releasing the trap (see [7], [29]) and comparing it to a distribution of particles in thermal equilibrium. Hence, it is an interesting question to prove that a system still exhibits complete Bose-Einstein condensation after a time $t > 0$ if it exhibits condensation initially at $t = 0$. We address this question in Chapter 3 and present our main result in Section 1.5

[9] We remark that, in the thermodynamic limit, the c-number substitution as such is mathematically valid independently of whether the system possesses Bose-Einstein condensation or not, see [72] and its slightly extended version [70, Appendix D]. As mentioned there, however, the c-number substitution is only useful in view of computing the energies if the system indeed exhibits condensation.

Once condensation has been proved, one may try to derive explicit expressions for the ground state energy of H_N and its excitation spectrum. In Chapter 4 we verify Bogoliubov's predictions about the low-energy spectrum of H_N with explicit error bounds. We in fact prove that the Hamiltonian H_N is, up to errors which are well under control in suitable low-energy spectral subspaces of H_N, unitarily equivalent to a diagonal quadratic Fock space Hamiltonian as predicted by Bogoliubov theory and that the excitation energies depend indeed only on the scattering length $\mathfrak{a}_0$ of the interaction potential. Our main result is presented in Section 1.6.

In Chapter 4, we not only determine the low-lying eigenvalues of H_N, but we also deduce $L^2_s(\Lambda^N)$-norm approximations for the corresponding eigenvectors. Up to the application of a unitary transformation, these approximations consist of the eigenvectors of a quadratic operator having the form (1.30), and acting on states in the Fock space. As in Chapter 3, we consider in the last Chapter 5 the time evolution of states initially close to the ground state of the system. In analogy to the norm approximations for low-energy eigenvectors in terms of eigenvectors of a quadratic Fock space Hamiltonian, we ask if also the full many-body time evolution can be described effectively by a unitary dynamics generated by a quadratic operator on the Fock space. Here, in contrast with the other parts of the thesis, we will not work in the Gross-Pitaevskii regime, but instead in slightly less singular scaling regimes as explained below.

1.4 Complete Bose-Einstein Condensation in the Gross-Pitaevskii Limit

In this section we present our main result on the complete Bose-Einstein condensation in the Gross-Pitaevskii regime. Our result is proved in [13]. The manuscript of [13] is provided in Chapter 2 and also appeared in the doctoral thesis [12] of Chiara Boccato in 2017, see [12, Chapter 3]. All authors contributed equally to the article [13]. More specific details about the individual contributions are provided in the introduction of Chapter 2. The summary of our main result, given in this section, is a modified and rephrased version of the introduction in [13].

We consider the Gross-Pitaevskii regime for N bosons moving in the box of volume one with periodic boundary conditions. The Hamilton operator H_N of the system reads

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + \kappa \sum_{1\leq i<j\leq N} N^2 V(N(x_i - x_j)) \tag{1.35}$$

and acts on (a dense subspace of) $L^2_s(\Lambda^N)$. We assume for positive coupling constant $\kappa > 0$ that the unscaled map $V \in L^3(\mathbb{R}^3)$ is non-negative, spherically symmetric and compactly supported[10]. By scaling, the scattering length of the interaction $\kappa N^2 V(N.)$ appearing in (1.35) is given by $\mathfrak{a}_0/N$ where $\mathfrak{a}_0$ denotes the scattering length of κV.

[10] Notice that for $N \gg r$, with r denoting the range of the unscaled map V, the scaled function $N^2V(N.)$ extends to a periodic function lying in $N^2V(N.) \in L^3(\mathbb{T}^3)$.

In the Gross-Pitaevskii regime, several important results concerning the ground state energy and the existence of Bose-Einstein condensation have been proved in the works [66, 73, 71, 79]. First of all, the results obtained in [73, 71, 79] show that the ground state energy E_N of (1.35) satisfies

$$\lim_{N\to\infty} \frac{E_N}{N} = 4\pi\mathfrak{a}_0 \tag{1.36}$$

In addition to the first order approximation of the ground state energy (1.36), it was shown in [66, 79] that the ground state of (1.35) exhibits Bose-Einstein condensation in the constant one-particle state $\varphi_0 \equiv 1_{|\Lambda}$. More precisely, if ψ_N denotes a normalized ground state vector for (1.35), and if $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$ denotes its one-particle reduced density, it follows from [66] that

$$\mathrm{tr}\,\big|\gamma_N^{(1)} - |\varphi_0\rangle\langle\varphi_0|\big| \to 0 \qquad (N\to\infty)$$

Recall from equation (1.16) in Section 1.2 that this is equivalent to

$$1 - \langle\varphi_0, \gamma_N^{(1)}\varphi_0\rangle \to 0 \qquad (N\to\infty) \tag{1.37}$$

We remark that the results in [71, 66] were actually more general and also applied to inhomogenous systems in the Gross-Pitaevskii regime, i.e. systems where the particles are usually trapped in a bounded subset in $\mathbb{R}^3$, see also the next Section 1.5. Similar results have been derived for rotating Bose gases in [67]. Finally, we point out that by following the arguments given in [66], one can also deduce a rate for the convergence in (1.37). This rate, however, is far from the optimal one.

Our main result in [13] is a proof of Bose-Einstein condensation via (1.37), assuming the coupling constant $\kappa \geq 0$ to be sufficiently small, with optimal rate of convergence. We have the following theorem, see [13, Theorem 1.1].

Theorem 1.4.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, spherically symmetric and compactly supported and assume the coupling constant $\kappa \geq 0$ to be small enough. Let $\psi_N \in L^2_s(\Lambda^N)$ be a sequence with $\|\psi_N\| = 1$ and such that*

$$\langle\psi_N, H_N\psi_N\rangle \leq 4\pi\mathfrak{a}_0 N + K \tag{1.38}$$

for some $K > 0$ for all $N \in \mathbb{N}$. Let $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}\,|\psi_N\rangle\langle\psi_N|$ be the one-particle reduced density associated with ψ_N. Then there exists a constant $C > 0$, depending on V and on κ, but independent of K, such that

$$1 - \langle\varphi_0, \gamma_N^{(1)}\varphi_0\rangle \leq \frac{C(K+1)}{N} \tag{1.39}$$

where $\varphi_0(x) = 1$ for all $x \in \Lambda$.

Furthermore, the ground state energy E_N of (1.35) is such that

$$|E_N - 4\pi\mathfrak{a}_0 N| \leq D \tag{1.40}$$

for a $D > 0$ independent of N (depending only on V and κ). Hence, the one-particle reduced density associated with the ground state of (1.35) satisfies (1.39), with K replaced by the constant D.

Before explaining the strategy of its proof, let us add a few comments on Theorem 1.4.1. First of all, Theorem 1.4.1 shows the validity of complete Bose-Einstein condensation in the Gross-Pitaveskii regime with optimal[11] condensation rate of order $\mathcal{O}(N^{-1})$. With regard to Bogoliubov theory, discussed in Section 1.3, this justifies rigorously Bogoliubov's basic assumption of condensation in the zero momentum state. From this point of view, our result can be seen as a first step towards a better mathematical understanding of Bogoliubov theory in the Gross-Pitaevskii limit.

Next, let us point out that the proof of Theorem 1.4.1 in [13] actually shows that the Hamiltonian H_N can be bounded from below by

$$H_N - 4\pi\mathfrak{a}_0 N \geq 4\pi^2 c \sum_{i=1}^{N} \big(1 - |\varphi_0\rangle\langle\varphi_0|_i\big) - C \tag{1.41}$$

for two constants $c, C > 0$. This inequality immediately implies (1.39) and it shows moreover that the ground state energy E_N of the Hamilton operator H_N is bounded from below by $E_N \geq 4\pi\mathfrak{a}_0 N + \mathcal{O}(1)$. Together with a similar upper bound which also follows from our analysis, this implies that $E_N = 4\pi\mathfrak{a}_0 N + \mathcal{O}(1)$. This improves the corresponding results of [71, 79] for the ground state energy in the Gross-Pitaevskii regime and is as such of independent interest.

Finally, we also mention that Theorem 1.4.1 can be generalized to the inhomogeneous setting in which the particles are trapped in a finite region in $\Lambda = \mathbb{R}^3$. This has been worked out in detail in the master thesis [92].

After these remarks on Theorem 1.4.1, let us now briefly explain the strategy for its proof. First of all, we follow [64] and use the map $U_N(\varphi_0) : L^2_s(\Lambda^N) \to \mathcal{F}^{\leq N}_{\perp\varphi_0}$, defined in (1.15), to work in the Fock space $\mathcal{F}^{\leq N}_{\perp\varphi_0}$ of excited particles. The map $U_N(\varphi_0)$ is unitary and enables us to define the excitation Hamiltonian

$$\mathcal{L}_N = U_N(\varphi_0) H_N U_N(\varphi_0)^*$$

mapping its dense domain in $\mathcal{F}^{\leq N}_{\perp\varphi_0}$ to $\mathcal{F}^{\leq N}_{\perp\varphi_0}$. As explained in detail in Section 2.3, the Hamiltonian $\mathcal{L}_N$ is given by the sum of the constant $\frac{N}{2}\kappa\widehat{V}(0) = \frac{N}{2}\int dx\; V(x)$ and of several operators which are either quadratic, cubic or quartic in the creation and annihilation operators. From the bound $\kappa\widehat{V}(0) > 8\pi\mathfrak{a}_0$, we observe that the constant $\frac{N}{2}\widehat{V}(0)$, explicitly contained in the excitation Hamiltonian $\mathcal{L}_N$, differs from the true ground state energy $E_N = 4\pi\mathfrak{a}_0 N + \mathcal{O}(1)$ by a quantity proportional to N. This indicates that important contributions to the ground state energy of $\mathcal{L}_N$ must still be contained in the quadratic, cubic and quartic operators mentioned above. In particular, this means that low-energy states of $\mathcal{L}_N$ are not close to the vacuum $\Omega = U_N(\varphi_0)\varphi_0^{\otimes N}$. It follows

[11] Optimality follows indeed from (1.69) in Chapter 4, where we actually compute the condensate depletion in the ground state of H_N, as $N \to \infty$.

already from [73, 71, 36, 37, 40, 39], that short-scale pair correlations, modeled by the solution f of the zero-energy scattering equation (1.27), play a crucial role in the correct description of low-energy states ψ_N of H_N. To implement such correlations, we follow the strategy of [20], which is based on important ideas from [11], and use *generalized Bogoliubov transformations*. For a suitable kernel $\eta \in L^2_{\perp\varphi_0}(\Lambda) \otimes_s L^2_{\perp\varphi_0}(\Lambda)$, these maps are unitary operators of the form

$$T(\eta) = \exp\left\{\frac{1}{2}\sum_{p\in\Lambda^*\setminus\{0\}} \eta_p \left[b^*_p b^*_{-p} - b_p b_{-p}\right]\right\} : \mathcal{F}^{\leq N}_{\perp\varphi_0} \to \mathcal{F}^{\leq N}_{\perp\varphi_0} \tag{1.42}$$

with b_p, b^*_p defined as in (1.22).

Using $T(\eta)$, one may hope that states of the form $\psi_N \simeq U_N(\varphi_0)^* T(\eta)\Omega$ correctly describe low-energy states of H_N, in the sense that the expectation of H_N with respect to such states yields the correct ground state energy $E_N = 4\pi\mathfrak{a}_0 N + \mathcal{O}(1)$. Notice that, compared to the product state $\varphi_0^{\otimes N} = U_N(\varphi_0)^*\Omega$, such states have the form

$$U_N(\varphi_0)^* T(\eta)\Omega = \exp\left[\frac{1}{2}\int dxdy\; \eta(x-y)\varphi_0(x)\varphi_0(y) a^*_x a^*_y a_0 a_0 - \text{h.c.}\right]\varphi_0^{\otimes N}$$

That is, starting from a pure product state $\varphi_0^{\otimes N}$, we may think of the map $T(\eta)$ as replacing uncorrelated particle pairs of the form $(x,y) \mapsto \varphi_0(x)\varphi_0(y)$ by correlated pairs of the form $(x,y) \mapsto \eta(x-y)\varphi_0(x)\varphi_0(y)$. As turns out indeed in Chapter 2, any low-energy state ψ_N satisfying (1.38) can be written as $\psi_N = U_N(\varphi_0)^* T(\eta)\xi_N$ where the excitation vector ξ_N satisfies $\langle \xi_N, \mathcal{N}_{\perp\varphi_0}\xi_N\rangle \leq C + K$, uniformly as $N \to \infty$. This motivates us to define a new excitation Hamiltonian

$$\mathcal{G}_N = T(\eta)^* \mathcal{L}_N T(\eta) = T(\eta)^* U_N(\varphi_0) H_N U_N(\varphi_0)^* T(\eta)$$

Similarly to $\mathcal{L}_N$, also $\mathcal{G}_N$ consists of a sum of several terms, but in this case the constant term that it contains is given by $4\pi\mathfrak{a}_0 N + \mathcal{O}(1)$. Moreover, we show that, for sufficiently small coupling constant $\kappa > 0$ of the interaction, there exist positive constants $c, C > 0$ such that $\mathcal{G}_N$ satisfies the operator bound

$$\mathcal{G}_N - 4\pi\mathfrak{a}_0 N \geq c\mathcal{N}_{\perp\varphi_0} - C$$

Together with the fact that generalized Bogoliubov transformations $T(\eta)$ do not substantially change the expectation of the number of particles operator $\mathcal{N}_{\perp\varphi_0}$, this enables us to deduce (1.39). Moreover, by using a similar upper bound on $\mathcal{G}_N$, we complete the proof of Theorem 1.4.1.

1.5 Dynamics of Bose-Einstein Condensates in the Gross-Pitaevskii Limit

In this section we summarize the results of Chapter 3, which reproduces the manuscript [20], on the dynamical stability of Bose-Einstein condensates in the Gross-Pitaevskii regime. The manuscript of [20] is provided in Chapter 3. The introduction given in this section is a modified and rephrased version of the introduction in [20].

In Section 1.4, we have seen that, in the Gross-Pitaevskii regime, low-energy states exhibit complete Bose-Einstein condensation. Since Bose-Einstein condensates are observed in experiments where particles are initially trapped at extremely low temperatures and measurements carried out after releasing the traps, see [7, 29], it is natural to ask whether the evolution of initial data close to the ground state of a trapped Hamiltonian still exhibits condensation after a positive time $t > 0$. To describe experiments where particles are initially trapped and evolve in time after releasing the traps, we consider first the Hamilton operator

$$H_N^{\text{trap}} = \sum_{i=1}^{N} [-\Delta_{x_i} + V_{\text{ext}}(x_j)] + \sum_{1 \le i < j \le N} N^2 V(N(x_i - x_j)) \tag{1.43}$$

Here, V_{ext} is a confining external potential and we assume the interaction potential V to be non-negative, spherically symmetric and compactly supported.

As shown in [66, 71, 79], the ground state energy E_N of (1.43) satisfies

$$\lim_{N\to\infty} \frac{E_N}{N} = \min_{\varphi \in \mathcal{D}_{\text{GP}} : \|\varphi\|_2 = 1} \mathcal{E}_{\text{GP}}^{\text{trap}}(\varphi) \tag{1.44}$$

where $\mathcal{D}_{\text{GP}} = L^2(\mathbb{R}^3, V_{\text{ext}}(x) \ dx) \cap H^1(\mathbb{R}^3)$ and with the Gross-Pitaevskii energy functional $\mathcal{E}_{\text{GP}}^{\text{trap}} : \mathcal{D}_{\text{GP}} \to \mathbb{R}$, defined by

$$\mathcal{E}_{\text{GP}}^{\text{trap}}(\varphi) = \int \left[|\nabla \varphi(x)|^2 + V_{\text{ext}}(x) |\varphi(x)|^2 + 4\pi \mathfrak{a}_0 |\varphi(x)|^4 \right] dx \tag{1.45}$$

Moreover, it was proven in [66, 79] that the ground state of (1.43) exhibits complete Bose-Einstein condensation in the minimizer $\phi_{\text{GP}} \in L^2(\mathbb{R}^3)$ of (1.45). In other words, if $\gamma_N^{(1)} = \text{tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|$ denotes the one-particle reduced density associated with the ground state of (1.43), then

$$\text{tr} \left| \gamma_N^{(1)} - |\phi_{\text{GP}}\rangle\langle\phi_{\text{GP}}| \right| \to 0 \qquad (N \to \infty) \tag{1.46}$$

The first main result of this section describes the evolution of initial data close to the ground state of (1.43) (in the sense of (1.48)), with respect to the Hamilton operator

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + \sum_{1 \le i < j \le N} N^2 V(N(x_i - x_j)) \tag{1.47}$$

obtained from (1.43) by switching off the traps. We state our result as the following theorem, see [20, Theorem 1.1].

Theorem 1.5.1. *Let $V_{ext} : \mathbb{R}^3 \to \mathbb{R}$ be locally bounded with $V_{ext}(x) \to \infty$ as $|x| \to \infty$. Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric. Let*

ψ_N be a sequence in $L^2_s(\mathbb{R}^{3N})$, with one-particle reduced density $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$. We assume that, as $N \to \infty$,

$$\begin{aligned} a_N &= 1 - \langle \phi_{GP}, \gamma_N^{(1)} \phi_{GP}\rangle \to 0 \qquad \text{and} \\ b_N &= \left| N^{-1}\langle \psi_N, H_N^{trap}\psi_N\rangle - \mathcal{E}_{GP}^{trap}(\phi_{GP})\right| \to 0 \end{aligned} \tag{1.48}$$

where $\phi_{GP} \in H^4(\mathbb{R}^3)$ is the unique non-negative minimizer of the Gross-Pitaevskii energy functional (1.45). Let $\psi_{N,t} = e^{-iH_N t}\psi_N$ be the solution of the Schrödinger equation (1.3) with initial data ψ_N and with H_N defined in (1.47). Denote by $\gamma_{N,t}^{(1)}$ the one-particle reduced density associated with $\psi_{N,t}$. Then there are constants $C, c > 0$ such that

$$1 - \langle \varphi_t, \gamma_{N,t}^{(1)} \varphi_t\rangle \leq C\left[a_N + b_N + N^{-1}\right] \exp\left(c \exp\left(c|t|\right)\right) \tag{1.49}$$

for all $t \in \mathbb{R}$. Here, $t \mapsto \varphi_t$ solves the time-dependent Gross-Pitaevskii equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t \tag{1.50}$$

with the initial data $\varphi_{t=0} = \phi_{GP}$.

Remarks:

1) Existence and uniqueness of the minimizer ϕ_{GP} of the Gross-Pitaevskii functional (1.45) was proved in [71]. In Theorem 1.5.1 we also make the assumption that $\phi_{\mathrm{GP}} \in H^4(\mathbb{R}^3)$. Under suitable assumptions on the external potential V_{ext} (it is enough to assume that $V_{\mathrm{ext}} \in C^2(\mathbb{R}^3)$ as well as its derivatives grow at most exponentially at infinity), this follows from elliptic regularity and [51].

2) It follows from (1.44) and (1.46) that the assumptions $a_N, b_N \to 0$ are satisfied if ψ_N is chosen as the ground state of (1.43). Furthermore, if we assume the interaction V to be sufficiently weak, the assumptions (1.48) are satisfied for any low-energy state in the sense of (1.38) (with H_N replaced by H_N^{trap}). This follows from the third remark after Theorem (1.4.1) of Section 1.4 showing that $a_N, b_N = \mathcal{O}(N^{-1})$. In this case, the rate $a_N = \mathcal{O}(N^{-1})$ is optimal so that Theorem 1.5.1 shows that the optimal condensation rate is preserved in time after releasing the traps.

Theorem 1.5.1 describes the time-evolution of initially trapped low-energy states of the Hamiltonian (1.43). More generally, one may ask if Bose-Einstein condensation is also preserved in time if the system initially exhibits condensation in an arbitrary one-particle condensate wavefunction $\varphi \in H^1(\mathbb{R}^3)$ (which does not necessarily minimize the Gross-Pitaevskii functional). Our second main result of [20] shows that this is indeed the case if the assumption $a_N \to 0$ (as $N \to \infty$) is replaced by a slightly stronger condition. We state it as follows, see [20, Theorem 1.2].

Theorem 1.5.2. *Assume that $V \in L^3(\mathbb{R}^3)$ is non-negative, compactly supported and spherically symmetric. Let ψ_N be a sequence in $L^2_s(\mathbb{R}^{3N})$, with one-particle reduced density $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$. Assume that, for a $\varphi \in H^4(\mathbb{R}^3)$,*

$$\begin{aligned} \widetilde{a}_N &= \mathrm{tr}\,\big|\gamma_N^{(1)} - |\varphi\rangle\langle\varphi|\big| \to 0 \qquad \textit{and} \\ \widetilde{b}_N &= \Big|N^{-1}\langle\psi_N, H_N\psi_N\rangle - \mathcal{E}_{GP}(\varphi)\Big| \to 0 \end{aligned} \tag{1.51}$$

as $N \to \infty$. Here $\mathcal{E}_{GP}$ is the translation invariant Gross-Pitaevskii functional

$$\mathcal{E}_{GP}(\varphi) = \int \big[|\nabla\varphi|^2 + 4\pi a_0|\varphi|^4\big]\,dx \tag{1.52}$$

Let $\psi_{N,t} = e^{-iH_Nt}\psi_N$ be the solution of the Schrödinger equation (1.3) with initial data ψ_N and where H_N denotes the translation invariant Hamiltonian defined in (1.47). Let $\gamma_{N,t}^{(1)}$ denote the one-particle reduced density associated with $\psi_{N,t}$. Then

$$1 - \langle\varphi_t, \gamma_{N,t}^{(1)}\varphi_t\rangle \leq C\left[\widetilde{a}_N + \widetilde{b}_N + N^{-1}\right]\exp\left(c\exp\left(c|t|\right)\right) \tag{1.53}$$

where $t \mapsto \varphi_t$ solves the time-dependent Gross-Pitaevskii equation (1.50).

Before explaining the strategy for the proof of Theorems 1.5.1 and 1.5.2, let us comment on earlier derivations of the Gross-Pitaevskii equation (1.50). The first rigorous derivation of the time-dependent Gross-Pitaevskii equation (1.50), starting from the microscopic Schrödinger dynamics, was given in [36, 37, 40, 39]. Part of the proof was simplified in [25], using also ideas from [56]. In these derivations, the convergence of the reduced one-particle density of the Schrödinger evolution towards the rank-one projection associated with the solution of the Gross-Pitaevskii equation was proved without control of the convergence rate. A different derivation of the Gross-Pitaevskii equation was given in [85] where it was shown that the convergence rate is of the order $\mathcal{O}(N^{-\eta})$, for some $\eta > 0$. The results of [85] were generalized in [74, 83, 54] and the methods of [85] were also used in [53] to derive the time-dependent Gross-Pitaevskii equation in two space dimensions. In a slightly different setting, the recent work [11] proves the convergence with a rate similar to (1.49), (1.53). In this work, Bose-Einstein condensates are modeled by approximately coherent states on the full bosonic Fock space. Compared to these previously known results, the novelty of (1.49), (1.53) is that they provide an explicit and optimal rate determined by the properties of the N-particle initial data.

Let us now briefly explain the strategy for the proofs of Theorem 1.5.1 and Theorem 1.5.2. In our approach, we combine ideas from [64] and [11]. First of all, we follow [64] and use the map $U_N(\varphi_t) : L^2_s(\mathbb{R}^{3N}) \to \mathcal{F}^{\leq N}_{\perp\varphi_t}$, defined in (1.15), to work in the Fock space $\mathcal{F}^{\leq N}_{\perp\varphi_t}$ of excited particles. Note that $\mathcal{F}^{\leq N}_{\perp\varphi_t}$ now depends on time, since $t \mapsto \varphi_t$ denotes the solution of the Gross-Pitaevskii equation (1.50). As explained in detail in Chapter 3, the bound (1.49), and similarly (1.53), follows if we can show that

$$\langle\widetilde{\mathcal{W}}_t\widetilde{\xi}_N, \mathcal{N}_{\perp\varphi_t}\widetilde{\mathcal{W}}_t\widetilde{\xi}_N\rangle \leq C_t(Na_N + Nb_N + 1) \tag{1.54}$$

where $\mathcal{N}_{\perp\varphi_t}$ denotes the number of particles operator in $\mathcal{F}_{\perp\varphi_t}^{\leq N}$, $C_t > 0$ denotes a constant which is independent of N, $\widetilde{\xi}_N$ denotes the excitation vector $\widetilde{\xi}_N = U_N(\varphi_{t=0})\psi_N$ and $\widetilde{\mathcal{W}}_t$ denotes the unitary fluctuation dynamics

$$\widetilde{\mathcal{W}}_t = U_N(\varphi_t)e^{-iH_N t}U_N(\varphi_{t=0})^* : \mathcal{F}_{\perp\varphi_{t=0}}^{\leq N} \to \mathcal{F}_{\perp\varphi_t}^{\leq N} \tag{1.55}$$

As a typical strategy to prove a bound like (1.54), one may try to apply a Gronwall estimate to the time-dependent function $t \mapsto \langle \widetilde{\mathcal{W}}_t\widetilde{\xi}_N, \mathcal{N}_{\perp\varphi_t}\widetilde{\mathcal{W}}_t\widetilde{\xi}_N\rangle$. If one follows this idea, a short computation shows, however, that the time derivative of the function $t \mapsto \langle \widetilde{\mathcal{W}}_t\widetilde{\xi}_N, \mathcal{N}_{\perp\varphi_t}\widetilde{\mathcal{W}}_t\widetilde{\xi}_N\rangle$ contains large terms which are hard to control uniformly in N. Recalling the definition of the map $U_N(\varphi_t)$ in (1.14), the reason for this is that the decomposition (1.15) expands the evolved state $\psi_{N,t} = e^{-iH_N t}\psi_N$ around the uncorrelated product state $\varphi_t^{\otimes N}$. As mentioned in the previous Section 1.4, however, short-scale pair correlations play a crucial role in the Gross-Pitaevskii regime. Consequently, we should modify the fluctuation dynamics $\widetilde{\mathcal{W}}_t$ in order to incorporate pair correlations.

To reach this goal, we adapt the approach of [11], where correlations were introduced by means of Bogoliubov transformations having the form

$$\widetilde{T}_t = \exp\left[\frac{1}{2}\int dxdy\,\left(k_t(x;y)a_x^*a_y^* - \text{h.c.}\right)\right] : \mathcal{F} \to \mathcal{F} \tag{1.56}$$

whose action on creation and annihilation operators is explicitly given by

$$\widetilde{T}_t^* a^*(f)\widetilde{T}_t = a^*(\cosh_{k_t}(f)) + a(\sinh_{k_t}(\bar{f})) \tag{1.57}$$

Here, $k_t \in L_s^2(\mathbb{R}^3 \times \mathbb{R}^3)$ is an appropriate symmetric kernel and $\cosh_{k_t}$ and $\sinh_{k_t}$ are the bounded operator series

$$\cosh_{k_t} = \sum_{n=0}^{\infty} \frac{(k_t\bar{k}_t)^n}{(2n)!}, \qquad \sinh_{k_t} = \sum_{n=0}^{\infty} \frac{(k_t\bar{k}_t)^n k_t}{(2n+1)!}$$

Unfortunately, transformations of the form (1.56) do not respect the truncation of the excitation Fock space $\mathcal{F}_{\perp\varphi_t}^{\leq N}$, i.e. states with at most N particles are mapped to vectors on the full Fock space $\mathcal{F}$, without restriction on the number of particles. To circumvent this problem, we define generalized Bogoliubov transformations of the form

$$T_t = \exp\left[\frac{1}{2}\int dxdy\,\left(\eta_t(x;y)b_x^*b_y^* - \text{h.c.}\right)\right] : \mathcal{F}_{\perp\varphi_t}^{\leq N} \to \mathcal{F}_{\perp\varphi_t}^{\leq N} \tag{1.58}$$

for $\eta_t \in L^2_{\perp\varphi_t}(\mathbb{R}^3)\otimes_s L^2_{\perp\varphi_t}(\mathbb{R}^3)$. While the operators T_t map $\mathcal{F}_{\perp\varphi_t}^{\leq N}$ to itself, their action on creation and annihilation operators is not explicit. Hence, to prove Theorem 1.5.1 and Theorem 1.5.2, we have to show that

$$T_t^* b^*(f)T_t = b^*(\cosh_{\eta_t}(f)) + b(\sinh_{\eta_t}(\bar{f})) + d_\eta(f,\bar{f}) \tag{1.59}$$

where the operator $d_\eta(f,\bar f) : \mathcal{F}^{\leq N}_{\perp\varphi_t} \to \mathcal{F}^{\leq N}_{\perp\varphi_t}$ is a bounded error operator which satisfies

$$\big\|d_\eta(f,\bar f)\xi\big\| \leq \frac{C_{\eta_t}\|f\|_2\|\mathcal{N}^{3/2}_{\perp\varphi_t}\xi\|}{N} \qquad (\xi \in \mathcal{F}^{\leq N}_{\perp\varphi_t})$$

We postpone more details to Sections 3.2 and 3.4.1 of Chapter 3.

With the generalized Bogoliubov transformations T_t, defined in (1.58), we can define the modified unitary fluctuation dynamics $\mathcal{W}_t$ by

$$\mathcal{W}_t = T_t^* U_N(\varphi_t) e^{-iH_N t} U_N(\varphi_{t=0})^* T_{t=0} : \mathcal{F}^{\leq N}_{\perp\varphi_{t=0}} \to \mathcal{F}^{\leq N}_{\perp\varphi_t} \tag{1.60}$$

Using the fact that $T_t^*\mathcal{N}_{\perp\varphi_t}T_t \leq C(\mathcal{N}_{\perp\varphi_t}+1)$, uniformly in N, we may then try to apply a Gronwall estimate on the expectation $t \mapsto \langle \mathcal{W}_t\xi_N, \mathcal{N}_{\perp\varphi_t}\mathcal{W}_t\xi_N\rangle$, where $\xi_N = T^*_{t=0}U_N(\varphi_{t=0})\psi_N$. To be more precise, it turns out that it possible to apply Gronwall's inequality on the sum of the time dependent generator $\mathcal{G}_t$ of the fluctuation dynamics $\mathcal{W}_t$, defined by $i\partial_t\mathcal{W}_t = \mathcal{G}_t\mathcal{W}_t$, and the number of particles operator $\mathcal{N}_{\perp\varphi_t}$. This enables us to prove the existence of a constant $C_t > 0$, which is independent of N, such that

$$\langle \mathcal{W}_t, \mathcal{N}_{\perp\varphi_t}\mathcal{W}_t\rangle \leq C_t\langle \mathcal{W}_{t=0}, \big[\mathcal{G}_{t=0} + \mathcal{N}_{\perp\varphi_{t=0}}\big]\mathcal{W}_{t=0}\rangle \tag{1.61}$$

Inserting the initial conditions (1.48) and (1.51), respectively, into the right hand side of (1.61), we conclude (1.54) and deduce Theorem 1.5.1 and Theorem 1.5.2.

1.6 The Excitation Spectrum of Bose Gases in the Gross-Pitaevskii Limit

In this section, we present our main result on the low-energy spectrum of Bose gases in the Gross-Pitaevskii regime. Our results confirm with mathematical rigor the predictions of Bogoliubov for the ground state energy and the low-energy excitation spectrum, as described in Section 1.3. The results are proved in the article [15], provided in Chapter 4. The introduction given in this section is adapted from [15, Section 1].

As in Section 1.4, we consider systems of N bosons moving in the unit box with periodic boundary conditions, i.e. we set $\Lambda = \mathbb{T}^3$. The Hamiltonian reads

$$H_N = \sum_{i=1}^N -\Delta_{x_i} + \kappa \sum_{1\leq i<j\leq N} N^2 V(N(x_i-x_j)) \tag{1.62}$$

The coupling constant $\kappa > 0$ is later chosen to be sufficiently small and we assume $V \in L^3(\mathbb{R}^3)$ to be non-negative, radially symmetric and compactly supported. The scattering length of the unscaled potential κV is denoted by $\mathfrak{a}_0$. As a consequence, the scattering length of the two-body interaction $\kappa N^2 V(N.)$ in (1.62) is given by $\mathfrak{a}_0/N$.

In Section 1.4, we have seen that, for sufficiently small values of the coupling constant $\kappa > 0$, the ground state energy E_N of the Hamilton operator H_N is given by

$$E_N = 4\pi\mathfrak{a}_0 N + \mathcal{O}(1) \tag{1.63}$$

In addition, the one-particle reduced density $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}\, |\psi_N\rangle\langle\psi_N|$ associated to any low-energy state $\psi_N \in L^2_s(\Lambda^N)$, satisfying $\langle \psi_N, H_N \psi_N\rangle \leq 4\pi\mathfrak{a}_0 N + C$, is such that

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0\rangle \leq C N^{-1} \tag{1.64}$$

for a constant $C > 0$, which is independent of N. Recall that $\varphi_0(x) = 1$ for all $x \in \Lambda$. While equation (1.63) determines the ground state energy of H_N up to leading order in N, equation (1.64) verifies Bogoliubov's fundamental assumption of complete Bose-Einstein condensation for low-energy states of the system, as explained in Section 1.3.

In this section we go one step further: Our main results verify the predictions of Bogoliubov theory for the low-energy spectrum of H_N. Up to errors that vanish in the limit $N \to \infty$, we verify Bogoliubov's predictions (1.31) and (1.32) on the $\mathcal{O}(1)$ contribution to the ground state energy E_N in (1.63) and on the low-energy excitation spectrum of H_N. We state our result as the following theorem, see [15, Theorem 1.1].

Theorem 1.6.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, spherically symmetric, compactly supported and assume that the coupling constant $\kappa > 0$ is small enough. Then, in the limit $N \to \infty$, the ground state energy E_N of the Hamilton operator H_N defined in (1.62) is given by*

$$\begin{aligned} E_N &= 4\pi(N-1)\mathfrak{a}_N \\ &\quad - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] + \mathcal{O}(N^{-1/4}) \end{aligned} \tag{1.65}$$

Here we introduced the notation $\Lambda_+^* = 2\pi\mathbb{Z}^3\backslash\{0\}$ and we defined

$$8\pi\mathfrak{a}_N = \kappa\widehat{V}(0) + \sum_{k=1}^{\infty}\frac{(-1)^k\kappa^{k+1}}{(2N)^k}\sum_{p_1,\dots,p_k\in\Lambda_+^*}\frac{\widehat{V}(p_1/N)}{p_1^2}\left(\prod_{i=1}^{k-1}\frac{\widehat{V}((p_i-p_{i+1})/N)}{p_{i+1}^2}\right)\widehat{V}(p_k/N) \tag{1.66}$$

Moreover, the spectrum of $H_N - E_N$ below a threshold ζ consists of eigenvalues given, in the limit $N \to \infty$, by

$$\sum_{p\in\Lambda_+^*} n_p\sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} + \mathcal{O}(N^{-1/4}(1+\zeta^3)) \tag{1.67}$$

Here $n_p \in \mathbb{N}$ for all $p \in \Lambda_+^*$ and $n_p \neq 0$ for finitely many $p \in \Lambda_+^*$ only.

Let us compare the second order ground state energy approximation (1.65) with Bogoliubov's prediction (1.31). In our setting, the side length L of the box is $L = 1$ which implies that the system's density is equal to $\rho = N$. The scattering length of the two-body interaction in the Hamiltonian (1.62) is given by $\mathfrak{a}_0/N$, where $\mathfrak{a}_0$ denotes the scatttering length of the unscaled potential κV. Hence, we have $(\mathfrak{a}_0/N)\rho = \mathfrak{a}_0$ such that the only difference between (1.65) and (1.31) lies in the leading order terms $4\pi(N-1)\mathfrak{a}_N$

and $4\pi(N-1)\mathfrak{a}_0$, respectively. The constant $\mathfrak{a}_N$ is in general different from the infinite volume scattering length $\mathfrak{a}_0$. In fact, we can compare the series (1.66) with the Born series of the scattering length $\mathfrak{a}_0$, given by

$$8\pi\mathfrak{a}_0 = \kappa\widehat{V}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k \kappa^{k+1}}{2^k(2\pi)^{3k}} \int_{\mathbb{R}^{3k}} dp_1 \dots dp_k \, \frac{\widehat{V}(p_1)}{p_1^2} \left(\prod_{i=1}^{k-1} \frac{\widehat{V}(p_i - p_{i+1})}{p_{i+1}^2} \right) \widehat{V}(p_k) \tag{1.68}$$

For sufficiently small $\kappa > 0$, both series are absolutely convergent and we recognize that the series in (1.66) is a Riemann sum for (1.68). We show in Chapter 4 that we can bound the difference between $\mathfrak{a}_N$ and $\mathfrak{a}_0$ by

$$|\mathfrak{a}_N - \mathfrak{a}_0| \le \frac{C\kappa^2}{N} \, .$$

Numerical computations for simple choices of V suggest, however, that the difference $|\mathfrak{a}_N - \mathfrak{a}_0|$ is really of the order $\mathcal{O}(N^{-1})$, and not smaller. Thus, in the Gross-Pitaevskii limit, we can not replace the constant $\mathfrak{a}_N$ by the infinite volume scattering length $\mathfrak{a}_0$.

Apart from this difference, let us stress that Bogoliubov's prediction for the excitation energies of $H_N - E_N$, given in (1.32), coincides exactly with our result (1.67), up to an error that vanishes in the limit $N \to \infty$. This is quite remarkable if one considers the fact that Bogoliubov, only relying on physical intuition, simply replaced several operators by real numbers, neglected some of them completely and, in addition to that, replaced the first and second Born approximations for the scattering length $\mathfrak{a}_0$, which differ from $\mathfrak{a}_0$ by a quantity of order $\mathcal{O}(1)$, by the scattering length $\mathfrak{a}_0$ itself. More precisely, from the mathematical point of view, Bogoliubov's approximation steps, in particular dropping cubic and quartic operators and replacing $\mathfrak{a}_0^{(1)}, \mathfrak{a}_0^{(2)} \to \mathfrak{a}_0$, produce error terms which are in fact of the order $\mathcal{O}(N)$. Therefore, the derivation proposed by Bogoliubov and described in Section 1.3 is certainly not correct in the Gross-Pitaevskii regime. Nevertheless, Bogoliubov ends up with the correct energy predictions. The mathematical reason for this is that the missing energy that Bogoliubov simply inserts by replacing the Born-approximations of $\mathfrak{a}_0$ by $\mathfrak{a}_0$ itself, are indeed hidden in the cubic and quartic operators which he discarded one step earlier! The mathematical difficulty in the proof of Theorem 1.6.1 consists therefore in the rigorous extraction of these energies from the cubic and quartic operators. We explain our strategy to achieve this goal below.

Finally, let us point out that Theorem 1.6.1 determines the low-energy eigenvalues of the Hamiltonian H_N. Using standard arguments, we derive in Chapter 4 also norm convergent approximations for the associated eigenvectors in $L^2_s(\Lambda^N)$. As is explained in detail in Chapter 4, these approximations are, up to a unitary transformation, in fact the eigenvectors of the quadratic operator (1.30) (where we set $L = 1$, $\rho\mathfrak{a}_0$ is replaced by $\mathfrak{a}_0$ and $E_{N,L}$ is replaced by E_N, the ground state energy of H_N), viewed as an operator on the Fock space $\mathcal{F}^{\le N}_{\perp\varphi_0}$ of excited particles. As an application, we are able to compute the condensate depletion, i.e. the expected number of excitations of the condensate, in the ground state ψ_N of (1.62). If $\gamma_N^{(1)}$ denotes the one-particle reduced density associated with the ground state ψ_N, we obtain

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle = \frac{1}{N} \sum_{p \in \Lambda_+^*} \left[\frac{p^2 + 8\pi \mathfrak{a}_0 - \sqrt{p^4 + 16\pi \mathfrak{a}_0 p^2}}{2\sqrt{p^4 + 16\pi \mathfrak{a}_0 p^2}} \right] + \mathcal{O}(N^{-9/8}) \tag{1.69}$$

While Theorem 1.6.1 describes the excitation spectrum for weakly interacting Bose gases in the Gross-Pitaevskii regime, previously known derivations are available for the following related scaling regimes. The first rigorous derivation of the excitation spectrum was obtained for mean-field systems on the unit torus in [96]. This result was generalized to inhomogeneous mean-field systems in [46] where the authors also conjectured the form of the excitation spectrum in the Gross-Pitaevskii regime (our result (1.67) indeed proves [46, Conjecture 1] in the setting where $\Lambda = \mathbb{T}^3$). A different derivation of the low-energy spectrum of mean-field bosons, covering the results of [96, 46], was then given in [64]. The work [30] introduced another approach for the derivation of the low-energy spectrum, valid in a combined mean-field and infinite volume limit, extending the results of [96]. In [89, 90, 91], the ground state energy of mean-field Hamiltonians with ultraviolet cutoff was derived as a function in powers of N^{-1}. Finally, the recent article [14] provides a derivation of the excitation spectrum of Bose gases interacting through singular potentials of the form $\kappa N^{3\beta-1} V(N^\beta(.))$ for all $\beta \in (0;1)$.

Let us now sketch the proof of Theorem 1.6.1. As in the previous Sections 1.4 and 1.5, we follow first of all [64] and use the map $U_N(\varphi_0)$, defined in Section 1.2, to switch from $L^2_s(\Lambda^N)$ to a setting in the excitation Fock space $\mathcal{F}_{\perp\varphi_0}^{\leq N}$. From the results of [13], discussed in Section 1.4, we know already that we can extract the leading order term $4\pi\mathfrak{a}_0 N$ of the ground state energy E_N of (1.62) by employing generalized Bogoliubov transformations of the form

$$T(\eta) = \exp\left[\frac{1}{2} \sum_{p \in \Lambda_+^*} \eta_p \left(b_p^* b_{-p}^* - b_p b_{-p}\right)\right] : \mathcal{F}_{\perp\varphi_0}^{\leq N} \to \mathcal{F}_{\perp\varphi_0}^{\leq N} \tag{1.70}$$

Here, the coefficients η_p are related to a suitable modification of the solution of the zero-energy scattering equation (1.27), and the operators b_p^*, b_p are the modified creation and annihilation operators introduced in (1.22). Although the action of $T(\eta)$ on creation and annihilation operators is not explicit, we know from [20, 13, 14] that

$$\begin{aligned} T(\eta)^* b_p T(\eta) &= \cosh(\eta_p) b_p + \sinh(\eta_p) b_{-p}^* + d_p \\ T(\eta)^* b_p^* T(\eta) &= \cosh(\eta_p) b_p^* + \sinh(\eta_p) b_{-p} + d_p^* \end{aligned} \tag{1.71}$$

for remainder operators d_p that are small on states with few excitations.

Using the generalized Bogoliubov transformation $T(\eta)$, we define the excitation Hamiltonian $\mathcal{G}_N$ by

$$\mathcal{G}_N = T(\eta)^* U_N(\varphi_0) H_N U_N(\varphi_0)^* T(\eta)$$

Hence, $\mathcal{G}_N$ is unitarily equivalent to H_N and maps its dense domain in $\mathcal{F}_{\perp\varphi_0}^{\leq N}$ into $\mathcal{F}_{\perp\varphi_0}^{\leq N}$. The main result of [13] shows that $\mathcal{G}_N$ can be written as

$$\mathcal{G}_N = 4\pi\mathfrak{a}_0 N + \mathcal{H}_N + \Delta_N \tag{1.72}$$

where $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N = \int dx\, a_x^*(-\Delta_x)a_x + \frac{\kappa}{2}\int dxdy\, N^2V(N(x-y))a_x^*a_y^*a_xa_y$, while the operator Δ_N is an error term. The latter has the property that for every $\delta > 0$ there exists a constant $C > 0$ such that

$$\pm\,\Delta_N \leq \delta\mathcal{H}_N + C\kappa(\mathcal{N}_{\perp\varphi_0} + 1) \tag{1.73}$$

where $\mathcal{N}_{\perp\varphi_0}$ is the number of particles operator on $\mathcal{F}_{\perp\varphi_0}^{\leq N}$. This result implies that low-energy states of (1.62) exhibit complete Bose-Einstein condensation.

In view of the proof of Theorem 1.6.1, the a priori information from (1.73) on low-energy states is not enough. Apart from information on the average number of excitations of low-energy states, we also need to control their energy. To this end, we combine (1.73) with analogous bounds for the commutator of $\mathcal{G}_N$ with $\mathcal{N}_{\perp\varphi_0}$. This enables us to prove that, if $\psi_N \in L^2_s(\Lambda^N)$ is in the spectral subspace of H_N with threshold ζ, i.e. $\psi_N = \chi(H_N - E_N \leq \zeta)\psi_N$, then its corresponding excitation vector $\xi_N = T(\eta)^*U_N(\varphi_0)\psi_N$ satisfies

$$\left\langle \xi_N, \left[(\mathcal{H}_N + 1)(\mathcal{N}_{\perp\varphi_0} + 1) + (\mathcal{N}_{\perp\varphi_0} + 1)^3\right]\xi_N\right\rangle \leq C(1+\zeta^3), \tag{1.74}$$

uniformly in N. With this substantially stronger a priori information on low-energy states we can take a closer look at $\mathcal{G}_N$ to discard several contributions which are negligible on low-energy states. Our analysis yields that $\mathcal{G}_N$ can be written as

$$\mathcal{G}_N = C_{\mathcal{G}_N} + \mathcal{Q}_{\mathcal{G}_N} + \mathcal{C}_N + \mathcal{H}_N + \mathcal{E}_{\mathcal{G}_N} \tag{1.75}$$

where $C_{\mathcal{G}_N}$ is a constant, $\mathcal{Q}_{\mathcal{G}_N}$ is quadratic in the modified creation and annihilation fields, $\mathcal{C}_N$ is the cubic operator, defined by

$$\mathcal{C}_N = \frac{\kappa}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*:q\neq -p}\widehat{V}(p/N)\left[b^*_{p+q}b^*_{-p}(b_q\cosh(\eta_q) + b^*_{-q}\sinh(\eta_q)) + \text{h.c.}\right], \tag{1.76}$$

and $\mathcal{E}_{\mathcal{G}_N}$ is an error operator which is negligible on low-energy states due to (1.74) and

$$\pm\,\mathcal{E}_{\mathcal{G}_N} \leq CN^{-1/2}\Big[(\mathcal{H}_N + 1)(\mathcal{N}_{\perp\varphi_0} + 1) + (\mathcal{N}_{\perp\varphi_0} + 1)^3\Big] \tag{1.77}$$

Up until the decomposition (1.75), our analysis is similar to the one of [14], where we considered intermediate regimes with interactions having the form $\kappa N^{3\beta-1}V(N^\beta(.))$, for $\beta \in (0;1)$. In [14], a similar bound to (1.77) was already enough to derive the excitation spectrum up to errors vanishing in the limit of large N. In the Gross-Pitaevskii limit where $\beta = 1$, this is no longer the case: Our analysis shows that the cubic operator $\mathcal{C}_N$ and the quartic potential energy operator $\mathcal{V}_N$ contain crucial contributions of order $\mathcal{O}(1)$ to the energy. To extract these important contributions, we conjugate $\mathcal{G}_N$ with a unitary transformation $S(\eta) = e^{A(\eta)}$. Compared to the generalized Bogoliubov transformations (1.70), the exponent $A(\eta)$ of the operator $S(\eta)$ is cubic rather than quadratic in the (modified) creation and annihilation operators. A similar idea was used in a different setting in [99] to derive an upper bound on the ground state energy of a dilute Bose gas in the thermodynamic limit, consistent with the Lee-Huang-Yang formula (1.33) up to second order.

On the mathematical level, the idea that guides us to find the correct cubic exponential $S(\eta) = e^{A(\eta)}$ is that conjugating the operator $\mathcal{C}_N + \mathcal{H}_N$ with $S(\eta)$ yields, after performing a second order Taylor expansion and ignoring for simplicity of this introductory discussion all higher order commutator terms, the sum

$$S(\eta)^*\big(\mathcal{C}_N + \mathcal{H}_N\big)S(\eta) \simeq \mathcal{C}_N + \mathcal{H}_N + [\mathcal{C}_N, A(\eta)] + [\mathcal{H}_N, A(\eta)] + \frac{1}{2}\big[[\mathcal{C}_N + \mathcal{H}_N, A(\eta)], A(\eta)\big]$$

On the one hand, it turns out that the sum $[\mathcal{C}_N, A(\eta)] + \frac{1}{2}\big[[\mathcal{H}_N, A(\eta)], A(\eta)\big]$ gives the important order $\mathcal{O}(1)$ contributions to the low-energy eigenvalues of H_N while the term $\frac{1}{2}\big[[\mathcal{C}_N, A(\eta)], A(\eta)\big]$ turns out to be negligible. On the other hand, the commutator $[\mathcal{H}_N, A(\eta)]$ contains cubic and non-normally ordered quintic terms which, after normal ordering, combine to a cubic contribution that cancels exactly the cubic operator $\mathcal{C}_N$, once the right choice for $A(\eta)$ has been found! In fact, a very similar mechanism is already caused by the generalized Bogoliubov transformation $T(\eta)$, enabling us to prove (1.75) in the first place. In view of this latter cancellation and the strategy for the cubic renormalization, one can find the correct choice for $S(\eta) = e^{A(\eta)}$ simply by analogy to the choice of $T(\eta)$. Of course, once this choice has been made, the main challenge consists in showing that all other terms produced from the conjugation of $\mathcal{G}_N$ by $S(\eta)$ are mathematically well under control.

With the appropriate choice for the unitary operator $S(\eta)$, we consider the new excitation Hamiltonian $\mathcal{J}_N$, defined by

$$\mathcal{J}_N = S(\eta)^*\mathcal{G}_N S(\eta) = S(\eta)^* T(\eta)^* U_N(\varphi_0) H_N U_N(\varphi_0)^* T(\eta) S(\eta)$$

and mapping its dense domain in $\mathcal{F}_{\perp\varphi_0}^{\leq N}$ into $\mathcal{F}_{\perp\varphi_0}^{\leq N}$. Since the conjugation of $S(\eta)$ leads to the cancellation of the cubic operator $\mathcal{C}_N$, we can now show that

$$\mathcal{J}_N = C_{\mathcal{J}_N} + \mathcal{Q}_{\mathcal{J}_N} + \mathcal{V}_N + \mathcal{E}_{\mathcal{J}_N} \tag{1.78}$$

where $C_{\mathcal{J}_N}$ and $\mathcal{Q}_{\mathcal{J}_N}$ are new constant and quadratic terms, while $\mathcal{E}_{\mathcal{J}_N}$ is a negligible error operator on low-energy states. Compared to (1.75), the important difference is now that $\mathcal{J}_N$ is, up to the potential energy $\mathcal{V}_N$, a quadratic operator which can be diagonalized. To conclude Theorem 1.6.1, we conjugate $\mathcal{J}_N$ with a last generalized Bogoliubov transformation $R(\eta)$ to diagonalize the quadratic operator $\mathcal{Q}_{\mathcal{J}_N}$. This leads to the operator

$$\mathcal{M}_N = R(\eta)^*\mathcal{J}_N R(\eta) = R(\eta)^* S(\eta)^* T(\eta)^* U_N(\varphi_0) H_N U_N(\varphi_0)^* T(\eta) S(\eta) R(\eta)$$

which can be written as

$$\begin{aligned}\mathcal{M}_N = 4\pi\mathfrak{a}_N(N-1) - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right]\\ + \sum_{p\in\Lambda_+^*}\sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}\, a_p^* a_p + \mathcal{V}_N + \mathcal{E}_{\mathcal{M}_N}\end{aligned} \tag{1.79}$$

for an error term $\mathcal{E}_{\mathcal{M}_N}$ that satisfies

$$\pm\mathcal{E}_{\mathcal{M}_N} \leq CN^{-1/4}\Big[(\mathcal{H}_N+1)(\mathcal{N}_{\perp\varphi_0}+1)+(\mathcal{N}_{\perp\varphi_0}+1)^3\Big]$$

and is therefore negligible on low-energy states. Applying the min-max principle, our main result Theorem 1.6.1 follows by comparing the eigenvalues of $\mathcal{M}_N$ with those of its quadratic part. The presence of the quartic operator $\mathcal{V}_N$ in (1.79) is not a problem, because $\mathcal{V}_N \geq 0$. This implies that $\mathcal{V}_N$ can be discarded to show lower bounds. To prove upper bounds, we only need to check that the expectation of $\mathcal{V}_N$ is small on eigenstates of the quadratic operator $\sum_{p\in\Lambda_+^*}\sqrt{|p|^4+16\pi\mathfrak{a}_0 p^2}\, a_p^* a_p$ in (1.79).

1.7 Fluctuation Dynamics of Bose-Einstein Condensates for Bose Gases Interacting through Singular Potentials

In this section, we explain our main result on the norm approximation of the exact Schrödinger evolution (1.3) of Bose-Einstein condensates interacting through singular potentials. Our approximation consists of an effective dynamical description of the condensate evolution and by approximating the fluctuation dynamics in terms of a unitary dynamics on the Fock space, generated by a quadratic Fock space Hamiltonian. Our results are proved in the article [19] which is reproduced in Chapter 5. The presentation in this section is a shortened and simplified version of the introduction [19, Section 1].

Here, we consider systems of N bosons moving in $\Lambda = \mathbb{R}^3$ and interacting through a strong short-range potential of the form $N^{3\beta-1}V(N^\beta.)$, where, in contrast to the previous sections (where $\beta = 1$), we let $\beta \in (0;1)$. The Hamilton operator has the form

$$H_N^\beta = \sum_{i=1}^N -\Delta_{x_i} + \frac{1}{N}\sum_{1\leq i<j\leq N} N^{3\beta}V(N^\beta(x_i-x_j)) \tag{1.80}$$

We assume the unscaled potential $V \geq 0$ to be non-negative, smooth, radially symmetric and compactly supported in $\mathbb{R}^3$.

In Section 1.5, we saw that, in the Gross-Pitaevskii regime, the time evolution of initial Bose-Einstein condensates continues to exhibit Bose-Einstein condensation, and that the evolution of the condensate wavefunction can be described in terms of the time-dependent Gross-Pitaevskii equation (1.50). It is easy to translate the results of Section 1.5 to the intermediate regimes with $\beta \in (0;1)$. In this case, we obtain that the evolution of the condensate wavefunction is determined by the cubic non-linear Schrödinger equation

$$\begin{cases} i\partial_t\varphi_t = -\Delta\varphi_t + \sigma|\varphi_t|^2\varphi_t, \\ (\varphi_t)_{|t=0} = \varphi_0 \end{cases} \tag{1.81}$$

with $\sigma = \widehat{V}(0) = \int dx\, V(x)$ (this follows from the observation that the scattering length of $N^{3\beta-1}V(N^\beta.)$ is equal to $\widehat{V}(0)$, up to errors vanishing as $N\to\infty$). Convergence towards the non-linear dynamics (1.81) holds in the sense of reduced densities. It is natural to ask whether one can derive a more precise approximation for the full evolution $\psi_{N,t} = e^{-iH_N^\beta t}\psi_N$, an approximation valid on the level of the $L^2_s(\mathbb{R}^{3N})$-norm.

To derive such a norm-approximation, one has to describe not only the evolution of the condensate wavefunction, as in (1.81), but one also has to find an effective description of the excitations of the full evolution $\psi_{N,t} = e^{-iH_N^\beta t}\psi_N$, that is, an effective description of the evolution of those particles which are not condensated. To this end, it is first of all convenient to switch to a Fock space representation, because the number of excitations, in contrast with the total number of particles, is not preserved. We therefore proceed as in the previous sections and use the map $U_N(\varphi_t) : L^2_s(\mathbb{R}^{3N}) \to \mathcal{F}^{\leq N}_{\perp\varphi_t}$ to switch to the excitation Fock space $\mathcal{F}^{\leq N}_{\perp\varphi_t}$ in which we describe fluctuations around the solution φ_t of (1.81). In the mean-field regime, i.e. for $\beta = 0$, it has been shown in [63] that the unitary fluctuation dynamics $U_N(\varphi_t)e^{-iH_N^{\beta=0}t}U_N(\varphi_{t=0})^* : \mathcal{F}^{\leq N}_{\perp\varphi_{t=0}} \to \mathcal{F}^{\leq N}_{\perp\varphi_t}$, corresponding to the full many-body evolution, can be approximated by a unitary evolution $\mathcal{U}_{2,\mathrm{mf}}(t;s) : \mathcal{F}_{\perp\varphi_s} \to \mathcal{F}_{\perp\varphi_t}$ on the full Fock space which is generated by a time-dependent generator, quadratic in creation and annihilation operators. That is, one has

$$\lim_{N\to\infty} \left\| U_N(\varphi_t)e^{-iH_N^{\beta=0}t}U_N(\varphi_{t=0})^*\xi_N - \mathcal{U}_{2,\mathrm{mf}}(t;0)\xi_N \right\| = 0 \tag{1.82}$$

for suitable initial states $\xi_N \in \mathcal{F}^{\leq N}_{\perp\varphi_{t=0}}$. In fact, the time-dependent quadratic generator $\mathcal{G}_{2,\mathrm{mf}}(t)$ of the effective dynamics $\mathcal{U}_{2,\mathrm{mf}}(t;s)$, defined by

$$i\partial_t\mathcal{U}_{2,\mathrm{mf}}(t;s) = \mathcal{G}_{2,\mathrm{mf}}(t)\mathcal{U}_{2,\mathrm{mf}}(t;s), \quad \mathcal{U}_{2,\mathrm{mf}}(s;s) = \mathbf{1}_{|\mathcal{F}},$$

is essentially given by the quadratic part of the generator of $U_N(\varphi_t)e^{-iH_N^{\beta=0}t}U_N(\varphi_{t=0})^*$. Notice that the latter contains also operators which are cubic and quartic in the creation and annihilation fields. Analogous to the results of [96, 46, 64] related to the spectrum and the eigenstates of mean-field systems, the convergence (1.82) can be interpreted as a rigorous verification of Bogoliubov theory in a dynamical setting.

Although the convergence in (1.82) could be extended in [76, 77, 58] to scaling regimes where $\beta \in \left[0; \frac{1}{2}\right)$, a heuristic argument of [58] suggests that this range can not be extended further to regimes where $\beta \geq \frac{1}{2}$. In fact, in view of the work [16], where an effective norm approximation of the full many-body evolution of an appropriate class of Fock space initial data was derived for all $\beta \in (0;1)$, this is not very surprising: With increasing scaling parameter β, the interactions among the particles get more and more singular. Hence, correlations among the particles play an important role for the correct description of the dynamics.

To introduce correlations among the particles, we follow therefore the approach of [11, 16] and use Bogoliubov transformations which are unitary operators of the form

$$T_{N,t} = \exp\left(\frac{1}{2}\int dxdy\ [k_{N,t}(x;y)a_xa_y - \mathrm{h.c.}]\right) : \mathcal{F}_{\perp\varphi_t} \to \mathcal{F}_{\perp\varphi_t} \tag{1.83}$$

for a suitable kernel $k_{N,t} \in L^2_{\perp\varphi_t}(\mathbb{R}^3) \otimes_s L^2_{\perp\varphi_t}(\mathbb{R}^3)$. As in the previous sections, the kernel $k_{N,t} \in L^2_{\perp\varphi_t}(\mathbb{R}^3) \otimes_s L^2_{\perp\varphi_t}(\mathbb{R}^3)$ is related to the solution f of the scattering equation (1.27).

More precisely, we relate $k_{N,t}$ to the Neumann ground state f_N of the problem

$$\Big[-\Delta + \frac{1}{2} N^{3\beta-1} V(N^\beta .)\Big] f_N = \lambda_N f_N \tag{1.84}$$

on the ball $|x| \leq \ell$, for a fixed $\ell > 0$. We fix $f_N(x) = 1$, for $|x| = \ell$, and we extend f_N to $\mathbb{R}^3$ requiring that $f_N(x) = 1$ for all $|x| \geq \ell$, see Section 5.1 for more details.

As mentioned earlier, in particular in Sections 1.4 and 1.5, Bogoliubov transformations of the form (1.83) do not preserve the truncation of the excitation Fock space $\mathcal{F}_{\perp\varphi_t}^{\leq N}$. In the previous sections we solved this problem by using generalized Bogoliubov transformations, tailored to stay in the truncated space. Here, we proceed differently and solve this problem by means of localization methods introduced in [64, 76, 77]. In particular, as an intermediate step in our analysis, we switch from $\mathcal{F}_{\perp\varphi_t}^{\leq N}$ to the full Fock space $\mathcal{F}_{\perp\varphi_t}$ by approximating the many-body fluctuation dynamics $U_N(\varphi_t) e^{-iH_N^\beta t} U_N(\varphi_{t=0})^* \, : \, \mathcal{F}_{\perp\varphi_{t=0}}^{\leq N} \to \mathcal{F}_{\perp\varphi_t}^{\leq N}$ by an auxiliary fluctuation dynamics $\widetilde{\mathcal{U}}_N(t;s) : \mathcal{F}_{\perp\varphi_s} \to \mathcal{F}_{\perp\varphi_t}$. Once, this approximation has been achieved, we can proceed similarly to [11, 16] and use (1.83) to implement correlations among particles. The main advantage of this approach is the fact that the action of Bogoliubov transformations on creation and annihilation operators is explicitly given by

$$\begin{aligned} T_{N,t} a(f) T_{N,t}^* &= a(\cosh_{k_{N,t}}(f)) + a^*(\sinh_{k_{N,t}}(\bar{f})) \\ T_{N,t} a^*(f) T_{N,t}^* &= a^*(\cosh_{k_{N,t}}(f)) + a(\sinh_{k_{N,t}}(\bar{f})) \end{aligned}$$

With $T_{N,t}$, defined as in (1.83) for an appropriate kernel $k_{N,t} \in L^2_{\perp\varphi_t}(\mathbb{R}^3) \otimes_s L^2_{\perp\varphi_t}(\mathbb{R}^3)$, we consider the modified fluctuation dynamics

$$T_{N,t} U_N(\varphi_t) e^{-iH_N^\beta t} U_N(\varphi_{t=0})^* \mathbf{1}^{\leq N} T_{N,0}^* : \mathcal{F}_{\perp\varphi_{t=0}} \to \mathcal{F}_{\perp\varphi_t}$$

and we prove that it can be approximated, as $N \to \infty$, by a unitary evolution

$$\mathcal{U}_2(t;s) : \mathcal{F}_{\perp\varphi_s} \to \mathcal{F}_{\perp\varphi_t} \tag{1.85}$$

with quadratic operator $\mathcal{G}_2(t)$ such that

$$i\partial_t \mathcal{U}_2(t;s) = \mathcal{G}_2(t) \mathcal{U}_2(t;s), \qquad \mathcal{U}_2(s;s) = \mathbf{1}_{|\mathcal{F}}$$

The precise definition of $\mathcal{G}_2(t)$ is given in Section 5.1 of Chapter 5 below. Let us now state our first main result as follows, see [19, Theorem 2].

Theorem 1.7.1. *Let $V : \mathbb{R}^3 \to \mathbb{R}$ be smooth, radially symmetric, compactly supported and pointwise non-negative. Let $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ with $\|\xi_N\| = 1$ and*

$$\langle \xi_N, (\mathcal{K} + \mathcal{N}) \xi_N \rangle \leq C \tag{1.86}$$

Let $\psi_{N,t} = e^{-iH_N^\beta t} \psi_N$ be the solution of the Schrödinger equation (1.3) with initial data

$$\psi_N = U_N(\varphi_0)^* 1^{\leq N} T_{N,0}^* \xi_N \tag{1.87}$$

and let $\mathcal{U}_2(t;0)$ be the unitary dynamics on $\mathcal{F}$ defined in (1.85). Then, for all $\alpha < \min(\beta/2, (1-\beta)/2)$, there exists a constant $C > 0$ and a time-dependent phase $e^{-i\int_0^t d\tau\, \eta_N(\tau)}$ such that

$$\left\| U_N(\varphi_t)\psi_{N,t} - e^{-i\int_0^t d\tau\, \eta_N(\tau)}\, T^*_{N,t}\mathcal{U}_2(t;0)\,\xi_N \right\|^2 \leq CN^{-\alpha}\exp(C\exp(C|t|)) \tag{1.88}$$

for all N sufficiently large and all $t \in \mathbb{R}$.

Theorem 1.7.1 applies to the study of the time-evolution of initial data of the form

$$\psi_N = U_N(\varphi_0)^* 1^{\leq N} T^*_{N,0}\xi_N \tag{1.89}$$

for a $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ satisfying the bound

$$\langle \xi_N, [\mathcal{K} + \mathcal{N}]\, \xi_N\rangle \leq C \tag{1.90}$$

uniformly in N. In our second main result we clarify what conditions need to be imposed on ψ_N so that it is possible to find $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ with (1.89) and (1.90). We state this as the following theorem, see [19, Theorem 3].

Theorem 1.7.2. *Let $V : \mathbb{R}^3 \to \mathbb{R}$ be smooth, radially symmetric, compactly supported and pointwise non-negative. Let $\psi_N \in L^2_s(\mathbb{R}^{3N})$ with reduced one-particle density matrix γ_N such that*

$$\operatorname{tr}\left|\gamma_N - |\varphi_0\rangle\langle\varphi_0|\right| \leq CN^{-1} \tag{1.91}$$

and

$$\left| \frac{1}{N}\langle \psi_N, H_N\psi_N\rangle - \left[\|\nabla\varphi_0\|^2 + \frac{1}{2}\langle \varphi_0, (N^{3\beta}V(N^\beta.)f_N * |\varphi_0|^2)\varphi_0\rangle \right] \right| \leq CN^{-1} \tag{1.92}$$

Let $\psi_{N,t} = e^{-iH^\beta_N t}\psi_N$ be the solution of the Schrödinger equation (1.3) with initial data $\psi_N \in L^2_s(\mathbb{R}^{3N})$ and let $\mathcal{U}_2(t;0)$ be the unitary dynamics on $\mathcal{F}$ defined in (1.85). Then, for all $\alpha < \min(\beta/2, (1-\beta)/2)$, there exists a constant $C > 0$ and a time-dependent phase $e^{-i\int_0^t d\tau\, \eta_N(\tau)}$ such that

$$\begin{aligned} &\left\| T_{N,t}U_N(\varphi_t)\psi_{N,t} - e^{-i\int_0^t d\tau\, \eta_N(\tau)}\,\mathcal{U}_2(t;0)\, T_{N,0}\, U_N(\varphi_{t=0})\Psi_{N,0} \right\|^2 \\ &\qquad\qquad \leq CN^{-\alpha}\exp(C\exp(C|t|)) \end{aligned} \tag{1.93}$$

for all N sufficiently large and all $t \in \mathbb{R}$.

Before closing this section, let us make a few remarks on Theorems 1.7.1 and 1.7.2. First of all, we assumed the initial data bounds (1.91) and (1.92) to hold with best possible rates N^{-1}, corresponding to initial data with bounded (i.e. N-independent) number of excitations and with bounded excitation energy. These assumptions could be slightly relaxed, allowing for more excitations and for a larger excitation energy. In this case, however, the rate on the right hand side of (1.93) would deteriorate.

Next, it is clear from the analysis of [20, Section 6], provided in Chapter 3 below, that one can also replace the condition (1.91) by the weaker bound

$$1 - \langle \varphi_0, \gamma_{N,0}\varphi_0 \rangle \le CN^{-1} \tag{1.94}$$

if one additionally assumes that there exists an external confining potential V_{ext} such that φ_0 minimizes the energy functional

$$\begin{aligned} \mathcal{E}(\varphi) = &\int \left[|\nabla \varphi(x)|^2 + V_{\text{ext}}(x)|\varphi(x)|^2 \right] dx \\ &+ \frac{1}{2} \int dxdy N^{3\beta} V(N^{\beta}.)(x-y) f_N(x-y) |\varphi(x)|^2 |\varphi(y)|^2 \end{aligned} \tag{1.95}$$

with the constraint $\|\varphi\|_2 = 1$ and if one replaces the condition (1.92) by the similar bound

$$\left| \frac{1}{N} \langle \psi_{N,0}, H_N^{\beta,\text{trap}} \psi_{N,0} \rangle - \mathcal{E}(\varphi_0) \right| \le CN^{-1}$$

for the Hamilton operator with confining potential $H_N^{\beta,\text{trap}} = H_N^{\beta} + \sum_{j=1}^{N} V_{\text{ext}}(x_j)$. The assumptions (1.94), (1.95) are expected to hold true if ψ_N is the ground state of the trapped Hamiltonian $H_N^{\beta,\text{trap}}$. They describe experiments where particles are initially trapped by external fields and they are cooled down at temperatures so low that they essentially relax to the ground state.

Finally, let us point out that the conditions (1.94), (1.95), and hence (1.89), (1.90), have been proved rigorously for the ground states (more generally, low-lying eigenstates) of trapped systems when either $\beta = 0$ (mean-field regime) [96, 46, 64, 30, 82, 89], or $0 < \beta < 1$ and particles are trapped in a unit torus without an external potential [13, 14].

1.A Notation and Conventions

Let us briefly comment on some conventions used throughout the rest of this thesis.

As should be clear from the context, we denote by $\|\cdot\|$ the norm on the spaces $L^2(\Lambda)$, $\ell^2(\Lambda^*)$, $L^2_s(\Lambda^N)$ or on the Fock space $\mathcal{F}$ (and consequently on the excitation Fock spaces $\mathcal{F}_{\perp\varphi}^{\le N}$ which naturally embed into $\mathcal{F}$). Sometimes we denote the norm on $L^2(\Lambda)$ or $\ell^2(\Lambda^*)$ by $\|\cdot\|_2$. Similarly, by $\|\cdot\|_p$ we denote the norm on $L^p(\Lambda)$ or $\ell^p(\Lambda^*)$ for $p \ge 1$. We also denote by $\langle \cdot, \cdot \rangle$ the inner product on the spaces $L^2(\Lambda)$, $\ell^2(\Lambda^*)$ or on the Fock space $\mathcal{F}$.

By $\|\cdot\|_{H^m}$, $m \in \mathbb{N}\backslash\{0\}$, we denote the norms of the spaces $H^m(\Lambda)$ of square-integrable (equivalence classes of) functions with square-integrable k-th derivatives, $k = 1, 2, \dots, m$.

Given two densely defined operators A, B on a Hilbert space with domains $\mathcal{D}_A$ and $\mathcal{D}_B$, respectively, by $B \le A$ we mean that $\mathcal{D}_A \subset \mathcal{D}_B$ and that $\langle \xi, B\xi \rangle \le \langle \xi, A\xi \rangle$ (denoting by $\langle \cdot, \cdot \rangle$ the inner product of the Hilbert space) for all $\xi \in \mathcal{D}_A$. We abbreviate by $\pm B \le A$ the operator inequalities $B \le A$ and $-B \le A$.

Given a normalized one-particle wavefunction $\varphi \in L^2(\Lambda)$ and the map $U_N(\varphi) : L^2_s(\Lambda^N) \to \mathcal{F}_{\perp\varphi}^{\le N}$, introduced in Section 1.2, we may write $U_N(\varphi) = U(\varphi) = U_\varphi$.

Similarly, we sometimes write $U_N(\varphi) = U_N$, $\mathcal{N}_{\perp\varphi} = \mathcal{N}_{\perp} = \mathcal{N}_{+} = \mathcal{N}$ (notice that indeed $\mathcal{N}_{\perp\varphi} = (\mathcal{N})_{|\mathcal{F}_{\perp\varphi}})$ as well as $\mathcal{F}_{\perp\varphi}^{\leq N} = \mathcal{F}_{+}^{\leq N}$ once the choice of $\varphi \in L^2(\Lambda)$ has been made.

Finally, given a potential $v : \mathbb{R}^3 \to \mathbb{R}$, we generally denote its scattering length by the letter $\mathfrak{a}_0(v) = \mathfrak{a}_0$. In the orginal articles [20, 14, 19], we denoted the scattering length by a_0. In the manuscripts provided in Chapters 2, 3 and 5 we changed the symbol a_0 to $\mathfrak{a}_0$ (since a_0 also denotes the operator $a(\varphi_0)$, at least in [14]) in order to have a uniform notation for the scattering length.

Any further conventions and notational remarks are deferred to the manuscripts of [20, 14, 15, 19] provided in the following Chapters 2, 3, 4 and 5.

Chapter 2

Complete Bose-Einstein Condensation in the Gross-Pitaevskii Limit

In this chapter, we give the details for the proof of Theorem 1.4.1. As discussed in Section 1.4, Theorem 1.4.1 shows complete Bose-Einstein condensation of low-energy states in the Gross-Pitaevskii regime. Our result was proved in [13], which already appeared in the doctoral thesis [12]. The coauthors C. Boccato, S. Cenatiempo, B. Schlein and myself contributed equally to the article [13]. In particular, up to editorial corrections of all coauthors, Sections 4.4 and 4.5 were worked out by myself while the remaining sections were worked out by the remaining coauthors.

The following manuscript is a slightly modified version of the article [13]. First, Section 2.1 is a partly rephrased and shortened verion of the introduction [13, Section 1]. Second, Section 2.2 is a shortened version of [13, Section 2], since we already introduced the general mathematical setting in which we work and most of the related standard results in Section 1.2. Up to the notational modifications already mentioned in Section 1.A, the remaining parts of the paper appear as in the original article [13].

2.1 Main Result

Recall from Section 1.4 that we consider systems of N bosons on the unit torus $\Lambda = \mathbb{T}^3$, i.e. the particles move in a box of volume one with periodic boundary conditions. We are interested in the Gross-Pitaevskii regime so that the Hamilton operator has the form

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \kappa \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) \tag{2.1}$$

We will assume $V \in L^3(\mathbb{R}^3)$ to be non-negative, spherically symmetric and compactly supported. In (2.1), we also introduced a coupling constant $\kappa > 0$, which we will later assume to be small enough. The scattering length $\mathfrak{a}_0$ of the potential κV is defined through the zero-energy scattering equation

$$\left[-\Delta + \frac{\kappa}{2}V\right] f = 0 \tag{2.2}$$

with the boundary condition $f(x) \to 1$ as $|x| \to \infty$ (note that (2.2) is an equation on $\mathbb{R}^3$, despite the fact that we consider particles moving on the torus Λ). Outside the support of V, f has the form

$$f(x) = 1 - \frac{\mathfrak{a}_0}{|x|} \tag{2.3}$$

The constant $\mathfrak{a}_0$ is known as the scattering length of κV. By scaling, the scattering length of the interaction $\kappa N^2 V(Nx)$ appearing in (2.1) is given by $a_N = \mathfrak{a}_0/N$.

It follows from [73, 71, 79] that the ground state energy E_N of (2.1) is such that

$$\lim_{N\to\infty} \frac{E_N}{N} = 4\pi\mathfrak{a}_0 \tag{2.4}$$

Moreover, it has been shown in [66, 79] that the ground state of (2.1) exhibits Bose-Einstein condensation in the one-particle orbital $\varphi_0(x) \equiv 1$ on Λ. In other words, if ψ_N is a normalized ground state vector for (2.1), and if $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$ denotes its one-particle reduced density, it was proven in [66] that

$$\gamma_N^{(1)} \to |\varphi_0\rangle\langle\varphi_0| \tag{2.5}$$

as $N \to \infty$ (for example, in the trace-norm topology). Actually, results in [71, 66] were more general and also applied to non-translation invariant bosonic systems in the Gross-Pitaevskii regime, where particles are trapped in a volume of order one by an external confining potential. For rotating gases similar results have been obtained in [67]. In fact, following the arguments of [66], it is also possible to give a bound on the rate of the convergence (2.5), which is, however, far from optimal.

The main result of our paper is a proof of Bose-Einstein condensation (2.5), valid for sufficiently small values of the coupling constant $\kappa \geq 0$, with a presumably optimal bound on the rate of the convergence. This is the content of the next theorem.

Theorem 2.1.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, spherically symmetric and compactly supported and assume the coupling constant $\kappa \geq 0$ to be small enough. Let $\psi_N \in L^2_s(\Lambda^N)$ be a sequence with $\|\psi_N\| = 1$ and such that*

$$\langle \psi_N, H_N \psi_N\rangle \leq 4\pi\mathfrak{a}_0 N + K \tag{2.6}$$

for some $K > 0$. Let $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}\,|\psi_N\rangle\langle\psi_N|$ be the one-particle reduced density associated with ψ_N. Then there exists a constant $C > 0$, depending on V and on κ but independent of K such that

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0\rangle \leq \frac{C(K+1)}{N} \tag{2.7}$$

where $\varphi_0(x) = 1$ for all $x \in \Lambda$.

Furthermore, the ground state energy E_N of (2.1) is such that

$$|E_N - 4\pi\mathfrak{a}_0 N| \leq D \tag{2.8}$$

for a $D > 0$ independent of N (depending only on V and κ). Hence, the one-particle reduced density associated with the ground state of (2.1) satisfies (2.7), with K replaced by the constant D.

Remarks:

1) The inequality (2.7) immediately implies convergence of the reduced density $\gamma_N^{(1)}$ towards the orthogonal projection $|\varphi_0\rangle\langle\varphi_0|$ in the trace-class topology, since

$$\operatorname{tr}\left|\gamma_N^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right| \leq 2\,\|\gamma_N^{(1)} - |\varphi_0\rangle\langle\varphi_0|\|_{\mathrm{HS}} \leq 2^{3/2}\left(1 - \langle\varphi_0, \gamma_N^{(1)}\varphi_0\rangle\right)^{1/2} \leq \frac{C}{\sqrt{N}}$$

   Together with

$$1 - \langle\varphi_0, \gamma_N^{(1)}\varphi_0\rangle = \operatorname{tr}\left[|\varphi_0\rangle\langle\varphi_0|\big(|\varphi_0\rangle\langle\varphi_0| - \gamma_N^{(1)}\big)\right] \leq \operatorname{tr}\left|\gamma_N^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right|$$

   this remark shows in particular the equivalence of the criteria (1.5) and (1.16).

2) We think that the smallness assumption on $\kappa > 0$ is technical; we expect the results of Theorem 2.1.1 to remain true, independently of the strength of the interaction (of course, assuming the interaction to scale as in (2.1)).

Bounds similar to (2.7) have been obtained in [96, 46, 30, 91] for N-boson systems in the mean field limit, described by the Hamilton operator

$$H_N^{\mathrm{mf}} = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^{N} V(x_i - x_j) \tag{2.9}$$

acting again on $L^2_s(\Lambda^{3N})$. In [96, 46, 64, 30] establishing an estimate on the number of particles orthogonal to the condensate was an important ingredient to show the validity of Bogoliubov theory for the mean-field Hamiltonian (2.9). In this sense, (2.7) can be thought of as a first step towards a better mathematical understanding of the excitation spectrum of Bose gases in the Gross-Pitaevskii regime corresponding to (2.1).

To prove Theorem 2.1.1 we combine techniques from [64] with ideas developed in [11] and recently in [20] to study the time-evolution in the Gross-Pitaevskii regime. First of all, following [64], we observe that every normalized $\psi_N \in L^2_s(\Lambda^N)$ can be represented uniquely as

$$\psi_N = \sum_{n=0}^{N} \psi_N^{(n)} \otimes_s \varphi_0^{\otimes(N-n)} \tag{2.10}$$

for a sequence $\psi_N^{(n)} \in L^2_\perp(\Lambda)^{\otimes_s n}$. Here $L^2_\perp(\Lambda)^{\otimes_s n}$ denotes the symmetric tensor product of n copies of the orthogonal complement $L^2_\perp(\Lambda)$ of φ_0 in $L^2(\Lambda)$. This remark allows us to define a unitary map

$$U_N : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N} \quad \text{through} \quad U_N\psi_N = \{\psi_N^{(0)}, \psi_N^{(1)}, \dots, \psi_N^{(N)}\}. \tag{2.11}$$

Here $\mathcal{F}_+^{\leq N} = \bigoplus_{n=0}^N L^2_\perp(\Lambda)^{\otimes n}$ denotes the bosonic Fock space constructed over $L^2_\perp(\Lambda)$, truncated to sectors with at most N particles. The unitary map U_N factors out the Bose-Einstein condensate described by φ_0 and it let us focus on its orthogonal excitations, described on $\mathcal{F}_+^{\leq N}$.

With U_N, we can define a first excitation Hamiltonian $\mathcal{L}_N = U_N H_N U_N^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$. To compute $\mathcal{L}_N$, it is convenient to rewrite the original Hamiltonian (2.1) in second quantized form as

$$H_N = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{N} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N) a^*_{p+r} a^*_q a_p a_{q+r} \tag{2.12}$$

where $\Lambda^* = 2\pi\mathbb{Z}^3$ is momentum space and where, for every $p \in \Lambda^*$, a_p^*, a_p are the usual Fock space operators, creating and annihilating a particle with momentum p (precise definitions will be given in Section 2.2). Roughly speaking, $\mathcal{L}_N$ can be obtained from (2.12) by replacing creation and annihilation operators a_0^*, a_0 in the zero-momentum mode by factors of $(N-\mathcal{N}_+)^{1/2}$, where $\mathcal{N}_+ = \sum_{p\in\Lambda^*\setminus\{0\}} a_p^* a_p$ is the number of particles operator on the excitation space $\mathcal{F}_+^{\leq N}$. This procedure can be thought of as a rigorous version of the Bogoliubov approximation, proposed already in [17]. Conjugating H_N with U_N we effectively extract, from the original interaction term in (2.12) (quartic in creation and annihilation operators), contributions that are constant (commuting numbers), quadratic and cubic in creation and annihilation operators (the precise form of $\mathcal{L}_N$ is given in (2.47) and (2.48)).

In the mean field regime described by the Hamilton operator (2.9), assuming that V is positive definite it turns out that, up to errors of order one,

i) the constant term in $\mathcal{L}_N^{\text{mf}} = U_N H_N^{\text{mf}} U_N^*$ is given by $N\widehat{V}(0)/2$, which is (again up to errors of order one) the ground state energy of (2.9),

ii) the sum of all other contributions in $\mathcal{L}_N^{\text{mf}}$ can be bounded below on $\mathcal{F}_+^{\leq N}$ by the number of particles operator $\mathcal{N}_+$.

We conclude that

$$\mathcal{L}_N^{\text{mf}} - N\widehat{V}(0)/2 \geq c\mathcal{N}_+ - C \tag{2.13}$$

for appropriate constants $C, c > 0$. This bound shows that states with small excitation energy can be written as $\psi_N = U_N^*\xi_N$ for an excitation vector $\xi_N \in \mathcal{F}_+^{\leq N}$ with $\langle \xi_N, \mathcal{N}_+\xi_N\rangle \leq C$, uniformly in N. It is easy to check that this estimate implies (2.7).

In the Gross-Pitaevskii regime, on the other hand, conjugating with U_N is not enough. The difference between the constant term in $\mathcal{L}_N$ and the ground state energy of (2.1) is still of order N and, moreover, the sum of the other contributions to $\mathcal{L}_N$ cannot be bounded below by the number of particles operator. The problem, in the Gross-Pitaevskii regime, is the fact that the completely factorized wave function $U_N^* \Omega = \varphi_0^{\otimes N}$ (with $\Omega = \{1, 0, \ldots, 0\}$ the vacuum vector in $\mathcal{F}_+^{\leq N}$) is not a good approximation for the ground state vector of (2.1) or, more generally, for low-energy states. Instead, states with small energies in the Gross-Pitaevskii limit are characterized by a short scale correlation structure, which already played a crucial role in [71, 66] and also in the analysis of the time-evolution; see [36, 37, 40, 39, 85, 34, 11, 26, 20]. To take into account correlations we proceed as in [20], conjugating $\mathcal{L}_N = U_N H_N U_N^*$ with a generalized Bogoliubov transformation T. This idea stems from [11], where Bogoliubov transformations of the form

$$\widetilde{T} = \exp \left\{ \frac{1}{2} \sum_{q \in \Lambda_+^*} \eta_q \left[a_q^* a_{-q}^* - a_q a_{-q} \right] \right\} \tag{2.14}$$

with coefficients $\eta_q \in \mathbb{R}$ related to the solution of the zero energy scattering equation (2.2), have been used to model correlations (in fact, since [11] studied the time-evolution in non-translation-invariant systems, a slightly more general version of (2.14) was used there). A nice property of the unitary map (2.14) is the fact that its action on creation and annihilation operators can be computed explicitly, i.e.

$$\widetilde{T}^* a_p \widetilde{T} = \cosh(\eta_p) \, a_p + \sinh(\eta_p) \, a_{-p}^*$$

for all $p \in \Lambda_+^*$. Unfortunately, however, the Bogoliubov transformation $\widetilde{T}$ does not map $\mathcal{F}^{\leq N}$ into itself (it does not preserve the constraint on the number of particles). To circumvent this obstacle, we follow [20] and introduce generalized Bogoliubov transformations, having the form

$$T = \exp \left\{ \frac{1}{2} \sum_{p \in \Lambda_+^*} \eta_p \left[b_p^* b_{-p}^* - b_p b_{-p} \right] \right\} \tag{2.15}$$

with the modified creation and annihilation operators

$$b_p = \sqrt{\frac{N - \mathcal{N}_+}{N}} \, a_p \qquad \text{and} \qquad b_p^* = a_p^* \sqrt{\frac{N - \mathcal{N}_+}{N}}$$

We will choose $\eta_p = -N^{-2} \widehat{w}_\ell(p/N)$, where $\widehat{w}_\ell$ are the Fourier coefficients of $w_\ell = 1 - f_\ell$ and f_ℓ is a modification of the solution f of the zero-energy scattering equation (2.2) (more precisely, f_ℓ is going to be the Neumann ground state on the ball of radius $N\ell$, for an ℓ of order one). We will show in Lemma 2.3.1 that, with this definition, $\eta_p \simeq C\kappa/|p|^2$ for $|p| \ll N$, with fast decay for $|p| \gtrsim N$ guaranteeing that $\sum_p p^2 \eta_p^2 \simeq CN$ (the large p behavior of η_p corresponds to the $|x|^{-1}$ singularity of (2.3), regularized on a length scale of order N^{-1}).

Let us point out that the idea of using unitary operators of the form (2.15) already appeared in [96], in the analysis of the excitation spectrum of mean-field Hamiltonians. In [96], however, these generalized Bogoliubov transformations were used to diagonalize the quadratic part of the excitation Hamiltonian $\mathcal{L}_N^{\text{mf}}$, and not, as we do here, to extract additional contributions from cubic and quartic terms in $\mathcal{L}_N$; as a consequence, in [96] the choice of the coefficients η_p was very different than in (2.15).

Since T maps $\mathcal{F}_+^{\leq N}$ back into itself, we can use it to define a new, modified, excitation Hamiltonian $\mathcal{G}_N = T^* U_N H_N U_N^* T : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$. While conjugation with T only creates a finite number of excitations (because η_p is square summable; see Lemma 2.2.2), it extracts an additional energy of order N (because $\sum_p p^2 \eta_p^2 \simeq CN$). Choosing η_p as indicated above makes sure that the constant term in $\mathcal{G}_N$ is exactly $4\pi\mathfrak{a}_0 N$ and that all other contributions can be bounded below by the number of particles operator, up to errors of order one. In Proposition 2.3.2 we will conclude that, similarly to (2.13),

$$\mathcal{G}_N - 4\pi\mathfrak{a}_0 N \geq c\mathcal{N}_+ - C \tag{2.16}$$

for appropriate constants $C, c > 0$ (the proof of Proposition 2.3.2 is given in Section 2.4 and represents the longest part of the paper). Conjugating (2.16) with T and U (and using the fact that, as discussed in Lemma 2.2.2, T only changes the number of particles by a multiplicative constant), we arrive at the estimate

$$H_N - 4\pi\mathfrak{a}_0 N \geq c\sum_{j=1}^{N}(1 - |\varphi_0\rangle\langle\varphi_0|)_j - C \tag{2.17}$$

between operators acting on the N-particle Hilbert space $L_s^2(\Lambda^N)$. For $j = 1, \dots, N$, $(1 - |\varphi_0\rangle\langle\varphi_0|)_j$ indicates the projection $1 - |\varphi_0\rangle\langle\varphi_0|$ onto the orthogonal complement of the condensate wave function φ_0 acting on the j-th particle. In other words, the operator on the r.h.s. of (2.17) measures the number of orthogonal excitations of the condensate. It is then easy to see that (2.17) implies complete Bose-Einstein condensation in the precise sense of (2.7).

Technically, the main challenge that we have to face is the fact that the action of the generalized Bogoliubov transformations (2.15) on creation and annihilation operators is not explicit, as it was for (2.14). Instead, we will have to expand operators of the form $T^* a_p T$ in absolutely convergent infinite series and we will need to bound several contributions. The main tool we use to control these expansions is Lemma 2.2.3 below, which we take from [20]. s

2.2 Generalized Bogoliubov Transformations for Translation Invariant Systems

As explained in the previous section, we will have to analyse the action of generalized Bogoliubov transformations of the form (2.15) on the creation and annihilation operators. To carry out the analysis, we will need to consider products of several creation and annihilation operators. In particular, two types of monomials in creation and annihilation operators will play an important role in our analysis. For $f_1, \dots, f_n \in \ell_2(\Lambda_+^*)$, $\sharp = (\sharp_1, \dots, \sharp_n), \flat = (\flat_0, \dots, \flat_{n-1}) \in \{\cdot, *\}^n$, we set

$$\begin{aligned} &\Pi^{(2)}_{\sharp,\flat}(f_1, \dots, f_n) \\ &\quad = \sum_{p_1,\dots,p_n \in \Lambda^*} b^{\flat_0}_{\alpha_0 p_1} a^{\sharp_1}_{\beta_1 p_1} a^{\flat_1}_{\alpha_1 p_2} a^{\sharp_2}_{\beta_2 p_2} a^{\flat_2}_{\alpha_2 p_3} \dots a^{\sharp_{n-1}}_{\beta_{n-1} p_{n-1}} a^{\flat_{n-1}}_{\alpha_{n-1} p_n} b^{\sharp_n}_{\beta_n p_n} \prod_{\ell=1}^n f_\ell(p_\ell) \end{aligned} \tag{2.18}$$

where, for every $\ell = 0, 1, \dots, n$, we set $\alpha_\ell = 1$ if $\flat_\ell = *$, $\alpha_\ell = -1$ if $\flat_\ell = \cdot$, $\beta_\ell = 1$ if $\sharp_\ell = \cdot$ and $\beta_\ell = -1$ if $\sharp_\ell = *$. In (2.18), we impose the condition that for every $j = 1, \dots, n-1$, we have either $\sharp_j = \cdot$ and $\flat_j = *$ or $\sharp_j = *$ and $\flat_j = \cdot$ (so that the product $a^{\sharp_\ell}_{\beta_\ell p_\ell} a^{\flat_\ell}_{\alpha_\ell p_{\ell+1}}$ always preserves the number of particles, for all $\ell = 1, \dots, n-1$). With this assumption, we find that the operator $\Pi^{(2)}_{\sharp,\flat}(f_1, \dots, f_n)$ maps $\mathcal{F}_+^{\leq N}$ into itself. If, for some $\ell = 1, \dots, n$, $\flat_{\ell-1} = \cdot$ and $\sharp_\ell = *$ (i.e. if the product $a^{\flat_{\ell-1}}_{\alpha_{\ell-1} p_\ell} a^{\sharp_\ell}_{\beta_\ell p_\ell}$ for $\ell = 2, \dots, n$, or the product $b^{\flat_0}_{\alpha_0 p_1} a^{\sharp_1}_{\beta_1 p_1}$ for $\ell = 1$, is not normally ordered) we require additionally that $f_\ell \in \ell^1(\Lambda_+^*)$. In position space, the same operator can be written as

$$\Pi^{(2)}_{\sharp,\flat}(j_1, \dots, j_n) = \int \check{b}^{\flat_0}_{x_1} \check{a}^{\sharp_1}_{y_1} \check{a}^{\flat_1}_{x_2} \check{a}^{\sharp_2}_{y_2} \check{a}^{\flat_2}_{x_3} \dots \check{a}^{\sharp_{n-1}}_{y_{n-1}} \check{a}^{\flat_{n-1}}_{x_n} \check{b}^{\sharp_n}_{y_n} \prod_{\ell=1}^n \check{f}_\ell(x_\ell - y_\ell)\, dx_\ell dy_\ell \tag{2.19}$$

An operator of the form (2.18), (2.19) with all the properties listed above, will be called a $\Pi^{(2)}$-operator of order n.

For $g, f_1, \dots, f_n \in \ell_2(\Lambda_+^*)$, $\sharp = (\sharp_1, \dots, \sharp_n) \in \{\cdot, *\}^n$, $\flat = (\flat_0, \dots, \flat_n) \in \{\cdot, *\}^{n+1}$, we also define the operator

$$\begin{aligned} &\Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g) \\ &\quad = \sum_{p_1,\dots,p_n \in \Lambda^*} b^{\flat_0}_{\alpha_0, p_1} a^{\sharp_1}_{\beta_1 p_1} a^{\flat_1}_{\alpha_1 p_2} a^{\sharp_2}_{\beta_2 p_2} a^{\flat_2}_{\alpha_2 p_3} \dots a^{\sharp_{n-1}}_{\beta_{n-1} p_{n-1}} a^{\flat_{n-1}}_{\alpha_{n-1} p_n} a^{\sharp_n}_{\beta_n p_n} a^{\flat n}(g) \prod_{\ell=1}^n f_\ell(p_\ell) \end{aligned} \tag{2.20}$$

where α_ℓ and β_ℓ are defined as above. Also here, we impose the condition that, for all $\ell = 1, \dots, n$, either $\sharp_\ell = \cdot$ and $\flat_\ell = *$ or $\sharp_\ell = *$ and $\flat_\ell = \cdot$. This implies that $\Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g)$ maps $\mathcal{F}_+^{\leq N}$ back into $\mathcal{F}_+^{\leq N}$. Additionally, we assume that $f_\ell \in \ell^1(\Lambda^*)$, if $\flat_{\ell-1} = \cdot$ and $\sharp_\ell = *$ for some $\ell = 1, \dots, n$ (i.e. if the pair $a^{\flat_{\ell-1}}_{\alpha_{\ell-1} p_\ell} a^{\sharp_\ell}_{\beta_\ell p_\ell}$ is not normally ordered). In position space, the same operator can be written as

$$\Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g) = \int \check{b}^{\flat_0}_{x_1} \check{a}^{\sharp_1}_{y_1} \check{a}^{\flat_1}_{x_2} \check{a}^{\sharp_2}_{y_2} \check{a}^{\flat_2}_{x_3} \dots \check{a}^{\sharp_{n-1}}_{y_{n-1}} \check{a}^{\flat_{n-1}}_{x_n} \check{a}^{\sharp_n}_{y_n} \check{a}^{\flat n}(g) \prod_{\ell=1}^n \check{f}_\ell(x_\ell - y_\ell)\, dx_\ell dy_\ell \tag{2.21}$$

An operator of the form (2.20), (2.21) will be called a $\Pi^{(1)}$-operator of order n. Operators of the form $b(\check{f})$, $b^*(\check{f})$, for a $f \in \ell^2(\Lambda_+^*)$, will be called $\Pi^{(1)}$-operators of order zero.

In the next lemma we show how to bound $\Pi^{(2)}$- and $\Pi^{(1)}$-operators. The simple proof, based on Lemma 1.2.2, can be found in [20].

Lemma 2.2.1. *Let $n \in \mathbb{N}$, $g, f_1, \dots, f_n \in \ell^2(\Lambda_+^*)$, $\xi \in \mathcal{F}_+^{\leq N}$. Let $\Pi^{(2)}_{\sharp,\flat}(f_1, \dots, f_n)$ and $\Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g)$ be defined as in (2.18), (2.20). Then*

$$\begin{aligned} \left\| \Pi^{(2)}_{\sharp,\flat}(f_1, \dots, f_n)\xi \right\| &\leq 6^n \prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell} \left\| (\mathcal{N}_+ + 1)^n \left(1 - \frac{\mathcal{N}_+ - 2}{N}\right) \xi \right\| \\ \left\| \Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g)\xi \right\| &\leq 6^n \|g\| \prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell} \left\| (\mathcal{N}_+ + 1)^{n+1/2} \left(1 - \frac{\mathcal{N}_+ - 2}{N}\right)^{1/2} \xi \right\| \end{aligned} \tag{2.22}$$

where

$$K_\ell^{\flat_{\ell-1},\sharp_\ell} = \begin{cases} \|f_\ell\|_2 + \|f_\ell\|_1 & \text{if } \flat_{\ell-1} = \cdot \text{ and } \sharp_\ell = * \\ \|f_\ell\|_2 & \text{otherwise} \end{cases}$$

Since $\mathcal{N}_+ \leq N$ on $\mathcal{F}_+^{\leq N}$, it follows that

$$\begin{aligned} \left\| \Pi^{(2)}_{\sharp,\flat}(f_1, \dots, f_n) \right\| &\leq (12N)^n \prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell} \\ \left\| \Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g) \right\| &\leq (12N)^n \sqrt{N} \|g\| \prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell} \end{aligned}$$

After these preliminaries, we introduce generalized Bogoliubov transformations, and we discuss their main properties. For $\eta \in \ell^2(\Lambda_+^*)$ with $\eta_{-p} = \eta_p$ for all $p \in \Lambda_+^*$, we define

$$B(\eta) = \frac{1}{2} \sum_{q \in \Lambda_+^*} \left(\eta_q b_q^* b_{-q}^* - \overline{\eta}_q b_q b_{-q} \right) \tag{2.23}$$

and the unitary operator

$$e^{B(\eta)} = \exp\left\{ \frac{1}{2} \sum_{q \in \Lambda_+^*} \left(\eta_q b_q^* b_{-q}^* - \overline{\eta}_q b_q b_{-q} \right) \right\} \tag{2.24}$$

Notice that $B(\eta), e^{B(\eta)} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$. We will call unitary operators of the form (2.24) generalized Bogoliubov transformations. The name arises from the observation that, on states with $\mathcal{N}_+ \ll N$, we can expect that $b_q \simeq a_q$, $b_q^* \simeq a_q^*$ and therefore that

$$B(\eta) \simeq \widetilde{B}(\eta) = \frac{1}{2} \sum_{q \in \Lambda_+^*} \left(\eta_q a_q^* a_{-q}^* - \overline{\eta}_q a_q a_{-q} \right)$$

Since $\widetilde{B}(\eta)$ is quadratic in creation and annihilation operators, the unitary operator $\exp(\widetilde{B}(\eta))$ is a standard Bogoliubov transformation, whose action on creation and annihilation operators is explicitly given by

$$e^{-\widetilde{B}(\eta)} a_p e^{\widetilde{B}(\eta)} = \cosh(\eta_p) a_p + \sinh(\eta_p) a_{-p}^* \tag{2.25}$$

As explained in the introduction, since the Bogoliubov transformation in (2.25) does not map $\mathcal{F}_+^{\leq N}$ in itself, in the following it will be convenient for us to work with generalized Bogoliubov transformations of the form (2.24). The price we have to pay is the fact that there is no explicit expression like (2.25) for the action of (2.24). Hence, we need other tools to control the action of generalized Bogoliubov transformations. A first result, whose proof can be found in [20] and which will play an important role in the sequel, is the fact that conjugating with (2.24) does not change the momenta of the number of particles operator substantially, if $\eta \in \ell^2(\Lambda_+^*)$ (the same result was previously established in [96]).

Lemma 2.2.2. *Let $\eta \in \ell^2(\Lambda_+^*)$ and $B(\eta)$ as in (2.23). Then, for every $n_1, n_2 \in \mathbb{Z}$, there exists a constant $C > 0$ (depending also on $\|\eta\|$) such that*

$$e^{-B(\eta)}(\mathcal{N}_+ + 1)^{n_1}(N + 1 - \mathcal{N}_+)^{n_2} e^{B(\eta)} \leq C(\mathcal{N}_+ + 1)^{n_1}(N + 1 - \mathcal{N}_+)^{n_2}$$

on $\mathcal{F}_+^{\leq N}$.

Controlling the change of the number of particles operator is not enough for our purposes. Instead, we will often need to express the action of generalized Bogoliubov transformations by means of convergent series of nested commutators. We start by noticing that, for any $p \in \Lambda_+^*$,

$$\begin{aligned} e^{-B(\eta)} b_p e^{B(\eta)} &= b_p + \int_0^1 ds \, \frac{d}{ds} e^{-sB(\eta)} b_p e^{sB(\eta)} \\ &= b_p - \int_0^1 ds \, e^{-sB(\eta)} [B(\eta), b_p] e^{sB(\eta)} \\ &= b_p - [B(\eta), b_p] + \int_0^1 ds_1 \int_0^{s_1} ds_2 \, e^{-s_2 B(\eta)} [B(\eta), [B(\eta), b_p]] e^{s_2 B(\eta)} \end{aligned}$$

Iterating m times, we obtain

$$\begin{aligned} e^{-B(\eta)} b_p e^{B(\eta)} = &\sum_{n=1}^{m-1} (-1)^n \frac{\text{ad}^{(n)}_{B(\eta)}(b_p)}{n!} \\ &+ \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m \, e^{-s_m B(\eta)} \text{ad}^{(m)}_{B(\eta)}(b_p) e^{s_m B(\eta)} \end{aligned} \tag{2.26}$$

where we introduced the notation $\text{ad}^{(n)}_{B(\eta)}(A)$ defined recursively by

$$\text{ad}^{(0)}_{B(\eta)}(A) = A \quad \text{and} \quad \text{ad}^{(n)}_{B(\eta)}(A) = [B(\eta), \text{ad}^{(n-1)}_{B(\eta)}(A)]$$

We will show later that, under suitable assumptions on η, the error term on the r.h.s. of (2.26) is negligible in the limit $m \to \infty$. This means that the action of the generalized Bogoliubov transformation $e^{B(\eta)}$ on b_p and similarly on b_p^* can be described in terms of the nested commutators $\text{ad}^{(n)}_{B(\eta)}(b_p)$ and $\text{ad}^{(n)}_{B(\eta)}(b_p^*)$. In the next lemma, we give a detailed analysis of these operators.

Lemma 2.2.3. *Let $\eta \in \ell^2(\Lambda_+^*)$ be such that $\eta_p = \eta_{-p}$ for all $p \in \ell^2(\Lambda^*)$. To simplify the notation, assume also η to be real-valued (as it will be in applications). Let $B(\eta)$ be defined as in (2.23), $n \in \mathbb{N}$ and $p \in \Lambda_+^*$. Then the nested commutator $\mathrm{ad}^{(n)}_{B(\eta)}(b_p)$ can be written as the sum of exactly $2^n n!$ terms, with the following properties.*

i) Possibly up to a sign, each term has the form

$$\Lambda_1 \Lambda_2 \dots \Lambda_i \, N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta_p^s \varphi_{\alpha p}) \tag{2.27}$$

for some $i, k, s \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\backslash\{0\}$, $\sharp \in \{\cdot, *\}^k$, $\flat \in \{\cdot, *\}^{k+1}$ and $\alpha \in \{\pm 1\}$ chosen so that $\alpha = 1$ if $\flat_k = \cdot$ and $\alpha = -1$ if $\flat_k = *$ (recall here that $\varphi_p(x) = e^{-ip\cdot x}$). In (2.27), each operator $\Lambda_w : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, $w = 1, \dots, i$, is either a factor $(N - \mathcal{N}_+)/N$, a factor $(N + 1 - \mathcal{N}_+)/N$ or an operator of the form

$$N^{-h} \Pi^{(2)}_{\sharp',\flat'}(\eta^{z_1}, \eta^{z_2}, \dots, \eta^{z_h}) \tag{2.28}$$

for some $h, z_1, \dots, z_h \in \mathbb{N}\backslash\{0\}$, $\sharp, \flat \in \{\cdot, *\}^h$.

ii) If a term of the form (2.27) contains $m \in \mathbb{N}$ factors $(N - \mathcal{N}_+)/N$ or $(N+1-\mathcal{N}_+)/N$ and $j \in \mathbb{N}$ factors of the form (2.28) with $\Pi^{(2)}$-operators of order $h_1, \dots, h_j \in \mathbb{N}\backslash\{0\}$, then we have

$$m + (h_1 + 1) + \dots + (h_j + 1) + (k + 1) = n + 1 \tag{2.29}$$

iii) If a term of the form (2.27) contains (considering all Λ- and $\Pi^{(1)}$-operators) the arguments $\eta^{i_1}, \dots, \eta^{i_m}$ and the factor η_p^s for some $m, s \in \mathbb{N}$ and some $i_1, \dots, i_m \in \mathbb{N}\backslash\{0\}$, then

$$i_1 + \dots + i_m + s = n.$$

iv) There is exactly one term having the form

$$\left(\frac{N - \mathcal{N}_+}{N}\right)^{n/2} \left(\frac{N + 1 - \mathcal{N}_+}{N}\right)^{n/2} \eta_p^n b_p \tag{2.30}$$

if n is even, and

$$-\left(\frac{N - \mathcal{N}_+}{N}\right)^{(n+1)/2} \left(\frac{N + 1 - \mathcal{N}_+}{N}\right)^{(n-1)/2} \eta_p^n b^*_{-p} \tag{2.31}$$

if n is odd.

v) If the $\Pi^{(1)}$-operator in (2.27) is of order $k \in \mathbb{N}\backslash\{0\}$, it has either the form

$$\sum_{p_1, \dots, p_k} b^{\flat_0}_{\alpha_0 p_1} \prod_{i=1}^{k-1} a^{\sharp_i}_{\beta_i p_i} a^{\flat_i}_{\alpha_i p_{i+1}} a^*_{-p_k} \eta_p^{2r} a_p \prod_{i=1}^{k} \eta^{j_i}_{p_i}$$

or the form

$$\sum_{p_1,\dots,p_k} b^{\flat_0}_{\alpha_0 p_1} \prod_{i=1}^{k-1} a^{\sharp_i}_{\beta_i p_i} a^{\flat_i}_{\alpha_i p_{i+1}} a_{p_k} \eta_p^{2r+1} a^*_{-p} \prod_{i=1}^{k} \eta^{j_i}_{p_i}$$

for some $r \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\backslash\{0\}$. *If it is of order* $k = 0$, *then it is either given by* $\eta_p^{2r} b_p$ *or by* $\eta_p^{2r+1} b^*_{-p}$, *for some* $r \in \mathbb{N}$.

vi) For every non-normally ordered term of the form

$$\begin{aligned} &\sum_{q\in\Lambda^*} \eta^i_q a_q a^*_q, \quad \sum_{q\in\Lambda^*} \eta^i_q b_q a^*_q \\ &\sum_{q\in\Lambda^*} \eta^i_q a_q b^*_q, \quad \text{or} \quad \sum_{q\in\Lambda^*} \eta^i_q b_q b^*_q \end{aligned}$$

appearing either in the Λ*-operators or in the* $\Pi^{(1)}$*-operator in (2.27), we have* $i \geq 2$.

Proof. The proof is a translation in momentum space of the proof of Lemma 3.2 in [20]. For completeness, we repeat here the main steps. We proceed by induction. For $n = 0$ the claims are clear. For the induction from n to $n+1$ we will repeatedly use the relations

$$\begin{aligned} [B(\eta), b_p] &= -\frac{(N-\mathcal{N}_+)}{N} \eta_p b^*_{-p} + \frac{1}{N} \sum_{q\in\Lambda^*_+} b^*_q a^*_{-q} a_p \eta_q \\ &= -\eta_p b^*_{-p} \frac{(N-\mathcal{N}_+ + 1)}{N} + \frac{1}{N} \sum_{q\in\Lambda^*_+} a_p a^*_{-q} b^*_q \eta_q, \\ [B(\eta), b^*_p] &= -\eta_p b_{-p} \frac{(N-\mathcal{N}_+)}{N} + \frac{1}{N} \sum_{q\in\Lambda^*_+} a^*_p a_{-q} b_q \eta_q \\ &= -\frac{(N-\mathcal{N}_+ + 1)}{N} \eta_p b_{-p} + \frac{1}{N} \sum_{q\in\Lambda^*_+} b_q a_{-q} a^*_p \eta_q, \\ [B(\eta), a^*_p a_q] &= [B(\eta), a_q a^*_p] = -b^*_p b^*_{-q} \eta_q - \eta_p b_{-p} b_q, \\ [B(\eta), N - \mathcal{N}_+] &= \sum_{q\in\Lambda^*_+} \eta_q (b^*_q b^*_{-q} + b_q b_{-q}). \end{aligned} \tag{2.32}$$

Since $\mathrm{ad}^{(n+1)}_{B(\eta)}(b_p) = [B(\eta), \mathrm{ad}^{(n)}_{B(\eta)}(b_p)]$, by linearity it is enough to analyze

$$\Big[B(\eta), \Lambda_1 \Lambda_2 \dots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p})\Big] \tag{2.33}$$

with $\Lambda_1 \Lambda_2 \dots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p})$ satisfying properties (i) to (vi). By Leibniz, the commutator (2.33) is a sum of terms, where $B(\eta)$ is either commuted with a Λ-operator, or with the $\Pi^{(1)}$-operator.

First, consider the case that $B(\eta)$ is commuted with a Λ-operator. If Λ is either equal $(N-\mathcal{N}_+)/N$ or to $(N+1-\mathcal{N}_+)/N$, the last identity in (3.43) implies that, after commutation with $B(\eta)$, Λ should be replaced by

$$N^{-1}\Pi^{(2)}_{*,*}(\eta) + N^{-1}\Pi^{(2)}_{\cdot,\cdot}(\eta) \tag{2.34}$$

This generates two terms contributing to $\text{ad}^{(n+1)}_{B(\eta)}(b_p)$. Let us check that these new terms satisfy (i)-(vi), with n replaced by $(n+1)$. (i) is obviously true. Also (ii) remains true because, when replacing $(N-\mathcal{N}_+)/N$ or $(N+1-\mathcal{N}_+)/N$ by one of the two summands in (2.34), the index m decreases by one but, at the same time, we have one more $\Pi^{(2)}$-operator of order one (which means that j is replaced by $j+1$, and that there is an additional factor $h_{j+1}+1=2$ in the sum (2.29)). Since exactly one additional factor η is inserted, also (iii) remains true. The $\Pi^{(1)}$-operator is not affected by the replacement, so also (v) continues to hold true. Since both terms in (2.34) are normally ordered, (vi) remains valid as well, by the induction assumption. Finally, the two terms generated in (2.34) are not of the form appearing in (iv).

Next, we consider the commutator of $B(\eta)$ with an operator of the form (2.28) for some $h \in \mathbb{N}$, with $h \leq n$ by (ii). By definition

$$\Lambda = N^{-h} \sum_{p_1,\dots,p_h \in \Lambda^*} b^{\flat'_0}_{\alpha_0 p_1} a^{\sharp'_1}_{\beta_1 p_1} a^{\flat'_1}_{\alpha_1 p_2} a^{\sharp'_2}_{\beta_2 p_2} a^{\flat'_2}_{\alpha_2 p_3} \dots a^{\sharp'_{h-1}}_{\beta_{h-1} p_{h-1}} a^{\flat'_{h-1}}_{\alpha_{h-1} p_h} b^{\sharp'_h}_{\beta_h p_h} \prod_{\ell=1}^{h} \eta^{z_l}_{p_l} \tag{2.35}$$

When $[B(\eta), \cdot]$ hits $b^{\flat'_0}_{\alpha_0 p_1}$, the first two equations in (3.43) imply that Λ is replaced by the sum of two operators. The first operator is either

$$\begin{aligned} &-\frac{N-\mathcal{N}_+}{N} N^{-h} \Pi^{(2)}_{\sharp',\widetilde{\flat'}}(\eta^{z_1+1}, \eta^{z_2}, \dots, \eta^{z_h}) \quad \text{or} \\ &\qquad\qquad -\frac{N-\mathcal{N}_+ + 1}{N} N^{-h} \Pi^{(2)}_{\sharp',\widetilde{\flat'}}(\eta^{z_1+1}, \eta^{z_2}, \dots, \eta^{z_h}) \end{aligned} \tag{2.36}$$

depending on whether $\flat'_0 = \cdot$ or $\flat'_0 = *$ (here $\widetilde{\flat'} = (\bar{\flat}'_0, \flat'_1, \dots, \flat'_{h-1})$ with $\bar{\flat}'_0 = \cdot$ if $\flat'_0 = *$ and $\bar{\flat}'_0 = *$ if $\flat'_0 = \cdot$). The second operator is a $\Pi^{(2)}$-operator of order $(h+1)$, given by

$$N^{-(h+1)} \Pi^{(2)}_{\widetilde{\sharp'},\widetilde{\flat'}}(\eta, \eta^{z_1}, \eta^{z_2}, \dots, \eta^{z_h}) \tag{2.37}$$

where $\widetilde{\sharp} = (\bar{\flat}'_0, \sharp'_1, \dots, \sharp'_h)$, $\widetilde{\flat} = (\bar{\flat}'_0, \flat'_0, \dots, \flat'_{h-1})$.

In both cases (i) is clearly correct and (ii) remains true as well (when we replace (2.35) with (2.36), the number of $(N-\mathcal{N}_+)/N$ or $(N-\mathcal{N}_+ +1)/N$-operators increases by one, while everything else remains unchanged; similarly, when we replace (2.35) with (2.37), the order of the $\Pi^{(2)}$-operator increases by one, while the rest remains unchanged). (iii) also remains true, since in (2.36) the power z_1+1 of the first η-kernel is increased by one unit and in (2.37) there is one additional factor η, compared with (2.35). (v) remains valid, since the $\Pi^{(1)}$-operator on the right is not affected by this commutator.

(vi) remains true in (2.36), because $z_1 + 1 \geq 2$. It remains true also in (2.37). In fact, according to (3.43), when switching from (2.35) to (2.37), we are effectively replacing $b \to b^* a^* a$ or $b^* \to b a a^*$. Hence, the first pair of operators in (2.37) is always normally ordered. As for the second pair of creation and annihilation operators (the one associated with the function η^{z_1} in (2.37)), the first field is of the same type as the original b-field appearing in (2.35); non-normally ordered pairs cannot be created. Finally, we remark that the terms we generated here are certainly not of the form in (iv).

The same arguments can be applied if $B(\eta)$ hits the factor $b^{\sharp'_h}_{\beta_h p_h}$ on the right of (2.35) (in this case, we use the identities for the first two commutators in (3.43) having the b-field to the left of the factors $(N + 1 - \mathcal{N}_+)/N$ and $(N - \mathcal{N}_+)/N$ and to the right of the $a_p a^*_{-q}$ and $a^*_p a_{-q}$ operators).

If instead $B(\eta)$ hits a term $a^*_{p_r} a_{p_{r+1}}$ or $a_{p_r} a^*_{p_{r+1}}$ in (2.35), for an $r = 1, \dots, h-1$, then, by (3.43), Λ is replaced by the sum of the two terms, given by

$$- \left[N^{-r} \Pi^{(2)}_{\sharp'', \flat''} (\eta^{z_1}, \eta^{z_2}, \dots, \eta^{z_r + 1}) \right] \left[N^{-(h-r)} \Pi^{(2)}_{\sharp''', \flat'''} (\eta^{z_{r+1}}, \eta^{z_2}, \dots, \eta^{z_h}) \right] \tag{2.38}$$

and by

$$- \left[N^{-r} \Pi^{(2)}_{\sharp'''', \flat''} (\eta^{z_1}, \eta^{z_2}, \dots, \eta^{z_r}) \right] \left[N^{-(h-r)} \Pi^{(2)}_{\sharp''', \flat''''} (\eta^{z_{r+1}+1}, \eta^{z_2}, \dots, \eta^{z_h}) \right] \tag{2.39}$$

with $\flat'' = (\flat'_0, \dots, \flat'_{r-1})$, $\flat''' = (\flat'_r, \dots, \flat'_{h-1})$, $\flat'''' = (\bar{\flat'}_r, \flat'_{r+1}, \dots, \flat'_{h-1})$ and with $\sharp'' = (\sharp'_1, \dots, \sharp'_{r-1}, \bar{\sharp'}_r)$, $\sharp''' = (\sharp'_{r+1}, \dots, \sharp'_h)$, $\sharp'''' = (\sharp'_1, \dots, \sharp'_r)$ (here, we denote $\bar{\sharp'}_r = *$ if $\sharp'_r = \cdot$ and $\bar{\sharp'}_r = \cdot$ if $\sharp'_r = *$, and similarly for $\bar{\flat'}_{r-1}$). Obviously, the new terms containing (2.38) and (2.39) satisfy (i). (ii) remains valid since the contribution of the original Λ to the sum in (2.29), which was given by $(h+1)$ is now given by $(r+1) + (h-r+1) = h+2$. Also (iii) continues to be true, because for both terms (2.38) and (2.39), there is one new additional factor η. Moreover, the terms we generated do not have the form (iv). Since the $\Pi^{(1)}$-operator is unaffected, (v) remains true. As for (vi), we observe that non-normally ordered pairs can only be created where $\sharp'_r$ is changed to $\bar{\sharp'}_r$ (in the term where $\sharp''$ appears) or where $\flat'_r$ is changed to $\bar{\flat'}_r$ (in the term where $\flat'''$ appears). In both cases, however, the change $\sharp'_r \to \bar{\sharp'}_r$ and $\flat'_r \to \bar{\flat'}_r$ comes together with an increase in the power of η (i.e. η^{z_r} is changed to η^{z_r+1} in the first case, while $\eta^{z_{r+1}}$ is changed to $\eta^{z_{r+1}+1}$ in the second case). Since $z_r + 1, z_{r+1} + 1 \geq 2$, (vi) is still satisfied.

Next, let us consider the terms arising from commuting $B(\eta)$ with the operator

$$\begin{aligned} & N^{-k} \Pi^{(1)}_{\sharp, \flat} (\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p}) \\ & = \sum_{p_1, \dots, p_k \in \Lambda^*} b^{\flat_0}_{\alpha_0 p_1} a^{\sharp_1}_{\beta_1 p_1} a^{\flat_1}_{\alpha_1 p_2} a^{\sharp_2}_{\beta_2 p_2} a^{\flat_2}_{\alpha_2 p_3} \dots a^{\sharp_{k-1}}_{\beta_{k-1} p_{k-1}} a^{\flat_{k-1}}_{\alpha_{k-1} p_k} a^{\sharp_k}_{\beta_k p_k} a^{\flat_k}_{\alpha p} \eta^s_p \prod_{\ell=1}^n \eta^{j_l}_{p_l} \end{aligned} \tag{2.40}$$

The arguments are very similar to the case when $B(\eta)$ is commuted with a $\Pi^{(2)}$-operator of the form (2.35). In particular, if $B(\eta)$ hits $b^{\flat_0}_{\alpha_0 p_1}$, (2.40) is replaced by the sum of two terms, the first one being

$$-\frac{N-\mathcal{N}_+}{N}N^{-k}\Pi^{(1)}_{\sharp,\widetilde{\flat}}(\eta^{j_1+1},\dots,\eta^{j_k};\eta_p^s\varphi_{\alpha p}) \qquad \text{or}$$
$$-\frac{N-\mathcal{N}_++1}{N}N^{-k}\Pi^{(1)}_{\sharp,\widetilde{\flat}}(\eta^{j_1+1},\dots,\eta^{j_k};\eta_p^s\varphi_{\alpha p})$$

depending on whether $\flat_0=\cdot$ or $\flat_0=*$ (with $\widetilde{\flat}=(\bar{\flat}_0,\flat_1,\dots,\flat_{k-1})$) and the second one being

$$N^{-(k+1)}\Pi^{(1)}_{\widetilde{\sharp},\widetilde{\flat}}(\eta,\eta^{j_1},\dots,\eta^{j_k};\eta_p^s\varphi_{\alpha p})$$

with $\widetilde{\sharp}=(\bar{\flat}_0,\sharp_1,\dots,\sharp_k)$ and $\widetilde{\flat}=(\bar{\flat}_0,\flat_1,\dots,\flat_k)$. As for (2.36) and (2.37) above, one can show that (i), (ii), (iii), (v), (vi) remain valid. Property (iv) will be discussed below.

If $B(\eta)$ is commuted with one of the factors $a^{\sharp_r}_{p_r}a^{\flat_r}_{p_{r+1}}$ for an $r=1,\dots,k-1$, the resulting two terms will be given by

$$-\left[N^{-r}\Pi^{(2)}_{\sharp'',\flat''}(\eta^{j_1},\dots,\eta^{j_r+1};\eta_p^s\varphi_{\alpha p})\right]\left[N^{-(k-r)}\Pi^{(1)}_{\sharp''',\flat'''}(\eta^{j_{r+1}},\dots,\eta^{j_k};\eta_p^s\varphi_{\alpha p})\right] \quad (2.41)$$

and by

$$-\left[N^{-r}\Pi^{(2)}_{\sharp'''',\flat''}(\eta^{j_1},\dots,\eta^{j_r};\eta_p^s\varphi_{\alpha p})\right]\left[N^{-(k-r)}\Pi^{(1)}_{\sharp''',\flat''''}(\eta^{j_{r+1}+1},\dots,\eta^{j_k};\eta_p^s\varphi_{\alpha p})\right] \quad (2.42)$$

with $\sharp'',\sharp''',\sharp''''$ and $\flat'',\flat''',\flat''''$ as defined after (2.39). Proceeding analogously as for (2.39), these terms satisfy (i),(ii),(iii),(v),(vi).

Let us next consider the case that (2.40) hits the last pair of operators appearing in (2.40). From the induction assumption, this pair either equals $\eta^{2r}a^*_{p_k}a_p$ or $\eta^{2r+1}a_{p_k}a^*_{-p}$. In the first case, (2.40) is replaced by

$$-\Pi^{(2)}_{\sharp,\flat'}(\eta^{j_1},\dots,\eta^{j_k})\,\eta_p^{2r+1}b^*_{-p}-\Pi^{(2)}_{\sharp',\flat'}(\eta^{j_1},\dots,\eta^{j_k+1})\,\eta_p^{2r}b_p \quad (2.43)$$

In the second case, it is replaced by

$$-\Pi^{(2)}_{\sharp',\flat'}(\eta^{j_1},\dots,\eta^{j_k+1})\,\eta_p^{2r+1}b^*_{-p}-\Pi^{(2)}_{\sharp,\flat'}(\eta^{j_1},\dots,\eta^{j_k})\,\eta_p^{2r+2}b_p \quad (2.44)$$

In (2.43), (2.44), we used the notation $\flat'=(\flat_0,\dots,\flat_{k-1})$, $\sharp'=(\sharp_1,\dots,\bar{\sharp}_k)$. From the expression (2.43), (2.44), we infer that also here (i), (ii), (iii), (v), (vi) are satisfied.

As for (iv), from the induction assumption there is exactly one term, in the expansion for $\text{ad}^{(n)}_{B(\eta)}(b_p)$, given by (2.30) if n is even and by (2.31) if n is odd. As an example, let us consider (2.30). If we commute the zero-order $\Pi^{(1)}$-operator $\eta_p^nb_p$ in (2.30) with $B(\eta)$, we obtain exactly the term in (2.31), with n replaced by $(n+1)$ (together with a second term, containing a $\Pi^{(1)}$-operator of order one). Similarly, if we take (2.31) and commute the $\Pi^{(1)}$-operator $\eta_p^nb^*_{-p}$ with $B(\eta)$, we get (2.30), with n replaced by $(n+1)$. Considering the terms above, it is clear that there can be only exactly one term with this form. This shows that also in the expansion for $\text{ad}^{(n+1)}_{B(\eta)}(b_p)$, there is precisely one term of the form given in (iv).

We conclude the proof by counting the number of terms in the expansion for the nested commutator $\text{ad}^{(n+1)}_{B(\eta)}(b_p)$. By the inductive assumption, $\text{ad}^{(n)}_{B(\eta)}(b_p)$ can be expanded in a sum of exactly $2^n n!$ terms. (ii) implies that each of these terms is a product of exactly $(n+1)$ operators, each of them being either $(N-\mathcal{N}_+)$, $(N-(\mathcal{N}_+-1))$, a field operator $b^\sharp_q$ or a quadratic factor $a^\sharp_u a^\flat_q$ commuting with the number of particles operator. By (3.43), the commutator of $B(\eta)$ with each such factor gives a sum of two terms. Therefore, by the product rule, $\text{ad}^{(n+1)}_{B(\eta)}(b_p)$ contains $2^n(n!)\times 2(n+1) = 2^{(n+1)}((n+1)!)$ summands. □

Using Lemma 2.2.3 the remainder terms in the expansion (2.26) can be estimated in the same way as in Lemma [20, Lemma 3.3]. The outcome is stated in the next lemma, whose proof is a translation into momentum space of the proof of [20, Lemma 3.3].

Lemma 2.2.4. *Let $\eta \in \ell^2(\Lambda^*_+)$ be symmetric, with $\|\eta\|$ sufficiently small. Then we have*

$$\begin{aligned} e^{-B(\eta)} b_p e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} ad^{(n)}_{B(\eta)}(b_p) \\ e^{-B(\eta)} b^*_p e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} ad^{(n)}_{B(\eta)}(b^*_p) \end{aligned} \tag{2.45}$$

where the series on the r.h.s. are absolutely convergent.

2.3 The excitation Hamiltonian

We define the unitary operator $U_N : L^2_s(\Lambda^N) \to \mathcal{F}^{\leq N}_+$ as in (2.11). In terms of creation and annihilation operators, the map U_N is given by

$$U_N \psi_N = \bigoplus_{n=0}^{N} (1 - |\varphi_0\rangle\langle\varphi_0|)^{\otimes n} \frac{a_0^{N-n}}{\sqrt{(N-n)!}} \psi_N$$

for all $\psi_N \in L^2_s(\Lambda^N)$ (here we identify $\psi_N \in L^2_s(\Lambda^N)$ with the vector $\{\dots, 0, \psi_N, 0, \dots\} \in \mathcal{F}$). The map $U^*_N : \mathcal{F}^{\leq N}_+ \to L^2_s(\Lambda^N)$ is given by

$$U^*_N \{\psi^{(0)}, \dots, \psi^{(N)}\} = \sum_{n=0}^{N} \frac{a^*(\varphi_0)^{N-n}}{\sqrt{(N-n)!}} \psi^{(n)}$$

It is useful to compute the action of U_N on the product of a creation and an annihilation operators. We find (see [64]):

$$\begin{aligned} U_N a^*_0 a_0 U^*_N &= N - \mathcal{N}_+ \\ U_N a^*_p a_0 U^*_N &= a^*_p \sqrt{N - \mathcal{N}_+} \\ U_N a^*_0 a_p U^*_N &= \sqrt{N - \mathcal{N}_+} a_p \\ U_N a^*_p a_q U^*_N &= a^*_p a_q \end{aligned} \tag{2.46}$$

for all $p, q \in \Lambda_+^* = \Lambda^* \backslash \{0\}$. Writing the Hamiltonian (2.1) in momentum space, we find

$$H_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{2N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N) a_p^* a_q^* a_{q-r} a_{p+r}$$

With (2.46), we can conjugate H_N with the map U_N, defining $\mathcal{L}_N = U_N H_N U_N^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$. We find

$$\mathcal{L}_N = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)} \tag{2.47}$$

with

$$\begin{aligned}
\mathcal{L}_N^{(0)} &= \frac{(N-1)}{2N} \kappa \widehat{V}(0)(N - \mathcal{N}_+) + \frac{\kappa \widehat{V}(0)}{2N} \mathcal{N}_+ (N - \mathcal{N}_+) \\
\mathcal{L}_N^{(2)} &= \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p + \sum_{p \in \Lambda_+^*} \kappa \widehat{V}(p/N) \left[b_p^* b_p - \frac{1}{N} a_p^* a_p \right] + \frac{\kappa}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[b_p^* b_{-p}^* + b_p b_{-p} \right] \\
\mathcal{L}_N^{(3)} &= \frac{\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) \left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right] \\
\mathcal{L}_N^{(4)} &= \frac{\kappa}{2N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p,-q} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}
\end{aligned} \tag{2.48}$$

The superscript $j = 0, 2, 3, 4$ indicates the number of creation and annihilation operators appearing in $\mathcal{L}_N^{(j)}$. As explained in the introduction, in the mean-field regime the term $\mathcal{L}_N^{(0)}$ is the ground state energy of the Bose gas and the sum of the quadratic, cubic and quartic contributions can be bounded below by $\mathcal{N}_+$, up to errors of order one (at least for positive definite interaction). This is not the case in the Gross-Pitaevskii regime we are considering here. To extract the important contributions to the energy that are still hidden in $\mathcal{L}_N^{(2)}, \mathcal{L}_N^{(3)}, \mathcal{L}_N^{(4)}$, we need to conjugate $\mathcal{L}_N$ with a generalized Bogoliubov transformation, as defined in (2.24).

To choose the function $\eta \in \ell^2(\Lambda_+^*)$ entering (2.23) and (2.24), we consider the solution of the Neumann problem

$$\left(-\Delta + \frac{\kappa}{2} V \right) f_\ell = \lambda_\ell f_\ell \tag{2.49}$$

on the ball $|x| \leq N\ell$ (we omit the N-dependence in the notation for f_ℓ and for λ_ℓ; notice that λ_ℓ scales as N^{-3}), with the normalization $f_\ell(x) = 1$ if $|x| = N\ell$. It is also useful to define $w_\ell = 1 - f_\ell$ (so that $w_\ell(x) = 0$ if $|x| > N\ell$). By scaling, we observe that $f_\ell(N.)$ satisfies the equation

$$\left(-\Delta + \frac{\kappa N^2}{2} V(Nx) \right) f_\ell(Nx) = N^2 \lambda_\ell f_\ell(Nx)$$

on the ball $|x| \leq \ell$. We choose $0 < \ell < 1/2$, so that the ball of radius ℓ is contained in the box Λ. We extend then $f_\ell(N.)$ to Λ, by choosing $f_\ell(Nx) = 1$ for all $|x| > \ell$. Then

$$\left(-\Delta + \frac{\kappa N^2}{2} V(Nx) \right) f_\ell(Nx) = N^2 \lambda_\ell f_\ell(Nx) \chi_\ell(x) \tag{2.50}$$

where χ_ℓ is the characteristic function of the ball of radius ℓ. In particular, $x \to w_\ell(Nx)$ is compactly supported and it can be extended to a periodic function on the torus Λ. The Fourier coefficients of the function $x \to w_\ell(Nx)$ are given by

$$\frac{1}{(2\pi)^3}\int_\Lambda w_\ell(Nx)e^{-ip\cdot x}dx = \frac{1}{N^3}\widehat{w}_\ell(p/N)$$

where

$$\widehat{w}_\ell(p) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} w_\ell(x)e^{-ip\cdot x}dx$$

is the Fourier transform of the function w_ℓ. From (2.50), we find the following relation for the Fourier coefficients of $w_\ell(Nx)$:

$$\begin{aligned}-p^2\widehat{w}_\ell(p/N) + \frac{\kappa N^2}{2}\widehat{V}(p/N) - \frac{\kappa}{2N}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N)\widehat{w}_\ell(q/N)\\ = N^5\lambda_\ell\widehat{\chi}_\ell(p) - N^2\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\widehat{w}_\ell(q/N)\end{aligned} \tag{2.51}$$

In the next lemma we collect some important properties of w_ℓ, f_ℓ; its proof can be found in [36, Lemma A.1] and in [20, Lemma 4.1] (exchanging V with κV and following the κ-dependence of the bounds). Notice that this lemma is the reason why we require that $V \in L^3(\mathbb{R}^3)$; for the rest of the analysis $V \in L^2(\mathbb{R}^3)$ would be enough.

Lemma 2.3.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric. Fix $\ell > 0$ and let f_ℓ denote the solution of* (2.49)*.*

i) We have

$$\lambda_\ell = \frac{3\mathfrak{a}_0}{N^3\ell^3}\left(1 + \mathcal{O}(\mathfrak{a}_0/N\ell)\right)$$

ii) We have $0 \le f_\ell, w_\ell \le 1$ and

$$\left|\kappa\int V(x)f_\ell(x)dx - 8\pi\mathfrak{a}_0\right| \le \frac{C\kappa}{N}\,. \tag{2.52}$$

iii) There exists a constant $C > 0$ such that

$$w_\ell(x) \le \frac{C\kappa}{|x|+1} \qquad \text{and} \qquad |\nabla w_\ell(x)| \le \frac{C\kappa}{x^2+1}. \tag{2.53}$$

for all $|x| \le N\ell$.

iv) There exists a constant $C > 0$ such that

$$|\widehat{w}_\ell(p)| \le \frac{C\kappa}{p^2}$$

for all $p \in \Lambda_+^*$.

Using the solution f_ℓ of (2.49) and recalling that $w_\ell = 1 - f_\ell$, we define $\eta : \Lambda^* \to \mathbb{R}$ through

$$\eta_p = -\frac{1}{N^2}\widehat{w}_\ell(p/N) \tag{2.54}$$

From Lemma 2.3.1, it follows that

$$|\eta_p| \leq \frac{C\kappa}{p^2} \tag{2.55}$$

and also that

$$|\eta_0| \leq N^{-2}\int_{\mathbb{R}^3} w_\ell(x)dx \leq C\kappa \tag{2.56}$$

Hence $\eta \in \ell^2(\Lambda_+^*)$, uniformly in N. Another useful bound which can be proven with Lemma 2.3.1 (part iii)) is given by

$$\sum_{p\in\Lambda^*} p^2|\eta_p|^2 = \|\nabla\check{\eta}\|_2^2 \leq CN\kappa^2 \tag{2.57}$$

From (2.51), we obtain

$$\begin{aligned} p^2\eta_p + \frac{\kappa}{2}\widehat{V}(p/N) + \frac{\kappa}{2N}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N)\eta_q \\ = N^3\lambda_\ell\widehat{\chi}_\ell(p) + N^2\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q \end{aligned} \tag{2.58}$$

Using the coefficients η_p, for $p \neq 0$, we construct the generalized Bogoliubov transformation $e^{B(\eta)} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ as in (2.24). With it, we define the excitation Hamiltonian $\mathcal{G}_N : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ by setting (recall the definition (2.47) of the operator $\mathcal{L}_N$)

$$\mathcal{G}_N = e^{-B(\eta)}\mathcal{L}_N e^{B(\eta)} = e^{-B(\eta)}U_N H_N U_N^* e^{B(\eta)} \tag{2.59}$$

In the next proposition, we collect important properties of the self-adjoint operator $\mathcal{G}_N$.

Proposition 2.3.2. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa \geq 0$ is small enough. Then there exists a constant $C > 0$ such that, on $\mathcal{F}_+^{\leq N}$,*

$$2\pi^2\mathcal{N}_+ - C \leq \frac{1}{2}(\mathcal{K} + \mathcal{V}_N) - C \leq \mathcal{G}_N - 4\pi\mathfrak{a}_0 N \leq C(\mathcal{K} + \mathcal{V}_N + 1) \tag{2.60}$$

where we used the notation

$$\mathcal{K} = \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p \qquad \text{and} \qquad \mathcal{V}_N = \frac{\kappa}{2N}\sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^* \\ r\neq -p,-q}} \widehat{V}(r/N)\, a_{p+r}^* a_q^* a_p a_{q+r}\,.$$

The proof of Proposition 2.3.2 is, from the technical point of view, the main part of our paper. It is deferred to Section 2.4 below. Using Proposition 2.3.2 we can now complete the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. From the upper bound in (2.60), taking the expectation in the vacuum $\Omega = \{1, 0, \dots, 0\} \in \mathcal{F}_+^{\leq N}$, we find

$$\langle U_N^* e^{B(\eta)} \Omega, H_N U_N^* e^{B(\eta)} \Omega \rangle = \langle \Omega, \mathcal{G}_N \Omega \rangle \leq 4\pi \mathfrak{a}_0 N + C$$

In particular, this implies that the ground state energy E_N of H_N is such that

$$E_N \leq 4\pi \mathfrak{a}_0 N + C\,. \tag{2.61}$$

From the lower bound

$$2\pi^2 \mathcal{N}_+ - C \leq \mathcal{G}_N - 4\pi \mathfrak{a}_0 N$$

in (2.60), conjugating with $e^{B(\eta)}$ and then with U_N^* we find, using Lemma 2.2.2, the inequality

$$H_N \geq 4\pi \mathfrak{a}_0 N + c\, U_N^* \mathcal{N}_+ U_N - C = 4\pi \mathfrak{a}_0 N + c \sum_{j=1}^N (1 - |\varphi_0\rangle\langle\varphi_0|)_j - C \tag{2.62}$$

between operators on $L^2_s(\Lambda^N)$. Here $(1 - |\varphi_0\rangle\langle\varphi_0|)_j$ denotes the orthogonal projection $1 - |\varphi_0\rangle\langle\varphi_0|$ acting on the j-th particle. On the one hand, (2.62) implies that $H_N \geq 4\pi \mathfrak{a}_0 N - C$ and therefore that

$$E_N \geq 4\pi \mathfrak{a}_0 N - C\,.$$

Combined with (2.61), this bound implies (2.8). On the other hand, (2.62) implies that for a normalized $\psi_N \in L^2_s(\Lambda^N)$ with

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi \mathfrak{a}_0 N + K$$

and with one-particle reduced density $\gamma_N^{(1)}$ we must have

$$K + C \geq c \sum_{j=1}^N \langle \psi_N, (1 - |\varphi_0\rangle\langle\varphi_0|)_j \psi_N \rangle = c\, N \left[1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \right]$$

which implies that

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq \frac{C(K+1)}{N}$$

for an appropriate $C > 0$. This shows (2.7) and concludes the proof of Theorem 2.1.1. $\square$

2.4 Analysis of the excitation Hamiltonian $\mathcal{G}_N$

In this section, we prove Proposition 2.3.2. To this end, we use (2.47) to decompose the excitation Hamiltonian (2.59) as

$$\mathcal{G}_N = \mathcal{G}_N^{(0)} + \mathcal{G}_N^{(2)} + \mathcal{G}_N^{(3)} + \mathcal{G}_N^{(4)} \tag{2.63}$$

with

$$\mathcal{G}_N^{(j)} = e^{-B(\eta)} \mathcal{L}_N^{(j)} e^{B(\eta)}$$

and with $\mathcal{L}_N^{(j)}$ as defined in (2.48), for $j = 0, 2, 3, 4$.

2.4.1 Preliminary results

Before analyzing the operators on the r.h.s. of (2.63), we collect in the following Lemma some preliminary bounds that will be used frequently in the next subsections.

Lemma 2.4.1. *Let* $\xi \in \mathcal{F}_+^{\leq N}$, $p, q \in \Lambda_+^*$, $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$, $j_1, \dots, j_{k_1}$, $m_1, \dots, m_{k_2} \in \mathbb{N}\backslash\{0\}$ *and* $\alpha_i = (-1)^{\ell_i}$ *for* $i = 1, 2$. *For* $s \in \{1, \dots, i_1\}, s' \in \{1, \dots, i_2\}$, *let* $\Lambda_s, \Lambda'_{s'}$ *be either a factor* $(N - \mathcal{N}_+)/N$, *a factor* $(N + 1 - \mathcal{N}_+)/N$ *or a* $\Pi^{(2)}$*-operator of the form*

$$N^{-h}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{z_1}, \dots, \eta^{z_h}) \tag{2.64}$$

for some $h \in \mathbb{N}\backslash\{0\}$, $z_1, \dots, z_h \in \mathbb{N}\backslash\{0\}$ *and* $\underline{\sharp}, \underline{\flat} \in \{\cdot, *\}^h$. *Suppose that the operators*

$$\begin{aligned} &\Lambda_1 \dots \Lambda_{i_1} N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1}\varphi_{\alpha_1 p}) \\ &\Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1}, \dots, \eta^{m_{k_2}}; \eta_q^{\ell_2}\varphi_{\alpha_2 q}) \end{aligned} \tag{2.65}$$

with some $\sharp \in \{\cdot, *\}^{k_1}, \flat \in \{\cdot, *\}^{k_1+1}, \sharp' \in \{\cdot, *\}^{k_2}, \flat' \in \{\cdot, *\}^{k_2+1}$, *appear in the expansion of* $ad^{(n)}_{B(\eta)}(b_p)$ *and of* $ad^{(k)}_{B(\eta)}(b_q)$ *for some* $n, k \in \mathbb{N}$, *as described in Lemma 2.2.3.*

i) For any $\beta \in \mathbb{Z}$, *let*

$$D = (\mathcal{N}_+ + 1)^{(\beta-1)/2}\Lambda_1 \dots \Lambda_{i_1} N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1}\varphi_{\alpha_1 p})\xi$$

and

$$\widetilde{D} = (\mathcal{N}_+ + 1)^{(\beta-1)/2}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1}\varphi_{\alpha_1 p})^*\Lambda^*_{i_1} \dots \Lambda^*_1\xi$$

Then, we have

$$\|D\|, \|\widetilde{D}\| \leq C^n\kappa^n p^{-2\ell_1}\|(\mathcal{N}_+ + 1)^{\beta/2}\xi\| \tag{2.66}$$

If ℓ_1 *is even, we also find*

$$\|D\| \leq C^n\kappa^n p^{-2\ell_1}\|a_p(\mathcal{N}_+ + 1)^{(\beta-1)/2}\xi\| \tag{2.67}$$

ii) For $\beta \in \mathbb{Z}$, *let*

$$\begin{aligned} E = (\mathcal{N}_+ + 1)^{(\beta-1)/2}&\Lambda_1 \dots \Lambda_{i_1} N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1}\varphi_{\alpha_1 p}) \\ &\times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1}, \dots, \eta^{m_{k_2}}; \eta_q^{\ell_2}\varphi_{\alpha_2 q})\xi \end{aligned}$$

Then, we have

$$\|E\| \leq C^{n+k}\kappa^{n+k}p^{-2\ell_1}q^{-2\ell_2}\|(\mathcal{N}_+ + 1)^{(\beta+1)/2}\xi\| \tag{2.68}$$

If ℓ_2 *is even, we find*

$$\|E\| \leq C^{n+k}\kappa^{n+k}p^{-2\ell_1}q^{-2\ell_2}\|a_q(\mathcal{N}_+ + 1)^{\beta/2}\xi\| \tag{2.69}$$

If ℓ_1 is even, we have

$$\begin{aligned} \|E\| \leq\; & C^{n+k} k N^{-1} \kappa^{n+k} p^{-2(\ell_1+1)} q^{-2\ell_2} \|(\mathcal{N}_+ + 1)^{(\beta+1)/2} \xi\| \\ & + C^{n+k} \kappa^{n+k} p^{-2(\ell_1+\ell_2)} \mu_{\ell_2} \delta_{p,-q} \|(\mathcal{N}_+ + 1)^{(\beta-1)/2} \xi\| \\ & + C^{n+k} \kappa^{n+k} p^{-2\ell_1} q^{-2\ell_2} \|a_p (\mathcal{N}_+ + 1)^{\beta/2} \xi\| \end{aligned} \tag{2.70}$$

where $\mu_{\ell_2} = 1$ if ℓ_2 is odd and $\mu_{\ell_2} = 0$ if ℓ_2 is even. If ℓ_1 is even and either $k_1 > 0$ or $k_2 > 0$ or there is at least one Λ- or Λ'-operator having the form (2.64), we obtain the improved bound

$$\begin{aligned} \|E\| \leq\; & C^{n+k} k N^{-1} \kappa^{n+k} p^{-2(\ell_1+1)} q^{-2\ell_2} \|(\mathcal{N}_+ + 1)^{(\beta+1)/2} \xi\| \\ & + C^{n+k} N^{-1} \kappa^{n+k} p^{-2(\ell_1+\ell_2)} \mu_{\ell_2} \delta_{p,-q} \|(\mathcal{N}_+ + 1)^{(\beta+1)/2} \xi\| \\ & + C^{n+k} \kappa^{n+k} p^{-2\ell_1} q^{-2\ell_2} \|a_p (\mathcal{N}_+ + 1)^{\beta/2} \xi\| \end{aligned} \tag{2.71}$$

Finally, if $\ell_1 = \ell_2 = 0$, we can write

$$E = E_1(p,q) + E_2\, a_p a_q \xi \tag{2.72}$$

where

$$\|E_1(p,q)\| \leq C^{n+k} k N^{-1} \kappa^{n+k} p^{-2} \|a_q (\mathcal{N}_+ + 1)^{\beta/2} \xi\|$$

and E_2 is a bounded operator on $\mathcal{F}_+^{\leq N}$ with

$$\|E_2^\natural \zeta\| \leq C^{n+k} \kappa^{n+k} \|(\mathcal{N}_+ + 1)^{(\beta-1)/2} \zeta\| \tag{2.73}$$

for $\natural \in \{\cdot, *\}$ and for all $\zeta \in \mathcal{F}_+^{\leq N}$. If $k_1 > 0$ or $k_2 > 0$ or at least one of the Λ- or Λ'-operators has the form (2.64), we also have the improved bound

$$\|E_2^\natural \zeta\| \leq C^{n+k} N^{-1} \kappa^{n+k} \|(\mathcal{N}_+ + 1)^{(\beta+1)/2} \zeta\| \tag{2.74}$$

for $\natural \in \{\cdot, *\}$ and all $\zeta \in \mathcal{F}_+^{\leq N}$.

Proof. Let us start with part i). If Λ_1 is either the operator $(N - \mathcal{N}_+)/N$ or $(N - \mathcal{N}_+ + 1)/N$, then, on $\mathcal{F}_+^{\leq N}$,

$$\begin{aligned} & \Big\| (\mathcal{N}_+ + 1)^{(\beta-1)/2} \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat} (\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_{\alpha_1 p}) \xi \Big\| \\ & \qquad \leq 2 \Big\| (\mathcal{N}_+ + 1)^{(\beta-1)/2} \Lambda_2 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat} (\eta^{j_1}, \dots, \eta^{j_k}; \eta_p^{\ell_1} \varphi_{\alpha_1 p}) \xi \Big\| \end{aligned} \tag{2.75}$$

If instead Λ_1 has the form (2.64) for a $h \geq 1$, we apply Lemma 2.2.1 and we find (using part vi) in Lemma 2.2.3)

$$\begin{aligned} & \Big\| (\mathcal{N}_+ + 1)^{(\beta-1)/2} \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat} (\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_{\alpha_1 p}) \xi \Big\| \\ & \qquad \leq C^h \kappa^{\bar{h}} \|(\mathcal{N}_+ + 1)^{(\beta-1)/2} \Lambda_2 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat} (\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_{\alpha_1 p}) \xi\| \end{aligned} \tag{2.76}$$

where we used the notation $\bar{h} = z_1 + \dots + z_h$ for the total number of factors η's appearing in (2.64). Iterating the bounds (2.75) and (2.76), we find

$$\begin{aligned} &\|(\mathcal{N}_+ + 1)^{(\beta-1)/2} \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta_p^{\ell_1} \varphi_{\alpha_1 p})\xi\| \\ &\leq C^{r+h_1+\dots+h_s} \kappa^{\bar{h}_1+\dots+\bar{h}_s} \|(\mathcal{N}_+ + 1)^{(\beta-1)/2} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_{\alpha_1 p})\xi\| \end{aligned} \tag{2.77}$$

if r of the operators $\Lambda_1, \dots, \Lambda_{i_1}$ have either the form $(N - \mathcal{N}_+)/N$ or the form $(N - \mathcal{N}_+ + 1)/N$, and the other $s = i_1 - r$ are $\Pi^{(2)}$-operators of the form (2.64) of order $h_1, \dots, h_s$, containing $\bar{h}_1, \dots \bar{h}_s$ factors η. Again with Lemma 2.2.1 and with (2.55), we obtain (using also Lemma 2.2.3, part iii), and the fact that $(\mathcal{N}_+ + 1)^{(\beta-1)/2} \Pi^{(1)}_{\sharp,\flat}(\dots) = \Pi^{(1)}_{\sharp,\flat}(\dots)(\mathcal{N}_+ + 1 \pm 1)^{(\beta-1)/2}$)

$$\begin{aligned} &\|(\mathcal{N}_+ + 1)^{(\beta-1)/2} \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_{\alpha_1 p})\xi\| \\ &\qquad \leq C^{r+h_1+\dots+h_s+j_1+\dots+j_{k_1}+\ell_1} \kappa^{\bar{h}_1+\dots+\bar{h}_s+j_1+\dots+j_{k_1}+\ell_1} p^{-2\ell_1} \|(\mathcal{N}_+ + 1)^{\beta/2}\xi\| \\ &\qquad \leq C^n \kappa^n p^{-2\ell_1} \|(\mathcal{N}_+ + 1)^{\beta/2}\xi\| \,. \end{aligned} \tag{2.78}$$

This shows the bound (2.66) for $\|D\|$. The bound (2.66) for $\|\widetilde{D}\|$ can be proven similarly. If we now assume that ℓ_1 is even, the last field on the right in the $\Pi^{(1)}$ operator in the term D must be an annihilation operator a_p (see Lemma 2.2.3, part v)). Proceeding as above, but estimating

$$\begin{aligned} &\|(\mathcal{N}_+ + 1)^{(\beta-1)}/2 N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_p)\xi\| \\ &\qquad\qquad \leq C^{j_1+\dots+j_{k_1}+\ell_1} \kappa^{j_1+\dots+j_{k_1}+\ell_1} p^{-2\ell_1} \|a_p (\mathcal{N}_+ + 1)^{(\beta-1)/2}\xi\| \end{aligned}$$

we also obtain (2.67).

Let us now consider part ii). The bounds (2.68) and (2.69) follow applying (2.66) twice and, respectively, (2.66) and then (2.67). We focus therefore on (2.70). Here, we assume that ℓ_1 is even. This implies that the field operator on the right of the first $\Pi^{(1)}$-operator is an annihilation operator a_p. To bound $\|\mathrm{E}\|$, we have to commute a_p to the right, until it hits ξ. To commute a_p through factors of $\mathcal{N}_+$, we use the pull-through formula $a_p \mathcal{N}_+ = (\mathcal{N}_+ + 1)a_p$. On the other hand, when we commute a_p through a pair of creation and/or annihilation operators associated with a function η^j for some $j \geq 1$ (like the pairs appearing in the $\Pi^{(2)}$-operators of the form (2.64) or in the $\Pi^{(1)}$-operators in (2.65)), we generate a creation or an annihilation operator a_p or a^*_{-p} together with an additional factor η^j_p. Furthermore, since the commutator erases a creation and an annihilation operator, we can save a factor N^{-1} (taken from the factor N^{-h} in (2.64) or from the factor N^{-k_2} in (2.65)). For example,

$$\left[a_p, \sum_{r \in \Lambda^*} \eta^j_r a^*_r a_r \right] = \eta^j_p a_p$$

There are at most k pairs of creation and/or annihilation operators through which a_p needs to be commuted (because every such pair carries a factor η^j, and the total number of η factors on the right of a_p is k). At the end, we also have to pass a_p through the field operator appearing on the right of the second $\Pi^{(1)}$-operator; this is either the annihilation operator a_q if ℓ_2 is even, or the creation operator a^*_{-q}, if ℓ_2 is odd. Hence, the commutator vanishes if ℓ_2 is even, while it is given by

$$[a_p, a^*_{-q}] = \delta_{p,-q} \tag{2.79}$$

if ℓ_2 is odd. This leads to the estimate (2.70). If we additionally assume that either $k_1 > 0$ or $k_2 > 0$ or that there is at least one Λ- or Λ'-operator having the form (2.64), in the contribution arising from the commutator of a_p and a^*_{-q} (which is only present if ℓ_2 is odd), we can extract an additional factor $(\mathcal{N}_+ + 1)/N$ (this additional factor can be used here and not elsewhere, because in this term, after commuting a_p and a^*_{-q}, there is one less factor of $\mathcal{N}_+$). This observation leads to (2.71). Finally, let us consider $\ell_1 = \ell_2 = 0$. In this case we proceed as before, commuting the annihilation operator a_p to the right. The contribution of the commutators of a_p with the pairs of creation and annihilation fields appearing in the $\Pi^{(1)}$-operator and possibly in the $\Pi^{(2)}$-operators lying on the right of a_p is collected in the term E_1 (this term can be estimated as on the first line on the r.h.s. of (2.70) or (2.71)). After commuting a_p all the way to the right, we are left with the second term on the r.h.s. of (2.72), with the operator E_2 containing all Λ- and Λ'-operators as well as all pairs of annihilation and/or creation operators appearing in the two $\Pi^{(1)}$-operator which can be estimated, following Lemma 2.2.1 as in (2.73) or (2.74). $\square$

2.4.2 Analysis of $\mathcal{G}_N^{(0)}$

From (2.48), we have

$$\mathcal{G}_N^{(0)} = e^{-B(\eta)} \mathcal{L}_N^{(0)} e^{B(\eta)} = \frac{(N-1)}{2} \kappa \widehat{V}(0) + \mathcal{E}_N^{(0)}$$

with

$$\mathcal{E}_N^{(0)} = \frac{\kappa \widehat{V}(0)}{2N} e^{-B(\eta)} \mathcal{N}_+ e^{B(\eta)} - \frac{\kappa \widehat{V}(0)}{2N} e^{-B(\eta)} \mathcal{N}_+^2 e^{B(\eta)}$$

In the next Proposition, we estimate the error term $\mathcal{E}_N^{(0)}$.

Proposition 2.4.2. *Let the assumptions of Proposition 2.3.2 be satisfied. Then there exists a constant $C > 0$ such that*

$$\pm \mathcal{E}_N^{(0)} \leq C\kappa (\mathcal{N}_+ + 1) \tag{2.80}$$

as operator inequality on $\mathcal{F}_+^{\leq N}$.

Proof. Eq. (2.80) follows from Lemma 2.2.2 and the fact that, on $\mathcal{F}_+^{\leq N}$, $\mathcal{N}_+ \leq N$. $\square$

2.4.3 Analysis of $\mathcal{G}_N^{(2)}$

From (2.48), we recall that

$$\mathcal{L}_N^{(2)} = \mathcal{K} + \widetilde{\mathcal{L}}_N^{(2)}$$

where

$$\mathcal{K} = \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p$$

is the kinetic energy operator and

$$\widetilde{\mathcal{L}}_N^{(2)} = \sum_{p\in\Lambda_+^*} \kappa\widehat{V}(p/N)\left[b_p^* b_p - \frac{1}{N}a_p^* a_p\right] + \frac{\kappa}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\left[b_p^* b_{-p}^* + b_p b_{-p}\right] \tag{2.81}$$

Analysis of $e^{-B(\eta)}\mathcal{K}e^{B(\eta)}$

We write

$$e^{-B(\eta)}\mathcal{K}e^{B(\eta)} = \mathcal{K} + \sum_{p\in\Lambda_+^*} p^2\eta_p^2 + \sum_{p\in\Lambda_+^*} p^2\eta_p\left[b_p^* b_{-p}^* + b_p b_{-p}\right] + \mathcal{E}_N^{(K)} \tag{2.82}$$

In the next proposition, we bound the error term $\mathcal{E}_N^{(K)}$.

Proposition 2.4.3. *Let the assumptions of Proposition 2.3.2 be satisfied (in particular, suppose $\kappa \geq 0$ is small enough). Then, for every $\delta > 0$ there exists a constant $C > 0$ such that, on $\mathcal{F}_+^{\leq N}$,*

$$\pm\mathcal{E}_N^{(K)} \leq \delta(\mathcal{K} + \mathcal{V}_N) + C\kappa(\mathcal{N}_+ + 1)\,.$$

Proof. We write

$$\begin{aligned} e^{-B(\eta)}\mathcal{K}e^{B(\eta)} &= \mathcal{K} + \int_0^1 e^{-sB(\eta)}[\mathcal{K}, B(\eta)]e^{sB(\eta)}ds \\ &= \mathcal{K} + \int_0^1 \sum_{p\in\Lambda_+^*} p^2\eta_p\left[e^{-sB(\eta)}b_p b_{-p}e^{sB(\eta)} + e^{-sB(\eta)}b_p^* b_{-p}^* e^{sB(\eta)}\right] ds \end{aligned}$$

Lemma 2.2.4, together with $\mathrm{ad}_{sB(\eta)}^{(n)}(A) = s^n \mathrm{ad}_{B(\eta)}^{(n)}(A)$, implies that

$$e^{-B(\eta)}\mathcal{K}e^{B(\eta)} = \mathcal{K} + \sum_{n,k\geq 0}\frac{(-1)^{n+k}}{n!k!(n+k+1)}\sum_{p\in\Lambda_+^*} p^2\eta_p\left[\mathrm{ad}_{B(\eta)}^{(n)}(b_p)\mathrm{ad}_{B(\eta)}^{(k)}(b_{-p}) + \text{h.c.}\right]$$

We separate the summands with $(n,k) = (0,0),(0,1)$; we find

$$\begin{aligned} e^{-B(\eta)}\mathcal{K}e^{B(\eta)} = \mathcal{K} &+ \sum_{p\in\Lambda_+^*} p^2\eta_p\left[b_p b_{-p} + b_p^* b_{-p}^*\right] - \frac{1}{2}\sum_{p\in\Lambda_+^*} p^2\eta_p\left(b_p[B(\eta), b_{-p}] + \text{h.c}\right) \\ &+ \sum_{n,k}^{*}\frac{(-1)^{n+k}}{n!k!(n+k+1)}\sum_{p\in\Lambda_+^*} p^2\eta_p\left[\mathrm{ad}_{B(\eta)}^{(n)}(b_p)\mathrm{ad}_{B(\eta)}^{(k)}(b_{-p}) + \text{h.c.}\right] \end{aligned}$$

where $\sum^*_{n,k}$ indicates the sum over all pairs $(n,k) \neq (0,0),(0,1)$. With (1.23) and (2.82) we obtain

$$\begin{aligned}
\mathcal{E}_N^{(K)} = & \sum_{p\in\Lambda_+^*} p^2\eta_p^2 \left[b_p^* b_p - \frac{1}{N} a_p^* a_p \right] - \frac{\mathcal{N}_+}{N} \sum_{p\in\Lambda_+^*} p^2 \eta_p^2 \\
& - \frac{1}{N} \sum_{p\in\Lambda_+^*} p^2 \eta_p^2 b_p \mathcal{N}_+ b_p^* - \frac{1}{2N} \sum_{p,q\in\Lambda_+^*} p^2 \eta_p \eta_q \left(b_p b_q^* a^*_{-q} a_{-p} + \text{h.c.} \right) \\
& + \sum^*_{n,k} \frac{(-1)^{n+k}}{n!k!(n+k+1)} \sum_{p\in\Lambda_+^*} p^2 \eta_p \left[\text{ad}^{(n)}_{B(\eta)}(b_p) \text{ad}^{(k)}_{B(\eta)}(b_{-p}) + \text{h.c.} \right] \\
=: & \, \text{G}_1 + \text{G}_2 + \text{G}_3 + \text{G}_4 \\
& + \sum^*_{n,k} \frac{(-1)^{n+k}}{n!k!(n+k+1)} \sum_{p\in\Lambda_+^*} p^2 \eta_p \left[\text{ad}^{(n)}_{B(\eta)}(b_p) \text{ad}^{(k)}_{B(\eta)}(b_{-p}) + \text{h.c.} \right]
\end{aligned} \tag{2.83}$$

The expectation of the first term on the r.h.s. of (2.83) can be estimated by

$$\begin{aligned}
|\langle \xi, \text{G}_1 \xi \rangle| & \leq \sum_{p\in\Lambda_+^*} p^2 \eta_p^2 \|b_p \xi\|^2 + \frac{1}{N} \sum_{p\in\Lambda_+^*} p^2 \eta_p^2 \|a_p \xi\|^2 \\
& \leq \sup_{p\in\Lambda_+^*} (p^2 \eta_p^2) \, \|\mathcal{N}_+^{1/2} \xi\|^2 \leq C\kappa^2 \|\mathcal{N}_+^{1/2} \xi\|^2
\end{aligned} \tag{2.84}$$

with (2.55). To bound the second term on the r.h.s. of (2.83) we remark that, by (2.57),

$$\sum_p p^2 \eta_p^2 = \|\nabla \check{\eta}\|^2 \leq CN\kappa^2 \tag{2.85}$$

This implies that

$$|\langle \xi, \text{G}_2 \xi \rangle| \leq C\kappa^2 \|\mathcal{N}_+^{1/2} \xi\|^2 \tag{2.86}$$

To estimate the contribution of the third term on the r.h.s. of (2.83), we commute b_p to the right of b_p^*. We find, using the fact that $\mathcal{N}_+ \leq N$ on $\mathcal{F}_+^{\leq N}$ and again (2.55), that

$$\begin{aligned}
|\langle \xi, \text{G}_3 \xi \rangle| & \leq \frac{2}{N} \sum_{p\in\Lambda_+^*} p^2 \eta_p^2 \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + \frac{1}{N} \sum_{p\in\Lambda_+^*} p^2 \eta_p^2 \|a_p (\mathcal{N}_+ + 1)^{1/2} \xi\|^2 \\
& \leq C\kappa^2 \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2
\end{aligned} \tag{2.87}$$

As for the fourth term on the r.h.s. of (2.83), we write it as

$$\begin{aligned}
\text{G}_4 = & -\frac{1}{2N} \sum_{p,q\in\Lambda_+^*} p^2 \eta_p \eta_q \left[b_q^* a^*_{-q} a_{-p} b_p + \text{h.c.} \right] + \frac{1}{2N^2} \sum_{p,q\in\Lambda_+^*} p^2 \eta_p \eta_q \left[a_q^* a_p a^*_{-q} a_{-p} + \text{h.c.} \right] \\
& - \frac{1}{N} \sum_{p\in\Lambda_+^*} p^2 \eta_p^2 \left[b_p^* b_p + \frac{N - \mathcal{N}_+}{N} a_p^* a_p \right] \\
=: & \, \text{G}_{41} + \text{G}_{42} + \text{G}_{43}
\end{aligned} \tag{2.88}$$

While it is easy to bound

$$
\begin{aligned}
|\langle \xi, \mathrm{G}_{42}\xi\rangle| &\leq \frac{1}{2N^2}\sum_{p,q\in\Lambda_+^*} p^2\eta_p\eta_q\|a_q(\mathcal{N}_++1)^{1/2}\xi\|\|a_p(\mathcal{N}_++1)^{1/2}\xi\| \\
&\leq \frac{1}{2N^2}\left[\sum_{p,q\in\Lambda_+^*} p^2\eta_p^2\|a_q(\mathcal{N}_++1)^{1/2}\xi\|^2\right]^{\frac12}\left[\sum_{p,q\in\Lambda_+^*} p^2\eta_q^2\|a_p(\mathcal{N}_++1)^{1/2}\xi\|^2\right]^{\frac12} \\
&\leq CN^{-1/2}\kappa^2\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|
\end{aligned}
\tag{2.89}
$$

and

$$
|\langle \xi, \mathrm{G}_{43}\xi\rangle| \leq CN^{-1}\kappa^2\|(\mathcal{N}_++1)^{1/2}\xi\|^2, \tag{2.90}
$$

in order to control the term G_{41} we need to use Eq. (2.58). We find

$$
\begin{aligned}
\mathrm{G}_{41} = &\frac{\kappa}{4N}\sum_{p,q\in\Lambda_+^*}\widehat{V}(p/N)\eta_q\left[b_q^*a_{-q}^*a_{-p}b_p + \text{h.c.}\right] \\
&-\frac{\kappa}{4N^2}\sum_{p,q\in\Lambda_+^*, r\in\Lambda^*}\widehat{V}((p-r)/N)\eta_r\eta_q\left[b_q^*a_{-q}^*a_{-p}b_p + \text{h.c.}\right] \\
&+N^2\lambda_\ell\sum_{p,q\in\Lambda_+^*}\widehat{\chi}_\ell(p)\eta_q\left[b_q^*a_{-q}^*a_{-p}b_p + \text{h.c.}\right] \\
&-N\lambda_\ell\sum_{p,q\in\Lambda_+^*, r\in\Lambda^*}\widehat{\chi}_\ell(p-r)\eta_r\eta_q\left[b_q^*a_{-q}^*a_{-p}b_p + \text{h.c.}\right] \\
=:\ &\mathrm{G}_{411}+\mathrm{G}_{412}+\mathrm{G}_{413}+\mathrm{G}_{414}
\end{aligned}
\tag{2.91}
$$

We estimate

$$
\begin{aligned}
|\langle \xi, \mathrm{G}_{413}\xi\rangle| &\leq \frac{C\kappa}{N}\sum_{p,q\in\Lambda_+^*}|\widehat{\chi}_\ell(p)||\eta_q|\|a_q(\mathcal{N}_++1)^{1/2}\xi\|\|a_p(\mathcal{N}_++1)^{1/2}\xi\| \\
&\leq \frac{C\kappa}{N}\|\widehat{\chi}_\ell\|_2\|\eta\|\|(\mathcal{N}_++1)\xi\|^2 \\
&\leq C\kappa^2\|(\mathcal{N}_++1)^{1/2}\xi\|^2
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
|\langle \xi, \mathrm{G}_{414}\xi\rangle| &\leq \frac{C\kappa}{N^2}\sum_{p,q\in\Lambda_+^*}|g(p)||\eta_q|\|a_q(\mathcal{N}_++1)^{1/2}\xi\|\|a_p(\mathcal{N}_++1)^{1/2}\xi\| \\
&\leq C\kappa\|\eta\|\|g\|\|(\mathcal{N}_++1)^{1/2}\xi\|^2
\end{aligned}
$$

where we defined $g(p) = \sum_{r\in\Lambda^*}\widehat{\chi}_\ell(p-r)\eta_r$. Since

$$
\|g\| = \|\chi_\ell\check{\eta}\| \leq \|\check{\eta}\| = \|\eta\| \leq C\kappa
$$

we conclude that

$$|\langle \xi, \mathrm{G}_{414}\xi\rangle| \leq C\kappa^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

Let us now consider the first term on the r.h.s. of (2.91). Switching to position space we find, on $\mathcal{F}_+^{\leq N}$,

$$\begin{aligned}
\mathrm{G}_{411} &= \frac{\kappa}{4N}\int_{\Lambda^{\times 4}} dxdydzdw \sum_{p,q\in\Lambda_+^*} \widehat{V}(p/N)\eta_q e^{iq(z-w)}e^{ip(x-y)}\check{b}_z^*\check{a}_w^*\check{a}_x\check{b}_y \\
&= \frac{\kappa}{4}\int_{\Lambda^{\times 4}} dxdydzdw\, N^2V(N(x-y))\check{\eta}(z-w)\check{b}_z^*\check{a}_w^*\check{a}_x\check{b}_y
\end{aligned}$$

Hence

$$\begin{aligned}
|\langle \xi, \mathrm{G}_{411}\xi\rangle| &\leq C\kappa\int_{\Lambda^{\times 4}} dxdydzdw\, N^2V(N(x-y))|\check{\eta}(z-w)|\|\check{a}_x\check{a}_y\xi\|\|\check{a}_w\check{a}_z\xi\| \\
&\leq C\kappa\left[\int_{\Lambda^{\times 4}} dxdydzdw\, N^2V(N(x-y))|\check{\eta}(z-w)|^2\|\check{a}_x\check{a}_y\xi\|^2\right]^{1/2} \\
&\qquad\qquad\times\left[\int_{\Lambda^{\times 4}} dxdydzdw\, N^2V(N(x-y))\|\check{a}_z\check{a}_w\xi\|^2\right]^{1/2} \\
&\leq C\kappa^{3/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|
\end{aligned}$$

The term G_{412} can also be estimated similarly. We conclude that

$$|\langle \xi, \mathrm{G}_{41}\xi\rangle| \leq C\kappa^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\kappa^{3/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|$$

and therefore, together with (2.89), (2.90), we find

$$|\langle \xi, \mathrm{G}_4\xi\rangle| \leq C\kappa^2\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{K}+\mathcal{N}_+ +1)^{1/2}\xi\| + C\kappa^{3/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\| \tag{2.92}$$

We consider next the last term in (2.83), namely the sum over all pairs $(n,k) \neq (0,0),(0,1)$. According to Lemma 2.2.4, the operator

$$\sum_{p\in\Lambda_+^*} p^2\eta_p\, \mathrm{ad}_{B(\eta)}^{(n)}(b_p)\mathrm{ad}^{(k)}(b_{-p}) \tag{2.93}$$

can be written as the sum of $2^{n+k}n!k!$ terms having the form

$$\begin{aligned}
\mathrm{G} = \sum_{p\in\Lambda_+^*} & p^2\eta_p\,\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\eta_p^{\ell_1}\varphi_{\alpha_1 p}) \\
&\times\Lambda_1'\dots\Lambda_{i_2}'N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1},\dots,\eta^{m_{k_2}};\eta_p^{\ell_2}\varphi_{-\alpha_2 p})
\end{aligned} \tag{2.94}$$

with $i_1,i_2,k_1,k_2,\ell_1,\ell_2\in\mathbb{N}$, $j_1,\dots,j_{k_1},m_1,\dots,m_{k_2}\in\mathbb{N}\backslash\{0\}$, $\alpha_i = (-1)^{\ell_i}$ for $i=1,2$, and where each Λ_r, Λ_r' is either a factor $(N-\mathcal{N}_+)/N$, $(N+1-\mathcal{N}_+)/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-h}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(z_1)},\dots,\eta^{(z_h)}) \tag{2.95}$$

with $h, z_1, \dots, z_h \in \mathbb{N}\backslash\{0\}$. We estimate the expectation of operators of the form (2.94).

Let us first assume that $\ell_1 + \ell_2 \geq 1$. With Lemma 2.4.1, part ii), we find (using the bounds (2.68) if $\ell_1 + \ell_2 \geq 2$, (2.69) if $(\ell_1, \ell_2) = (1, 0)$ and (2.71) if $(\ell_1, \ell_2) = (0, 1)$)

$$\begin{aligned} |\langle \xi, \mathrm{G}\xi\rangle| \leq C^{n+k} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ \times \sum_{p\in\Lambda_+^*} p^2\eta_p \Big\{ (1 + k/N)\eta_p^2 \kappa^{n+k-2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ + N^{-1}\eta_p\kappa^{n+k-1}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \eta_p\kappa^{n+k-1}\|a_p\xi\| \Big\} \end{aligned} \tag{2.96}$$

To apply (2.71) in the case $(\ell_1, \ell_2) = (0, 1)$, we use here the fact that the pairs $(n, k) = (0, 0), (0, 1)$ are excluded. The choice $(n, k) = (1, 0)$ is not compatible with $(\ell_1, \ell_2) = (0, 1)$ (by Lemma 2.2.3, $\ell_1 \leq n$ and $\ell_2 \leq k$). Hence $n + k \geq 2$, while $\ell_1 + \ell_2 = 1$; this implies by Lemma 2.2.3, part iii), that either $k_1 > 0$ or $k_2 > 0$ or at least one of the Λ- or Λ'-operators is a $\Pi^{(2)}$-operator of the form (2.95). With (2.55) and (2.57), we conclude from (2.96) that

$$|\langle \xi, \mathrm{G}\xi\rangle| \leq C^{k+n}(1 + k/N)\kappa^{n+k+1}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \tag{2.97}$$

Let us now consider the case $\ell_1 = \ell_2 = 0$. With (2.72) in Lemma 2.4.1, we can write

$$\langle \xi, \mathrm{G}\xi\rangle = \sum_{p\in\Lambda_+^*} p^2\eta_p\langle (\mathcal{N}_+ + 1)^{1/2}\xi, \mathrm{E}_1(p, -p)\rangle + \sum_{p\in\Lambda_+^*} p^2\eta_p\langle (\mathcal{N}_+ + 1)^{1/2}\xi, \mathrm{E}_2 a_p a_{-p}\xi\rangle \tag{2.98}$$

where the first term can be bounded by

$$\begin{aligned} \Big| \sum_{p\in\Lambda_+^*} p^2\eta_p\langle (\mathcal{N}_+ + 1)^{1/2}\xi, \mathrm{E}_1(p, -p)\rangle \Big| &\leq \sum_{p\in\Lambda_+^*} p^2\eta_p \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathrm{E}_1(p, -p)\| \\ &\leq C^{n+k} k N^{-1}\kappa^{n+k+1}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \sum_{p\in\Lambda_+^*} p^{-2}\|a_p\xi\| \\ &\leq C^{n+k} k N^{-1}\kappa^{n+k+1}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \end{aligned}$$

As for the second term on the r.h.s. of (2.98), we use the relation (2.58) to replace

$$p^2\eta_p = -\frac{\kappa}{2}\widehat{V}(p/N) - \frac{\kappa}{2N}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N)\eta_q + N^3\lambda_\ell\widehat{\chi}_\ell(p) + N^2\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q \tag{2.99}$$

To bound the contribution proportional to $\kappa\widehat{V}(p/N)$, we switch to position space. We find , for $\xi \in \mathcal{F}_+^{\leq N}$,

$$\begin{aligned} &\Big|\kappa \sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\langle (\mathcal{N}_+ + 1)^{1/2}\xi, E_2 a_p a_{-p}\xi\rangle\Big| \\ &\qquad = \Big|\kappa\int_{\Lambda\times\Lambda} dxdy N^3 V(N(x-y))\langle E_2^*(\mathcal{N}_+ + 1)^{1/2}\xi, \check{a}_x\check{a}_y\xi\rangle\Big| \\ &\qquad \leq \kappa\int_{\Lambda\times\Lambda} dxdy N^3 V(N(x-y))\|\mathrm{E}_2^*(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\check{a}_x\check{a}_y\xi\| \end{aligned}$$

Since we are excluding the term with $(n,k) = (0,0)$, we have either $k_1 > 0$ or $k_2 > 0$ or at least one of the Λ-operators has the form (2.95); this allows us to apply the bound (2.74). We obtain

$$\begin{aligned}
&\left| \kappa \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \langle \mathrm{E}_2^* (\mathcal{N}_+ + 1)^{1/2} \xi, a_p a_{-p} \xi \rangle \right| \\
&\qquad \leq C^{n+k} \kappa^{n+k+1} \int_{\Lambda \times \Lambda} dx dy N^{5/2} V(N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\check{a}_x \check{a}_y \xi\| \\
&\qquad \leq C^{n+k} \kappa^{n+k+1} \left[\int_{\Lambda \times \Lambda} dx dy N^2 V(N(x-y)) \|\check{a}_x \check{a}_y \xi\|^2 \right]^{1/2} \\
&\qquad\qquad \times \left[\int_{\Lambda \times \Lambda} dx dy N^3 V(N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 \right]^{1/2} \\
&\qquad \leq C^{n+k} \kappa^{n+k+1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\|
\end{aligned}$$

The contribution of the other terms on the r.h.s. of (2.99) can be bounded similarly. We conclude that, in the case $\ell_1 = \ell_2 = 0$,

$$\begin{aligned}
|\langle \xi, \mathrm{G} \xi \rangle| \leq\; & C^{k+n} (1 + k/N) \kappa^{n+k+1} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 \\
& + C^{k+n} \kappa^{n+k+1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\|
\end{aligned} \tag{2.100}$$

Combining this bound with (2.97) we obtain from (2.93), for sufficiently small κ,

$$\begin{aligned}
&\Big| \sum_{n,k}^{*} \frac{(-1)^{n+k}}{n!k!(n+k+1)} \sum_{p \in \Lambda_+^*} p^2 \eta_p \left\langle \xi, \left[\mathrm{ad}^{(n)}_{B(\eta)}(b_p) \mathrm{ad}^{(k)}_{B(\eta)}(b_{-p}) + \text{h.c.} \right] \xi \right\rangle \Big| \\
&\qquad \leq C\kappa \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + C\kappa^{1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\|
\end{aligned}$$

Together with (2.84), (2.86), (2.87), (2.92), we finally estimate (2.83) by

$$|\langle \xi, \mathcal{E}_N^{(K)} \xi \rangle| \leq C\kappa \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|(\mathcal{K} + \mathcal{N}_+ + 1)^{1/2} \xi\| + C\kappa^{1/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2} \xi\|$$

Hence, for any $\delta > 0$, we can find $C > 0$ such that

$$\pm \mathcal{E}_N^{(K)} \leq \delta(\mathcal{K} + \mathcal{V}_N) + C\kappa(\mathcal{N}_+ + 1)$$

as claimed. □

Analysis of $e^{-B(\eta)} \widetilde{\mathcal{L}}_N^{(2)} e^{B(\eta)}$

With $\widetilde{\mathcal{L}}_N^{(2)}$ as in (2.81), we write

$$e^{-B(\eta)} \widetilde{\mathcal{L}}_N^{(2)} e^{B(\eta)} = \kappa \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \eta_p + \frac{\kappa}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[b_p b_{-p} + b_p^* b_{-p}^* \right] + \mathcal{E}_N^{(2)} \tag{2.101}$$

In the next proposition, we estimate the error term $\mathcal{E}_N^{(2)}$.

Proposition 2.4.4. *Let the assumptions of Proposition 2.3.2 be satisfied (in particular, suppose $\kappa \geq 0$ is small enough). Then, for every $\delta > 0$, there exists a constant $C > 0$ such that, on $\mathcal{F}_+^{\leq N}$,*

$$\pm \mathcal{E}_N^{(2)} \leq \delta \mathcal{V}_N + C\kappa(\mathcal{N}_+ + 1)$$

Proof. Recall that

$$\widetilde{\mathcal{L}}_N^{(2)} = \kappa \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left(b_p^* b_p - \frac{1}{N} a_p^* a_p \right) + \frac{\kappa}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left(b_p b_{-p} + b_p^* b_{-p}^* \right) \tag{2.102}$$

The expectation of the conjugation of the first term can be estimated by

$$\begin{aligned} \left| \kappa \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \langle \xi, e^{-B(\eta)} b_p^* b_p e^{B(\eta)} \xi \rangle \right| &\leq \kappa \sum_{p \in \Lambda_+^*} |\widehat{V}(p/N)| \langle \xi, e^{-B(\eta)} b_p^* b_p e^{B(\eta)} \xi \rangle \\ &\leq C\kappa \langle \xi, e^{-B(\eta)} \mathcal{N}_+ e^{B(\eta)} \xi \rangle \\ &\leq C\kappa \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 \end{aligned} \tag{2.103}$$

The contribution proportional to $-N^{-1} a_p^* a_p$ on the r.h.s. of (2.102) can be bounded analogously. So, let us focus on the last sum on the r.h.s. of (2.102). According to Lemma 2.2.4, we can expand

$$\begin{aligned} &\kappa \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) e^{-B(\eta)} b_p b_{-p} e^{B(\eta)} \\ &\qquad = \sum_{n,k \geq 0} \frac{(-1)^{k+n}}{k!n!} \kappa \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \mathrm{ad}^{(n)}(b_p) \mathrm{ad}^{(k)}(b_{-p}) \\ &\qquad = \kappa \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) b_p b_{-p} - \kappa \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) b_p [B(\eta), b_{-p}] \\ &\qquad\quad + \sum_{n,k}^{*} \frac{(-1)^{k+n}}{k!n!} \kappa \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \mathrm{ad}^{(n)}(b_p) \mathrm{ad}^{(k)}(b_{-p}) \end{aligned} \tag{2.104}$$

where the sum $\sum^*$ runs over all pairs $(n,k) \neq (0,0), (0,1)$. The first term on the r.h.s. of (2.104) does not enter the definition (2.101) of the error term $\mathcal{E}_N^{(2)}$. The second term on the r.h.s. of (2.104) is given by

$$\begin{aligned}
-\kappa \sum_{p\in\Lambda_+^*} &\widehat{V}(p/N) b_p \left[B(\eta), b_{-p}\right] \\
&= \frac{N-\mathcal{N}_+}{N}\kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\eta_p b_b b_p^* - \frac{\kappa}{N}\sum_{p,q\in\Lambda_+^*} \widehat{V}(p/N)\eta_q b_p b_q^* a_{-q}^* a_{-p} \\
&= \left(\frac{N-\mathcal{N}_+}{N}\right)^2 \kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\eta_p + \frac{N-\mathcal{N}_+}{N}\kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\eta_p \left(b_p^* b_p - \frac{3}{N} a_p^* a_p\right) \\
&\quad - \frac{N-\mathcal{N}_+}{N^2}\kappa \sum_{p,q\in\Lambda^*} \widehat{V}(p/N)\eta_q a_q^* a_{-q}^* a_p a_{-p}
\end{aligned} \tag{2.105}$$

To bound the expectation of the last term, we observe that

$$\left|\frac{\kappa}{N}\sum_{p,q\in\Lambda^*} \widehat{V}(p/N)\eta_q \langle \xi, a_q^* a_{-q}^* a_p a_{-p}\xi\rangle\right| \leq \frac{\kappa}{N}\Big\|\sum_{q\in\Lambda_+^*} \eta_q a_q a_{-q}\xi\Big\| \Big\|\sum_{p\in\Lambda_+^*} \widehat{V}(p/N) a_p a_{-p}\xi\Big\| \tag{2.106}$$

On the one hand,

$$\begin{aligned}
\Big\|\sum_{q\in\Lambda_+^*} \eta_q a_q a_{-q}\xi\Big\| &\leq \sum_{q\in\Lambda_+^*} |\eta_q| \|a_q(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\leq C\kappa \|(\mathcal{N}_+ + 1)\xi\| \leq C N^{1/2}\kappa \|(\mathcal{N}_+ + 1)^{1/2}\xi\|
\end{aligned}$$

On the other hand, switching to position space,

$$\begin{aligned}
\kappa\Big\|\sum_{p\in\Lambda_+^*} \widehat{V}(p/N) a_p a_{-p}\xi\Big\| &\leq \kappa \int_{\Lambda\times\Lambda} dxdy N^3 V(N(x-y)) \|\check{a}_x \check{a}_y \xi\| \\
&\leq C N^{1/2}\left(\kappa^{1/2}\|\mathcal{V}_N^{1/2}\xi\| + C\kappa \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\right)
\end{aligned}$$

From (2.106), we find

$$\begin{aligned}
&\left|\frac{\kappa}{N}\sum_{p,q\in\Lambda^*} \widehat{V}(p/N)\eta_q \langle \xi, a_q^* a_{-q}^* a_p a_{-p}\xi\rangle\right| \\
&\qquad \leq C\kappa^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\kappa^{3/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|
\end{aligned} \tag{2.107}$$

To control the first and second term on the r.h.s. of (2.105), we observe that

$$\begin{aligned}
\frac{\kappa}{N}\sum_{p\in\Lambda_+^*} |\widehat{V}(p/N)|\eta_p &\leq \frac{C\kappa^2}{N}\sum_{p\in\Lambda_+^*} \frac{|\widehat{V}(p/N)|}{p^2} \\
&\leq C\kappa^2 \sum_{q\in\Lambda_+^*/N} \frac{1}{N^3}\frac{|\widehat{V}(q)|}{q^2} \leq C\kappa^2 \int_{\mathbb{R}^3} \frac{|\widehat{V}(q)|}{q^2} dq \leq C\kappa^2
\end{aligned} \tag{2.108}$$

since the sum over the rescaled lattice $N^{-1}\Lambda_+^*$ can be interpreted as a Riemann sum. Together with (2.107), this remark implies that

$$\begin{aligned}\Big|-\kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\langle\xi, b_p[B(\eta), b_{-p}]\xi\rangle - \kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\eta_p\Big| \\ \leq C\kappa^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\kappa^{3/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\end{aligned} \tag{2.109}$$

Let us now focus on the sum $\sum^*$ over all pairs $(n,k) \neq (0,0),(0,1)$ on the r.h.s. of (2.104). According to Lemma 2.2.4, the operator

$$\kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\mathrm{ad}^{(n)}(b_p)\mathrm{ad}^{(k)}(b_{-p}) \tag{2.110}$$

can be expanded as the sum of $2^{n+k}n!k!$ terms having the form

$$\begin{aligned}\mathrm{I} =& \kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\eta_p^{\ell_1}\varphi_{\alpha_1 p}) \\ &\times \Lambda_1'\dots\Lambda_{i_2}'N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1},\dots,\eta^{m_{k_2}};\eta_p^{\ell_2}\varphi_{-\alpha_2 p})\end{aligned}$$

where $i_1,i_2,k_1,k_2,\ell_1,\ell_2 \in \mathbb{N}$, $j_1,\dots,j_{k_1},m_1,\dots,m_{k_2} \in \mathbb{N}\backslash\{0\}$, $\alpha_i = (-1)^{\ell_i}$ for $i=1,2$ and where each operator Λ_i, Λ_i' is either a a factor $(N-\mathcal{N}_+)/N$, a factor $(N-\mathcal{N}_+ +1)/N$ or a $\Pi^{(2)}$-operator of order $h \in \mathbb{N}\backslash\{0\}$ having the form

$$N^{-h}\,\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{z_1},\dots\eta^{z_h}) \tag{2.111}$$

with $z_1,\dots,z_h \in \mathbb{N}\backslash\{0\}$. To bound the expectation of an operator of the form I we consider first the case $\ell_1+\ell_2 \geq 1$. Combining the bounds (2.68) (if $\ell_1+\ell_2\geq 2$), (2.69) (if $(\ell_1,\ell_2)=(1,0)$) and (2.71) (if $(\ell_1,\ell_2)=(0,1)$) from Lemma 2.4.1, we obtain

$$\begin{aligned}|\langle\xi,\mathrm{I}\xi\rangle| \leq\ & \kappa \sum_{p\in\Lambda_+^*} |\widehat{V}(p/N)|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}\Lambda_1\dots\Lambda_{i_1} \\ &\times N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\eta_p^{\ell_1}\varphi_p)\Lambda_1'\dots\Lambda_{i_2}'N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1},\dots,\eta^{m_{k_2}};\eta_p^{\ell_2}\varphi_p)\xi\| \\ \leq\ & C^{n+k}\kappa^{n+k+1}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\times \sum_{p\in\Lambda_+^*} |\widehat{V}(p/N)|\Big\{(1+k/N)p^{-4}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\qquad + p^{-2}\|a_p\xi\| + N^{-1}p^{-2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\Big\} \\ \leq\ & C^{k+n}(1+k/N)\kappa^{n+k+1}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\end{aligned} \tag{2.112}$$

where we used again the bound (2.108). If instead $\ell_1 = \ell_2 = 0$, we use (2.72) to decompose

$$\begin{aligned}\langle \xi, \mathrm{I}\,\xi\rangle &= \kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\langle(\mathcal{N}_+ +1)^{1/2}\xi, \mathrm{E}_1(p,-p)\rangle \\ &\quad + \kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\langle(\mathcal{N}_+ +1)^{1/2}\xi, \mathrm{E}_2 a_p a_{-p}\xi\rangle\end{aligned}$$

The r.h.s. of the last equation can be estimated exactly as we did with the r.h.s. of (2.98). We obtain, similarly to (2.100), that for $\ell_1 = \ell_2 = 0$,

$$\begin{aligned}|\langle \xi, \mathrm{I}\xi\rangle| &\leq C^{k+n}(1+k/N)\kappa^{n+k+1}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 \\ &\quad + C^{k+n}\kappa^{n+k+1/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\,.\end{aligned}$$

Combining this bound with (2.112), we find from (2.110) that for sufficiently small κ,

$$\begin{aligned}\Big|\sum_{n,k}^{*} \frac{(-1)^{k+n}}{k!n!}\kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\langle \xi, \mathrm{ad}^{(n)}(b_p)\mathrm{ad}^{(k)}(b_{-p})\xi\rangle\Big| \\ \leq C\kappa\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + C\kappa^{1/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\end{aligned}$$

Together with (2.103), (2.104) and (2.109), we conclude that

$$|\langle \xi, \mathcal{E}_N^{(2)}\xi\rangle| \leq C\kappa\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + C\kappa^{1/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|$$

Hence, for every $\delta > 0$ we can find a constant $C > 0$ such that

$$\pm\mathcal{E}_N^{(2)} \leq \delta\mathcal{V}_N + C\kappa\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2$$

□

2.4.4 Analysis of $\mathcal{G}_N^{(3)}$

From (2.48) and (2.63), we have

$$\mathcal{G}_N^{(3)} = \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/N) e^{-B(\eta)} b_{p+q}^* a_{-p}^* a_q e^{B(\eta)} + \text{h.c.} \tag{2.113}$$

In the next proposition, we show how to bound $\mathcal{G}_N^{(3)}$.

Proposition 2.4.5. *Let the assumptions of Proposition 2.3.2 be satisfied (in particular, suppose $\kappa \geq 0$ is small enough). Then, for every $\delta > 0$ there exists $C > 0$ such that, on $\mathcal{F}_+^*$,*

$$\pm\mathcal{G}_N^{(3)} \leq \delta\mathcal{V}_N + C\kappa(\mathcal{N}_+ +1)$$

Since some of the terms in $\mathcal{G}_N^{(3)}$ (and many terms in $\mathcal{G}_N^{(4)}$, which will be analyzed in the next subsection) have to be bounded with the potential energy operator, in the proof of Prop. 2.4.5 (and in the proof of Prop. 2.4.7 in the next subsection) we will often need to switch to position space. For this reason it is convenient to show a version of the estimates in Lemma 2.4.1 stated in position space. The proof of the following Lemma follows closely the proof of Lemma 5.2 in [20].

Lemma 2.4.6. *Let $\xi \in \mathcal{F}_+^{\leq N}$, $\beta \in \mathbb{N}$, $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$, $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \in \mathbb{N}\backslash\{0\}$, For every $s = 1, \dots, \max\{i_1, i_2\}$, let Λ_s, Λ'_s be either a factor $(N - \mathcal{N}_+)/N$, $(N - \mathcal{N}_+ + 1)/N$ or a $\Pi^{(2)}$-operator of the form*

$$N^{-h} \Pi^{(2)}_{\underline{\sharp}, \underline{\flat}}(\eta^{z_1}, \dots, \eta^{z_h}) \tag{2.114}$$

for some $h \in \mathbb{N}\backslash\{0\}$, $z_1, \dots, z_h \in \mathbb{N}\backslash\{0\}$ and $\underline{\sharp}, \underline{\flat} \in \{\cdot, *\}^h$. Suppose that the operators

$$\begin{gathered}\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp, \flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \check{\eta}^{\ell_1}_x) \\ \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp', \flat'}(\eta^{m_1}, \dots, \eta^{m_{k_2}}; \check{\eta}^{\ell_2}_y)\end{gathered}$$

for some $\sharp \in \{\cdot, *\}^{k_1}, \flat \in \{\cdot, *\}^{k_1+1}, \sharp' \in \{\cdot, *\}^{k_2}, \flat' \in \{\cdot, *\}^{k_2+1}$ appear in the expansion of $ad^{(n)}_{B(\eta)}(\check{b}_x)$ and of $ad^{(k)}_{B(\eta)}(\check{b}_y)$ for some $n, k \in \mathbb{N}$, as described in Lemma 2.2.3. Here we use the notation $\check{\eta}^{\ell_1}_x$ for the function $z \to \check{\eta}^{\ell_1}(x - z)$, where $\check{\eta}^{\ell_1}$ denotes the Fourier transform of the function η^{ℓ_1} defined on Λ_+^*. Let

$$\begin{aligned} S = \|(\mathcal{N}_+ + 1)^{\beta/2} \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp', \flat'}(\eta^{m_1}, \dots, \eta^{m_{k_2}}; \check{\eta}^{\ell_2}_y) \\ \times \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp, \flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \check{\eta}^{\ell_1}_x) \xi\| \end{aligned}$$

Then we have the following bounds. If $\ell_1, \ell_2 \geq 1$,

$$S \leq C^{n+k} \kappa^{n+k} \|(\mathcal{N}_+ + 1)^{(\beta+2)/2} \xi\| \tag{2.115}$$

If $\ell_1 = 0$ and $\ell_2 \geq 1$,

$$S \leq C^{n+k} \kappa^{n+k} \|\check{a}_x (\mathcal{N}_+ + 1)^{(\beta+1)/2} \xi\|$$

If $\ell_1 \geq 1$ and $\ell_2 = 0$,

$$\begin{aligned} S \leq\ & C^{n+k} \kappa^{n+k} n N^{-1} \|(\mathcal{N}_+ + 1)^{(\beta+2)/2} \xi\| \\ & + C^{n+k} \kappa^{n+k-\ell_1} \mu_{\ell_1} |\check{\eta}^{\ell_1}(x - y)| \|(\mathcal{N}_+ + 1)^{\beta/2} \xi\| \\ & + C^{n+k} \kappa^{n+k} \|\check{a}_y (\mathcal{N}_+ + 1)^{(\beta+1)/2} \xi\| \end{aligned} \tag{2.116}$$

where $\mu_{\ell_1} = 1$ if ℓ_1 is odd, while $\mu_{\ell_1} = 0$ if ℓ_1 is even. If $\ell_1 \geq 1$ and $\ell_2 = 0$ and we additionally assume that $k_1 > 0$ or $k_2 > 0$ or at least one of the Λ- or Λ'-operators is a $\Pi^{(2)}$-operator of the form (2.114), we obtain the improved estimate

$$\begin{aligned} S \leq\ & C^{n+k} \kappa^{n+k} n N^{-1} \|(\mathcal{N}_+ + 1)^{(\beta+2)/2} \xi\| \\ & + C^{n+k} \kappa^{n+k-\ell_1} \mu_{\ell_1} N^{-1} |\check{\eta}^{\ell_1}(x - y)| \|(\mathcal{N}_+ + 1)^{(\beta+2)/2} \xi\| \\ & + C^{n+k} \kappa^{n+k} \|\check{a}_y (\mathcal{N}_+ + 1)^{(\beta+1)/2} \xi\| \end{aligned} \tag{2.117}$$

Finally, if $\ell_1 = \ell_2 = 0$,

$$S \leq C^{n+k}\kappa^{n+k} n N^{-1} \|\check{a}_x(\mathcal{N}_+ + 1)^{(\beta+1)/2}\xi\| + C^{n+k}\kappa^{n+k}\|\check{a}_x\check{a}_y(\mathcal{N}_+ + 1)^{\beta/2}\xi\|$$

We are now ready to proceed with the proof of Prop. 2.4.5.

Proof of Prop. 2.4.5. We start by writing

$$\begin{aligned} e^{-B(\eta)} a^*_{-p} a_q e^{B(\eta)} &= a^*_{-p}a_q + \int_0^1 ds\, e^{-sB(\eta)}[a^*_{-p}a_q, B(\eta)]e^{sB(\eta)} \\ &= a^*_{-p}a_q + \int_0^1 e^{-sB(\eta)}(\eta_q b^*_{-p}b^*_{-q} + \eta_p b_q b_p)e^{sB(\eta)} \end{aligned}$$

With Lemma 2.2.4, we obtain

$$\begin{aligned} &e^{-B(\eta)} a^*_{-p} a_q e^{B(\eta)} \\ &\quad = a^*_{-p}a_q \\ &\qquad + \sum_{n,k\geq 0} \frac{(-1)^{n+k}}{n!k!(n+k+1)} \Big[\eta_q \mathrm{ad}^{(n)}_{B(\eta)}(b^*_{-p})\mathrm{ad}^{(k)}_{B(\eta)}(b^*_{-q}) + \eta_p \mathrm{ad}^{(n)}_{B(\eta)}(b_q)\mathrm{ad}^{(k)}_{B(\eta)}(b_p)\Big] \end{aligned}$$

From (2.113), we find

$$\begin{aligned} \mathcal{G}_N^{(3)} = &\sum_{r\geq 0} \frac{(-1)^r}{r!}\frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda^*_+ : p+q\neq 0} \widehat{V}(p/N)\mathrm{ad}^{(r)}_{B(\eta)}(b^*_{p+q})a^*_{-p}a_q \\ &+ \sum_{n,k,r\geq 0} \frac{(-1)^{n+k+r}}{n!k!r!(n_k+1)} \\ &\qquad \times \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda^*_+, p+q\neq 0} \widehat{V}(p/N)\eta_q\, \mathrm{ad}^{(r)}_{B(\eta)}(b^*_{p+q})\mathrm{ad}^{(n)}_{B(\eta)}(b^*_{-p})\mathrm{ad}^{(k)}_{B(\eta)}(b^*_{-q}) \\ &+ \sum_{n,k,r\geq 0} \frac{(-1)^{n+k+r}}{n!k!r!(n+k+1)} \\ &\qquad \times \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda^*_+, p+q\neq 0} \widehat{V}(p/N)\eta_p\, \mathrm{ad}^{(r)}_{B(\eta)}(b^*_{p+q})\mathrm{ad}^{(n)}_{B(\eta)}(b_p)\mathrm{ad}^{(k)}_{B(\eta)}(b_q) \\ &+ \text{h.c.} \end{aligned} \tag{2.118}$$

We start by analyzing the last sum on the r.h.s. of (2.118). From Lemma 2.2.3, each operator

$$\frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda^*_+, p+q\neq 0} \widehat{V}(p/N)\eta_p\, \mathrm{ad}^{(r)}_{B(\eta)}(b^*_{p+q})\mathrm{ad}^{(n)}_{B(\eta)}(b_p)\mathrm{ad}^{(k)}_{B(\eta)}(b_q) \tag{2.119}$$

can be expanded in the sum of $2^{n+k+r}n!k!r!$ terms having the form

$$\begin{aligned}\mathrm{L} = \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} & \widehat{V}(p/N)\eta_p\, \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\eta^{\ell_1}_{p+q}\varphi_{\alpha_1(p+q)})^*\Lambda^*_{i_1}\dots\Lambda^*_{i_1} \\ & \times \Lambda'_1\dots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1},\dots,\eta^{m_{k_2}};\eta^{\ell_2}_p\varphi_{\alpha_2 p}) \\ & \times \Lambda''_1\dots\Lambda''_{i_3}N^{-k_3}\Pi^{(1)}_{\sharp'',\flat''}(\eta^{s_1},\dots,\eta^{s_{k_3}};\eta^{\ell_3}\varphi_{\alpha_3 q})\end{aligned} \tag{2.120}$$

where $i_1,i_2,i_3,k_1,k_2,k_3,\ell_1,\ell_2,\ell_3\in\mathbb{N}$, $j_1,\dots,j_{k_1},m_1,\dots,m_{k_2},s_1,\dots,s_{k_3}\in\mathbb{N}\backslash\{0\}$, $\alpha_i = (-1)^{\ell_i}$ for $i=1,2$ and where each operator $\Lambda_i,\Lambda'_i,\Lambda''_i$ is either a factor $(N-\mathcal{N}_+)/N$, a factor $(N+1-\mathcal{N}_+)/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-h}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{z_1},\dots,\eta^{z_s})$$

for some $h,z_1\dots,z_h\in\mathbb{N}\backslash\{0\}$. The expectation of (2.120) can be bounded by

$$\begin{aligned}|\langle\xi,\mathrm{L}\xi\rangle| \leq \frac{\kappa}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*:\\ p\neq -q}} & |\widehat{V}(p/N)||\eta_p|\|\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\eta^{\ell_1}_{p+q}\varphi_{\alpha_1(p+q)})\xi\| \\ & \times \|\Lambda'_1\dots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1},\dots,\eta^{m_{k_2}};\eta^{\ell_2}_p\varphi_{\alpha_2 p}) \\ & \quad \times \Lambda''_1\dots\Lambda''_{i_3}N^{-k_3}\Pi^{(1)}_{\sharp'',\flat''}(\eta^{s_1},\dots,\eta^{s_{k_3}};\eta^{\ell_3}\varphi_{\alpha_3 q})\xi\|\end{aligned}$$

Combining the bounds (2.66) (if $\ell_1\geq 1$) and (2.67) (if $\ell_1 = 0$) on the one hand, and the bounds (2.68) (if $\ell_2,\ell_3\geq 1$), (2.69) (if $\ell_2\geq 1$ and $\ell_3 = 0$), (2.70) (if $\ell_2 = 0$ and $\ell_3\geq 1$) and (2.72) (if $\ell_1=\ell_2=0$) on the other hand, we conclude that

$$\begin{aligned}|\langle\xi,\mathrm{L}\xi\rangle| &\leq C^{n+k+r}\kappa^{n+k+r+2} \\ &\quad \times \frac{1}{\sqrt{N}}\sum_{\substack{p,q\in\Lambda_+^*:\\ p\neq -q}} \frac{1}{p^2}\Big\{\frac{1}{(p+q)^2}\|(\mathcal{N}_++1)^{1/2}\xi\| + \|a_{p+q}\xi\|\Big\} \\ &\quad\quad \times \Big\{\frac{(1+r/N)}{p^2q^2}\|(\mathcal{N}_++1)\xi\| + \frac{(1+r/N)}{p^2}\|a_q(\mathcal{N}_++1)^{1/2}\xi\| \\ &\quad\quad\quad + \frac{1}{q^2}\|a_p(\mathcal{N}_++1)^{1/2}\xi\| + \|a_pa_q\xi\|\Big\} \\ &\leq C^{n+k+r}(1+r/N)\kappa^{n+k+r+2}\|(\mathcal{N}_++1)^{1/2}\xi\|^2\end{aligned}$$

From (2.119), we obtain that the expectation of the last sum on the r.h.s. of (2.118) is bounded by

$$\begin{aligned}\Big|\sum_{n,k,r\geq 0}&\frac{(-1)^{n+k+r}}{n!k!r!(n+k+1)} \\ &\times\frac{\kappa}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*,p+q\neq 0}\widehat{V}(p/N)\eta_p\,\langle\xi,\mathrm{ad}^{(r)}_{B(\eta)}(b^*_{p+q})\mathrm{ad}^{(n)}_{B(\eta)}(b_p)\mathrm{ad}^{(k)}_{B(\eta)}(b_q)\xi\rangle\Big| \\ &\qquad\qquad\qquad\qquad \leq C\kappa^2\|(\mathcal{N}_++1)^{1/2}\xi\|^2\end{aligned} \tag{2.121}$$

Next, we consider the second sum on the r.h.s. of (2.118) (we take the hermitian conjugated operator). To bound the expectation of this term, we will need to use the potential energy operator. For this reason, it is convenient to switch to position space. We find

$$\begin{aligned}&\frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/N)\eta_q \, \mathrm{ad}^{(r)}_{B(\eta)}(b_{-q})\mathrm{ad}^{(n)}_{B(\eta)}(b_{-p})\mathrm{ad}^{(k)}_{B(\eta)}(b_{p+q}) \\ &\qquad = \kappa \int_{\Lambda\times\Lambda} dxdy\, N^{5/2}V(N(x-y))\mathrm{ad}^{(r)}_{B(\eta)}(b(\check{\eta}_x^{1+\ell_1})\mathrm{ad}^{(n)}_{B(\eta)}(\check{b}_y)\mathrm{ad}^{(k)}_{B(\eta)}(\check{b}_x)\end{aligned} \tag{2.122}$$

where we used the notation $\check{\eta}^s$ to indicate the Fourier transform of the sequence $\Lambda^* \ni p \to \eta_p^s$, and $\check{\eta}_x^s$ denotes the function (or the distribution, if $s=0$) $z \to \check{\eta}_x^s(z) = \check{\eta}^s(z-x)$. With Lemma 2.2.3, the r.h.s. of (2.122) can be written as the sum of $2^{n+k+r}n!k!r!$ terms, all having the form

$$\begin{aligned}\mathrm{M} = \kappa \int_{\Lambda\times\Lambda} dxdy\, N^{5/2}V(N(x-y))\, &\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\check{\eta}_x^{1+\ell_1}) \\ &\times \Lambda_1'\dots\Lambda_{i_2}'N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1},\dots,\eta^{m_{k_2}};\check{\eta}_y^{\ell_2}) \\ &\times \Lambda_1''\dots\Lambda_{i_3}''N^{-k_3}\Pi^{(1)}_{\sharp'',\flat''}(\eta^{s_1},\dots,\eta^{s_{k_3}};\check{\eta}_x^{\ell_3})\end{aligned} \tag{2.123}$$

where $i_1,i_2,i_3,k_1,k_2,k_3,\ell_1,\ell_2,\ell_3 \in \mathbb{N}$, $j_1,\dots,j_{k_1},m_1,\dots,m_{k_2},s_1,\dots,s_{k_3} \in \mathbb{N}\backslash\{0\}$ and where each operator $\Lambda_i,\Lambda_i',\Lambda_i''$ is either a factor $(N-\mathcal{N}_+)/N$, a factor $(N+1-\mathcal{N}_+)/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-h}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{z_1},\dots,\eta^{z_h}) \tag{2.124}$$

for some $h,z_1,\dots,z_h \in \mathbb{N}\backslash\{0\}$. To bound the expectation of (2.123), we first assume that $(n,k)\neq(0,1)$. Under this condition, we bound

$$\begin{aligned}|\langle\xi,\mathrm{M}\xi\rangle| \leq \kappa \int_{\Lambda\times\Lambda} dxdy\, N^{5/2}V(N(x-y))&\|N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\check{\eta}_x^{\ell_1+1})^*\Lambda_{i_1}^*\dots\Lambda_1^*\xi\| \\ \times \Big\|&\Lambda_1'\dots\Lambda_{i_2}'N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1},\dots,\eta^{m_{k_2}};\check{\eta}_y^{\ell_2}) \\ &\times \Lambda_1''\dots\Lambda_{i_3}''N^{-k_3}\Pi^{(1)}_{\sharp'',\flat''}(\eta^{s_1},\dots,\eta^{s_{k_3}};\check{\eta}_x^{\ell_3})\xi\Big\|\end{aligned} \tag{2.125}$$

With Lemma 2.4.6, we estimate

$$\|N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\check{\eta}_x^{\ell_1+1})^*\Lambda_{i_1}^*\dots\Lambda_1^*\xi\| \leq C^r\kappa^{r+1}\|(\mathcal{N}_++1)^{1/2}\xi\| \tag{2.126}$$

Considering separately all possible choices for the parameters ℓ_2,ℓ_3, Lemma 2.4.6 also implies that

$$
\begin{aligned}
&\Big\| \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1}, \dots, \eta^{m_{k_2}}; \check{\eta}^{\ell_2}_y) \Lambda''_1 \dots \Lambda''_{i_3} N^{-k_3} \Pi^{(1)}_{\sharp'',\flat''}(\eta^{s_1}, \dots, \eta^{s_{k_3}}; \check{\eta}^{\ell_3}_x) \xi \Big\| \\
&\qquad \leq C^{n+k} \kappa^{n+k} \Big\{ (1+k/N) \|(\mathcal{N}_+ + 1)\xi\| + (1+k/N) \|\check{a}_x (\mathcal{N}_+ + 1)^{1/2} \xi\| \\
&\qquad\qquad\qquad + \|\check{a}_y (\mathcal{N}_+ + 1)^{1/2} \xi\| + \|\check{a}_x \check{a}_y \xi\| \Big\}
\end{aligned}
\tag{2.127}
$$

When dealing with the choice $(\ell_2, \ell_3) = (0,1)$, we used here the exclusion of the pair $(n,k) = (0,1)$, which implies that $n + k \geq 1$ (because $n \geq \ell_2, k \geq \ell_3$) and therefore that either $k_2 > 0$ or $k_3 > 0$ or that at least one of the Λ'- or of the Λ''-operators is a $\Pi^{(2)}$-operator of the form (2.124); this observation allowed us to use the bound (2.117), which together with $|\check{\eta}(x-y)| \leq CN\|V\|_1$, led us to (2.127). Inserting (2.126) and (2.127) in (2.125), we arrive at

$$
\begin{aligned}
|\langle \xi, \mathrm{M} \xi \rangle| &\leq C^{n+k+r} (1+k/N) \kappa^{n+k+r+2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \int_{\Lambda \times \Lambda} dx dy \, N^{5/2} V(N(x-y)) \\
&\qquad \times \Big\{ \|(\mathcal{N}_+ + 1)\xi\| + \|\check{a}_x (\mathcal{N}_+ + 1)^{1/2}\| + \|\check{a}_y (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_x \check{a}_y \xi\| \Big\} \\
&\leq C^{n+k+r} (1+k/N) \kappa^{n+k+r+2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \\
&\quad + C^{n+k+r} (1+k/N) \kappa^{n+k+r+3/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|
\end{aligned}
\tag{2.128}
$$

Finally, let us consider the expectation of (2.123) in the case $(n,k) = (0,1)$. In fact, we can further restrict our attention to the choice $(\ell_2, \ell_3) = (0,1)$, because for all other choices of (ℓ_2, ℓ_3), the bound (2.127) remains true even if $(n,k) = (0,1)$. If $(\ell_2, \ell_3) = (n,k) = (0,1)$, by Lemma 2.2.3, part iii) and iv), the operator (2.123) has the form

$$
\begin{aligned}
\mathrm{M} &= \kappa \int_{\Lambda \times \Lambda} dx dy \, N^{5/2} V(N(x-y)) \\
&\qquad \times \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}, \check{\eta}^{1+\ell_1}_x) \check{b}_y \frac{(N - \mathcal{N}_+)}{N} b^*(\check{\eta}_x) \\
&= \kappa \int_{\Lambda \times \Lambda} dx dy \, N^{5/2} V(N(x-y)) \\
&\qquad \times \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}, \check{\eta}^{1+\ell_1}_x) a^*(\check{\eta}_x) \frac{(N + 1 - \mathcal{N}_+)}{N} \check{a}_y \\
&\quad + \kappa \int_{\Lambda \times \Lambda} dx dy \, N^{5/2} V(N(x-y)) \\
&\qquad \times \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}, \check{\eta}^{1+\ell_1}_x) \frac{(N + 1 - \mathcal{N}_+)}{N} \frac{(N - \mathcal{N}_+)}{N} \check{\eta}(x-y) \\
&=: \mathrm{M}_1 + \mathrm{M}_2
\end{aligned}
\tag{2.129}
$$

The expectation of the first term can be bounded by

$$\begin{aligned}|\langle \xi, \mathrm{M}_1 \xi\rangle| &\leq C^r \kappa^{r+2} \int_{\Lambda\times\Lambda} dxdy\, N^{5/2} V(N(x-y)) \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|\check{a}_y (\mathcal{N}_+ +1)^{1/2}\xi\| \\ &\leq C^r \kappa^{r+2} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2\end{aligned} \tag{2.130}$$

As for the second term on the r.h.s. of (2.129), its expectation vanishes on vectors $\xi \in \mathcal{F}_+^{\leq N}$ (because of the orthogonality to the constant orbital φ_0).

Combining (2.128) with (2.129) and (2.130), and summing over all $n,k,r \in \mathbb{N}$, we conclude that, if $\|V\|_1$ is small enough, the expectation of the second sum on the r.h.s. of (2.118) is bounded by

$$\begin{aligned}\Big| \sum_{n,k,r\geq 0} &\frac{(-1)^{n+k+r}}{n!k!r!(n+k+1)} \\ &\frac{\kappa}{\sqrt{N}} \times \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/N)\eta_q \,\langle \xi, \mathrm{ad}^{(r)}_{B(\eta)}(b^*_{p+q}) \mathrm{ad}^{(n)}_{B(\eta)}(b^*_{-p}) \mathrm{ad}^{(k)}_{B(\eta)}(b^*_{-q})\xi\rangle \Big| \\ &\qquad\qquad \leq C\kappa^2 \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + C\kappa^{3/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|\end{aligned} \tag{2.131}$$

Finally, we consider the first sum on the r.h.s. of (2.118). From Lemma 2.2.3, each operator

$$\frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^* : p+q\neq 0} \widehat{V}(p/N) \mathrm{ad}^{(r)}_{B(\eta)}(b^*_{p+q}) a^*_{-p} a_q \tag{2.132}$$

can be written as the sum of $2^r r!$ terms having the form

$$\mathrm{P} = \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^* : p+q\neq 0} \widehat{V}(p/N) N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\eta^{\ell_1}_{p+q}\varphi_{\alpha_1(p+q)})^* \Lambda^*_{i_1}\dots\Lambda^*_1 a^*_{-p} a_q \tag{2.133}$$

for $i_1, k_1, \ell_1 \in \mathbb{N}$, $j_1,\dots,j_{k_1} \in \mathbb{N}\backslash\{0\}$, $\alpha_1 = 1$ if ℓ_1 is even, $\alpha_1 = -1$ if ℓ_1 is odd. To bound the expectation of P we distinguish three cases.

If $\ell_1 \geq 2$, we bound (proceeding as in Lemma 2.4.1)

$$\begin{aligned}|\langle \xi, \mathrm{P}\xi\rangle| &\leq \frac{C\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^* : p\neq -q} |\eta_{p+q}|^{\ell_1} \, \|a_{-p}\Lambda_1..\Lambda_{i_1} N^{-k_1}\Pi^{(1)}(\eta^{j_1},\dots,\eta^{j_{k_1}};\varphi_{\alpha_1(p+q)})\xi\| \|a_q\xi\| \\ &\leq C^r\kappa^{r+1} \sum_{p,q\in\Lambda_+^* : p\neq -q} \frac{1}{(p+q)^4} \left\{ \|a_{-p}\xi\| + \frac{r}{Np^2}\|(\mathcal{N}_+ +1)^{1/2}\xi\| \right\} \|a_q\xi\| \\ &\leq C^r (1+r/N)\kappa^{r+1} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2\end{aligned}$$

If $\ell_1 = 1$, we commute the operator $a_{-(p+q)}$ (or the $b_{-(p+q)}$ operator) appearing in the $\Pi^{(1)}$-operator in (2.133) to the right, and the operator a^*_{-p} to the left (it is important to note that $[a_{-(p+q)}, a^*_{-p}] = 0$ since $q \neq 0$). We find

$$\begin{aligned}
|\langle \xi, \mathrm{P}\xi\rangle| &\leq \frac{C^r\kappa^{r+1}}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p\neq -q} |\widehat{V}(p/N)| \frac{1}{(p+q)^2} \Big\{ \frac{r}{Np^2} \|(\mathcal{N}_+ + 1)\xi\| \|a_q\xi\| \\
&\qquad\qquad + \frac{1}{N(p+q)^2} \|a_{-p}(\mathcal{N}_+ + 1)^{1/2}\xi\| \|a_q\xi\| + \|a_{-p}\xi\| \|a_{-(p+q)}a_q\xi\| \Big\} \\
&\leq C^r\kappa^{r+1} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2
\end{aligned}$$

Finally, if $\ell_1 = 0$ we only commute a^*_{-p} to the left. We find (similarly as in Lemma 2.4.1)

$$\begin{aligned}
|\langle \xi, \mathrm{P}\xi\rangle| \leq & \left| \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/N) \langle \mathrm{R}\, a_{p+q}a_{-p}\xi, a_q\xi\rangle \right| \\
& + \frac{C^r r\kappa^{r+1}}{N} \sum_{p,q\in\Lambda_+^*: p\neq -q} \frac{|\widehat{V}(p/N)|}{p^2} \|a_{p+q}\xi\| \|a_q\xi\|
\end{aligned} \tag{2.134}$$

for an operator R with $\|R\xi\| \leq C^r\kappa^r$. To bound the first term, we switch to position space. We find, similarly to (2.108),

$$\begin{aligned}
|\langle \xi, \mathrm{P}\xi\rangle| &\leq \kappa \int_{\Lambda\times\Lambda} dxdy\, N^{5/2}V(N(x-y)) \|\mathrm{R}\, \check{a}_x\check{a}_y\xi\| \|\check{a}_x\xi\| \\
&\qquad + \frac{C^r r\kappa^{r+1}}{N} \sum_{p,q\in\Lambda_+^*: p\neq -q} \frac{|\widehat{V}(p/N)|}{p^2} \|a_{p+q}\xi\| \|a_q\xi\| \\
&\leq C^r\kappa^{r+1/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| + C^r r\kappa^{r+1} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2
\end{aligned}$$

From (2.132), summing over all $r \in \mathbb{N}$, we conclude that the expectation of the first sum on the r.h.s. of (2.118) is bounded, if $\|V\|_1$ is small enough, by

$$\begin{aligned}
\Big| \sum_{r\geq 0} \frac{(-1)^r}{r!} \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/n) \langle \xi, \mathrm{ad}^{(r)}_{B(\eta)}(b^*_{p+q}) a^*_{-p}a_q\xi\rangle \Big| \\
\leq C\kappa \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\kappa^{1/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|
\end{aligned}$$

From (2.118), (2.121), (2.131) and the last equation, it follows that for every $\delta > 0$ there exists $C > 0$ such that

$$\pm \mathcal{G}_N^{(3)} \leq \delta \mathcal{V}_N + C\kappa(\mathcal{N}_+ + 1)$$

$\square$

2.4.5 Analysis of $\mathcal{G}_N^{(4)}$

With $\mathcal{L}_N^{(4)}$ as defined in (2.48), we write

$$\begin{aligned}\mathcal{G}_N^{(4)} &= e^{-B(\eta)}\mathcal{L}_N^{(4)}e^{B(\eta)} \\ &= \mathcal{V}_N + \frac{\kappa}{2N}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{q+r}\eta_q \\ &\quad + \frac{\kappa}{2N}\sum_{q,r\in\Lambda_+^*: r\neq -q} \widehat{V}(r/N)\,\eta_{q+r}\left(b_q b_{-q} + b_q^* b_{-q}^*\right) + \mathcal{E}_N^{(4)}\end{aligned} \tag{2.135}$$

In the next proposition, we estimate the error term $\mathcal{E}_N^{(4)}$.

Proposition 2.4.7. *Let the assumptions of Proposition 2.3.2 be satisfied (in particular, suppose $\kappa \geq 0$ is small enough). Then, for every $\delta > 0$ there exists $C > 0$ such that, on $\mathcal{F}_+^*$,*

$$\pm\mathcal{E}_N^{(4)} \leq \delta\mathcal{V}_N + C\kappa(\mathcal{N}_+ + 1)$$

Proof. We have

$$\begin{aligned}&e^{-B(\eta)}\mathcal{L}_N^{(4)}e^{B(\eta)} \\ &\quad= \frac{\kappa}{2N}\sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq -p,q} \widehat{V}(r/N)e^{-B(\eta)}a_p^*a_q^*a_{q-r}a_{p+r}e^{B(\eta)} \\ &\quad= \mathcal{V}_N + \frac{\kappa}{2N}\sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq -p,q} \widehat{V}(r/N)\int_0^1 ds\, e^{-sB(\eta)}\left[a_p^*a_q^*a_{q-r}a_{p+r}, B(\eta)\right]e^{sB(\eta)} \\ &\quad= \mathcal{V}_N + \frac{\kappa}{2N}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{q+r}\int_0^1 ds\,\left(e^{-sB(\eta)}b_q^*b_{-q}^*e^{sB(\eta)} + \text{h.c.}\right) \\ &\qquad + \frac{\kappa}{N}\sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq p,-q} \widehat{V}(r/N)\eta_{q+r}\int_0^1 ds\,\left(e^{-sB(\eta)}b_{p+r}^*b_q^*a_{-q-r}^*a_p e^{sB(\eta)} + \text{h.c.}\right)\end{aligned} \tag{2.136}$$

Now we observe that

$$\begin{aligned}e^{-sB(\eta)}a_{-q-r}^*a_p e^{sB(\eta)} &= a_{-q-r}^*a_p + \int_0^s d\tau\, e^{-\tau B(\eta)}\left[a_{-q-r}^*a_p, B(\eta)\right]e^{-\tau B(\eta)} \\ &= a_{-q-r}^*a_p + \int_0^s d\tau\, e^{-\tau B(\eta)}\left(\eta_p b_{-p}^*b_{-q-r}^* + \eta_{q+r}b_p b_{q+r}\right)e^{-\tau B(\eta)}\end{aligned}$$

Inserting in (2.136) and using Lemma 2.2.4, we obtain

$$e^{-B(\eta)}\mathcal{L}_N^{(4)}e^{B(\eta)} - \mathcal{V}_N = \mathrm{W}_1 + \mathrm{W}_2 + \mathrm{W}_3 + \mathrm{W}_4$$

where we defined

$$\begin{aligned}
\mathrm{W}_1 = & \sum_{n,k=0}^{\infty} \frac{(-1)^{n+k}}{n!k!(n+k+1)} \\
& \times \frac{\kappa}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{q+r} \left(\mathrm{ad}^{(n)}_{B(\eta)}(b_q)\mathrm{ad}^{(k)}_{B(\eta)}(b_{-q}) + \mathrm{h.c.}\right) \\
\mathrm{W}_2 = & \sum_{n,k=0}^{\infty} \frac{(-1)^{n+k}}{n!k!(n+k+1)} \\
& \times \frac{\kappa}{N} \sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq p,-q} \widehat{V}(r/N)\eta_{q+r} \left(\mathrm{ad}^{(n)}_{B(\eta)}(b^*_{p+r})\mathrm{ad}^{(k)}_{B(\eta)}(b^*_{q})a^*_{-q-r}a_p + \mathrm{h.c.}\right)
\end{aligned} \tag{2.137}$$

and

$$\begin{aligned}
\mathrm{W}_3 = & \sum_{n,k,i,j=0}^{\infty} \frac{(-1)^{n+k+i+j}}{n!k!i!j!(i+j+1)(n+k+i+j+2)} \\
& \frac{\kappa}{N} \sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq -p-q} \widehat{V}(r/N)\eta_{q+r}\eta_p \\
& \quad \times \left(\mathrm{ad}^{(n)}_{B(\eta)}(b^*_{p+r})\mathrm{ad}^{(k)}_{B(\eta)}(b^*_{q})\mathrm{ad}^{(i)}_{B(\eta)}(b^*_{-p})\mathrm{ad}^{(j)}_{B(\eta)}(b^*_{-q-r}) + \mathrm{h.c.}\right) \\
\mathrm{W}_4 = & \sum_{n,k,i,j=0}^{\infty} \frac{(-1)^{n+k+i+j}}{n!k!i!j!(i+j+1)(n+k+i+j+2)} \\
& \times \frac{\kappa}{N} \sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq -p-q} \widehat{V}(r/N)\eta^2_{q+r} \\
& \quad \times \left(\mathrm{ad}^{(n)}_{B(\eta)}(b^*_{p+r})\mathrm{ad}^{(k)}_{B(\eta)}(b^*_{q})\mathrm{ad}^{(i)}_{B(\eta)}(b_{p})\mathrm{ad}^{(j)}_{B(\eta)}(b_{q+r}) + \mathrm{h.c.}\right)
\end{aligned} \tag{2.138}$$

We consider, first of all, the expectation of the term W_2. Since we will need the potential energy operator to bound this term, it is convenient to switch to position space. On $\mathcal{F}_+^{\leq N}$, we find

$$\begin{aligned}
\mathrm{W}_2 = & \sum_{n,k=0}^{\infty} \frac{(-1)^{n+k}}{n!k!(n+k+1)} \\
& \times \kappa \int_{\Lambda\times\Lambda} dxdy N^2 V(N(x-y)) \left(\mathrm{ad}^{(n)}_{B(\eta)}(\check{b}^*_x)\mathrm{ad}^{(k)}_{B(\eta)}(\check{b}^*_y)a^*(\check{\eta}_x)\check{a}_y + \mathrm{h.c.}\right)
\end{aligned} \tag{2.139}$$

with the notation $\check{\eta}_x(z) = \check{\eta}(x-z)$. With Cauchy-Schwarz, we find

$$\begin{aligned}
& \Big| \kappa \int_{\Lambda\times\Lambda} dxdy\, N^2 V(N(x-y)) \langle \xi, \mathrm{ad}^{(n)}_{B(\eta)}(\check{b}^*_x)\mathrm{ad}^{(k)}_{B(\eta)}(\check{b}^*_y)a^*(\check{\eta}_x)\check{a}_y\xi\rangle \Big| \\
& \qquad \leq \kappa \int_{\Lambda\times\Lambda} dxdy\, N^2 V(N(x-y)) \\
& \qquad\qquad \times \|(\mathcal{N}_+ + 1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta)}(\check{b}_y)\mathrm{ad}^{(n)}_{B(\eta)}(\check{b}_x)\xi\| \|(\mathcal{N}_+ + 1)^{-1/2}a^*(\check{\eta}_x)\check{a}_y\xi\|
\end{aligned}$$

We bound

$$\|(\mathcal{N}_+ + 1)^{-1/2} a^*(\check{\eta}_x)\check{a}_y\xi\| \leq C\kappa\|\check{a}_y\xi\|$$

With Lemma 2.2.3, we estimate $\|(\mathcal{N}_+ + 1)^{1/2}\text{ad}^{(k)}_{B(\eta)}(\check{b}_y)\text{ad}^{(n)}_{B(\eta)}(\check{b}_x)\xi\|$ by the sum of $2^{n+k}n!k!$ terms of the form

$$\begin{aligned} \mathrm{T} = \Big\|(\mathcal{N}_+ + 1)^{1/2}\Lambda_1 \dots \Lambda_{i_1} N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \check{\eta}^{\ell_1}_y) \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2}\Pi^{(1)}_{\sharp,\flat}(\eta^{m_1}, \dots, \eta^{m_{k_2}}; \check{\eta}^{\ell_2}_x)\xi\Big\| \end{aligned} \tag{2.140}$$

with $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \geq 0$, $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \geq 0$ and where each Λ_i and Λ'_i operator is either a factor $(N - \mathcal{N}_+)/N$, $(N - \mathcal{N}_+ + 1)/N$ or a $\Pi^{(2)}$-operator (here $\check{\eta}^{\ell_1}$ indicates the function with Fourier coefficients given by $\eta^{\ell_1}_p$, for all $p \in \Lambda^*_+$).

With Lemma 2.4.6, we find

$$\begin{aligned} \mathrm{T} \leq (n+1)C^{k+n}\kappa^{k+n}\Big\{\|(\mathcal{N}_+ + 1)^{3/2}\xi\| + \|\check{a}_y(\mathcal{N}_+ + 1)\xi\| + \|\check{a}_x(\mathcal{N}_+ + 1)\xi\| \\ + N\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \sqrt{N}\|\check{a}_x\check{a}_y\xi\|\Big\} \end{aligned} \tag{2.141}$$

For $\xi \in \mathcal{F}^{\leq N}_+$, we obtain

$$\begin{aligned} &\Big|\kappa\int_{\Lambda\times\Lambda} dxdy N^2V(N(x-y))\langle\xi, \text{ad}^{(n)}_{B(\eta)}(\check{b}^*_x)\text{ad}^{(k)}_{B(\eta)}(\check{b}^*_y)a^*(\check{\eta}_x)\check{a}_y\xi\rangle\Big| \\ &\leq (n+1)!k!\, C^{n+k}\kappa^{n+k+2}\int_{\Lambda\times\Lambda} dxdy\, N^2V(N(x-y))\|a_y\xi\| \\ &\quad\times\Big\{N\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N\|\check{a}_x\xi\| + N\|\check{a}_y\xi\| + N^{1/2}\|\check{a}_x\check{a}_y\xi\|\Big\} \\ &\leq (n+1)!k!\, C^{n+k}\kappa^{n+k+3/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{V}_N + \mathcal{N}_+ + 1)^{1/2}\xi\| \end{aligned}$$

and therefore, if κ is small enough,

$$|\langle\xi, \mathrm{W}_2\xi\rangle| \leq C\kappa^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\kappa^{3/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}^{1/2}_N\xi\|\,. \tag{2.142}$$

Next, let us consider the term W_3, defined in (2.138). As above, we switch to position space. We find

$$\begin{aligned} \mathrm{W}_3 = &\sum_{n,k,i,j=0}^{\infty} \frac{(-1)^{n+k+i+j}}{n!k!i!j!(i+j+1)(n+k+i+j+2)} \\ &\times \kappa\int dxdy\, N^2V(N(x-y)) \\ &\quad\times\Big(\text{ad}^{(n)}(\check{b}^*_x)\text{ad}^{(k)}_{B(\eta)}(\check{b}^*_y)\text{ad}^{(i)}_{B(\eta)}(b^*(\check{\eta}_x))\text{ad}^{(j)}_{B(\eta)}(b^*(\check{\eta}_y)) + \text{h.c.}\Big) \end{aligned} \tag{2.143}$$

With Cauchy-Schwarz, we have

$$\begin{aligned}\Big|\kappa\int dxdy N^2V(N(x-y))\langle\xi,\mathrm{ad}^{(n)}_{B(\eta)}(\check{b}^*_x)\mathrm{ad}^{(k)}_{B(\eta)}(\check{b}^*_y)\mathrm{ad}^{(i)}_{B(\eta)}(\check{b}^*(\check{\eta}_x))\mathrm{ad}^{(j)}_{B(\eta)}(\check{b}(\check{\eta}_y))\xi\rangle\Big| \\ \leq \kappa\int dxdy\, N^2V(N(x-y))\,\|(\mathcal{N}_++1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta)}(\check{b}_y)\mathrm{ad}^{(n)}_{B(\eta)}(\check{b}_x)\xi\| \\ \times\|(\mathcal{N}_++1)^{-1/2}\mathrm{ad}^{(i)}_{B(\eta)}(b(\check{\eta}_x))\mathrm{ad}^{(j)}(b(\check{\eta}_y))\xi\|\end{aligned}$$

Expanding $\mathrm{ad}^{(i)}_{B(\eta_t)}(b(\check{\eta}_x))\mathrm{ad}^{(j)}(b(\check{\eta}_y))$ as in Lemma 2.2.3 and using Lemma 2.4.6 (with ℓ_1 and ℓ_2 replaced by ℓ_1+1 and ℓ_2+1, so that we can always use the inequality (2.115)), we obtain

$$\|(\mathcal{N}_++1)^{-1/2}\mathrm{ad}^{(i)}_{B(\eta)}(b(\check{\eta}_x))\mathrm{ad}^{(j)}(b(\check{\eta}_y))\xi\|\leq i!j!\,C^{i+j}\kappa^{i+j+2}\|(\mathcal{N}_++1)^{1/2}\xi\| \tag{2.144}$$

As for the norm $\|(\mathcal{N}_++1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta)}(\check{b}_y)\mathrm{ad}^{(n)}_{B(\eta)}(\check{b}_x)\xi\|$, we can estimate by the sum of $2^{n+k}n!k!$ contributions of the form (2.140). With (2.141), we conclude that, if κ is small enough,

$$|\langle\xi,\mathrm{W}_3\xi\rangle|\leq C\kappa^2\|(\mathcal{N}_++1)^{1/2}\xi\|^2+C\kappa^{3/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\| \tag{2.145}$$

The term W_4 in (2.138) can be bounded similarly. First, we switch to position space. We find

$$\begin{aligned}\mathrm{W}_4=&\sum_{n,k,i,j=0}^{\infty}\frac{(-1)^{n+k+i+j}}{n!k!i!j!(i+j+1)(n+k+i+j+2)}\\ &\times\kappa\int dxdy\,N^2V(N(x-y))\,\Big(\mathrm{ad}^{(n)}(\check{b}_x)\mathrm{ad}^{(k)}(\check{b}_y)\mathrm{ad}^{(i)}(b(\check{\eta}^2_x))\mathrm{ad}^{(j)}(\check{b}_y)+\mathrm{h.c.}\Big)\end{aligned} \tag{2.146}$$

The expectation of the operators on the r.h.s. of (2.146) can be bounded similarly as we did for the operators on the r.h.s. of (2.143). The only difference is the fact that now we have to replace the estimate (2.144) with

$$\|(\mathcal{N}_++1)^{-1/2}\mathrm{ad}^{(i)}(b(\check{\eta}^2_x))\mathrm{ad}^{(j)}(\check{b}_y)\xi\|\leq i!j!C^{i+j}\kappa^{i+j+2}\Big[\|(\mathcal{N}_++1)^{1/2}\xi\|+\|a_y\xi\|\Big]$$

We arrive at

$$|\langle\xi,\mathrm{W}_4\xi\rangle|\leq C\kappa^2\|(\mathcal{N}_++1)^{1/2}\xi\|^2+C\kappa^{3/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\| \tag{2.147}$$

Finally, we consider the term W_1 in (2.137). Here, we separate contributions with $(n,k)=(0,0),(0,1)$ by writing:

$$\begin{aligned}\mathrm{W}_1=&\frac{\kappa}{2N}\sum_{q\Lambda^*_+,r\in\Lambda^*:r\neq-q}\widehat{V}(r/N)\eta_{r+q}(b_qb_{-q}+\mathrm{h.c.})\\ &-\frac{\kappa}{4N}\sum_{q\in\Lambda^*_+,r\in\Lambda^*:r\neq-q}\widehat{V}(r/N)\eta_{q+r}\left(b_q\left[B(\eta),b_{-q}\right]+\mathrm{h.c.}\right)+\widetilde{W}_1\end{aligned} \tag{2.148}$$

where

$$\widetilde{W}_1 = \sum_{n,k}^{*} \frac{(-1)^{n+k}}{n!k!(n+k+1)} \frac{\kappa}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{q+r} \left(\mathrm{ad}^{(n)}_{B(\eta)}(b_q)\mathrm{ad}^{(k)}_{B(\eta)}(b_{-q}) + \text{h.c.}\right) \tag{2.149}$$

and where the sum $\sum_{n,k}^*$ runs over all pairs $(n,k) \neq (0,0),(0,1)$.

The first term on the r.h.s. of (2.148) does not enter the definition (2.135) of the error term $\mathcal{E}_N^{(4)}$. We do not have to estimate it. As for the second term on the r.h.s. of (2.148), we compute the commutator

$$[B(\eta), b_{-q}] = -\eta_q(1-\mathcal{N}_+/N)b_q^* + \frac{1}{N}\sum_{m\in\Lambda_+^*} \eta_m b_m^* a_{-m}^* a_{-q}$$

Hence

$$\begin{aligned}
&\frac{\kappa}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q} b_q \,[B(\eta), b_{-q}] \\
&\qquad = -\frac{\kappa}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_q b_q b_q^* \left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) \\
&\qquad\quad + \frac{\kappa}{N^2} \sum_{q,m\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_m b_q b_m^* a_{-m}^* a_{-q}
\end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{\kappa}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q} b_q \,[B(\eta), b_{-q}] \\
&\qquad = -\frac{\kappa}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_q \left(1 - \frac{\mathcal{N}_+}{N}\right)\left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) \\
&\qquad\quad + \frac{2\kappa}{N^2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_q a_q^* a_q \left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) \\
&\qquad\quad + \frac{\kappa}{N^3} \sum_{q,m\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_m a_m^* a_{-m}^* a_q a_{-q}
\end{aligned}$$

We conclude that

$$\begin{aligned}
\frac{\kappa}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q} b_q \,[B(\eta), b_{-q}] + \frac{\kappa}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_q \\
= \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3
\end{aligned}$$

with

$$\mathrm{T}_1 = \frac{\kappa}{N^2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_q(2\mathcal{N}_+ + 1 + \mathcal{N}_+/N + \mathcal{N}_+^2/N)$$

$$\mathrm{T}_2 = \frac{2\kappa}{N^2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_q a_q^* a_q \left(1 - \frac{\mathcal{N}_+ + 1}{N}\right)$$

$$\mathrm{T}_3 = \frac{\kappa}{N^3} \sum_{q,m\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_m a_m^* a_{-m}^* a_q a_{-q}$$

Since

$$\frac{\kappa}{N^2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\frac{1}{(r+q)^2 q^2} \leq C < \infty \tag{2.150}$$

uniformly in N, we easily find

$$|\langle \xi, \mathrm{T}_1\xi\rangle| \leq C\kappa \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

Furthermore,

$$|\langle \xi, \mathrm{T}_2\xi\rangle| \leq \frac{2\kappa^3}{N^2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} |\widehat{V}(r/N)| \frac{1}{(r+q)^2 q^2}\|a_q\xi\|^2 \leq CN^{-1}\kappa^3\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

Finally, we consider the term T_3. To this end, we switch to position space. We find

$$\begin{aligned}\mathrm{T}_3 &= \frac{\kappa}{N^3} \sum_{q,m\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{r+q}\eta_m a_m^* a_{-m}^* a_q a_{-q} \\ &= \kappa \int_{\Lambda\times\Lambda} dxdy\, V(N(x-y))\check{\eta}(x-y) B\check{a}_x\check{a}_y\end{aligned}$$

where $B = \sum_{m\in\Lambda_+^*} \eta_m a_m^* a_{-m}^*$. Since $\|B^*\xi\| \leq C\kappa\|(\mathcal{N}_+ + 1)\xi\|$, we obtain

$$\begin{aligned}|\langle \xi, \mathrm{T}_3\xi\rangle| &\leq C\kappa^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_{\Lambda\times\Lambda} dxdy\, N^{1/2}V(N(x-y))|\check{\eta}(x-y)|\|\check{a}_x\check{a}_y\xi\| \\ &\leq C\kappa^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_{\Lambda\times\Lambda} N^{3/2}V(N(x-y))\|a_x a_y\xi\| \\ &\leq CN^{-1}\kappa^{3/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\end{aligned}$$

Let us now focus on the expectation of (2.149). According to Lemma 2.2.3, the operator

$$\frac{\kappa}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{q+r}\mathrm{ad}^{(n)}(b_q)\mathrm{ad}^{(k)}(b_{-q})$$

can be written as the sum of $2^{n+k}n!k!$ terms having the form

$$\begin{aligned}\mathrm{X} = \frac{\kappa}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} &\widehat{V}(r/N)\eta_{q+r}\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_{k_1}};\eta_q^{\ell_1}\varphi_{\alpha_1 q}) \\ &\times \Lambda_1'\dots\Lambda_{i_2}'N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{m_1},\dots,\eta^{m_{k_2}};\eta_q^{\ell_2}\varphi_{-\alpha_2 q})\end{aligned}$$

where $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$, $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \in \mathbb{N}\backslash\{0\}$, $\alpha_i = 1$ if ℓ_i is even and $\alpha_i = -1$ if ℓ_i is odd. To bound the expectation of the operator X, we distinguish two cases.

If $\ell_1 + \ell_2 \geq 1$, we use Lemma 2.4.1 to estimate

$$\begin{aligned} |\langle \xi, \mathrm{X}\xi\rangle| \leq C^{n+k}\kappa^{n+k+2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| N^{-1} &\sum_{q,r\in\Lambda_+^*: r\neq -q} \frac{|\widehat{V}(r/N)|}{(q+r)^2} \\ &\times \left\{ \frac{1}{q^4}(1+k/N)\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \frac{1}{q^2}\|a_q\xi\| + \frac{1}{Nq^2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \right\} \end{aligned}$$

Here we used the fact that we excluded the pairs $(n,k) = (0,0), (0,1)$ to make sure that, if $\ell_1 = 0$ and $\ell_2 = 1$, then either $k_1 > 0$ or $k_2 > 0$ or at least one of the operators Λ or Λ' has to be a $\Pi^{(2)}$-operator. From (2.150) and from the similar bound

$$\sup_q \frac{1}{N} \sum_r |\widehat{V}(r/N)| \frac{1}{(q+r)^2} \leq C < \infty$$

uniformly in N, we conclude that, for $\ell_1 + \ell_2 \geq 1$,

$$|\langle \xi, \mathrm{X}\xi\rangle| \leq C\kappa^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \tag{2.151}$$

For $\ell_1 = \ell_2 = 0$, we use Lemma 2.4.1 to write

$$\mathrm{X} = \frac{\kappa}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*} \widehat{V}(r/N)\eta_{q+r}\left[A_q + Ba_q a_{-q}\right] =: \mathrm{X}_1 + \mathrm{X}_2$$

where

$$|\langle \xi, \mathrm{A}_q\xi\rangle| \leq C^{n+k}\kappa^{n+k} \frac{k}{Nq^2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

and (since we excluded the term with $(n,k) = (0,0)$)

$$\|B^*\xi\| \leq C^{n+k} N^{-1} \kappa^{n+k} \|(\mathcal{N}_+ + 1)\xi\|$$

We immediately obtain that

$$\begin{aligned} |\langle \xi, \mathrm{X}_1\xi\rangle| &\leq \frac{C^{n+k}\kappa^{n+k+2}}{N^2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*} \widehat{V}(r/N) \frac{1}{(q+r)^2 q^2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \\ &\leq C^{n+k}\kappa^{n+k+2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \end{aligned}$$

and, switching to position space,

$$\begin{aligned} |\langle \xi, \mathrm{X}_2\xi\rangle| &= \left| \kappa \int_{\Lambda\times\Lambda} dxdy\, N^2 V(N(x-y))\eta(x-y)\langle B^*\xi, \check{a}_x\check{a}_y\xi\rangle \right| \\ &\leq \kappa \int_{\Lambda\times\Lambda} dxdy N^2 V(N(x-y))|\eta(x-y)| \|\check{a}_x\check{a}_y\xi\| \|B^*\xi\| \\ &\leq C\kappa^{n+k+2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_{\Lambda\times\Lambda} dxdy N^{5/2} V(N(x-y)) \|\check{a}_x\check{a}_y\xi\| \\ &\leq C\kappa^{n+k+3/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| \end{aligned}$$

Combining the last two bounds with (2.151), and then summing over all n, k, we find

$$|\langle \xi, \widetilde{W}_1 \xi\rangle| \leq C\kappa^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\kappa^{3/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|$$

With (2.137), (2.138), (2.142), (2.145), (2.147), we conclude the proof of the proposition. $\square$

2.4.6 Proof of Proposition 2.3.2

Combining the results of Prop. 2.4.2, Prop. 2.4.3, Prop. 2.4.4, Prop. 2.4.5 and Prop. 2.4.7, we conclude that the excitation Hamiltonian $\mathcal{G}_N$ defined in (2.59) is such that

$$\begin{aligned}
\mathcal{G}_N = \; & \frac{(N-1)}{2}\kappa\widehat{V}(0) + \sum_{p\in\Lambda_+^*} \eta_p \left[p^2\eta_p + \kappa\widehat{V}(p/N) + \frac{\kappa}{2N}\sum_{r\in\Lambda^*: r\neq -p} \widehat{V}(r/N)\eta_{p+r}\right] \\
& + \sum_{p\in\Lambda_+^*} \left[p^2\eta_p + \kappa\frac{\widehat{V}(p/N)}{2} + \frac{\kappa}{2N}\sum_{r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_{p+r}\right]\left[b_p b_{-p} + b_p^* b_{-p}^*\right] \\
& + \mathcal{K} + \mathcal{V}_N + \mathcal{E}_N
\end{aligned}$$

where the operator $\mathcal{E}_N$ is such that, for all $\delta > 0$ there exists $C > 0$ with

$$\pm\mathcal{E}_N \leq \delta(\mathcal{K} + \mathcal{V}_N) + C\kappa(\mathcal{N}_+ + 1)$$

With (2.58), we obtain

$$\begin{aligned}
\mathcal{G}_N = \; & \frac{(N-1)}{2}\kappa\widehat{V}(0) + \frac{\kappa}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\eta_p + \mathcal{K} + \mathcal{V}_N + \mathcal{E}_N \\
& + \sum_{p\in\Lambda_+^*} \eta_p \left[-\kappa\frac{\widehat{V}(p/N)\eta_0}{2N} + \lambda_\ell N^3 \widehat{\chi}_\ell(p) + \lambda_\ell N^2 \sum_{q\in\Lambda^*} \widehat{\chi}_\ell(p-q)\eta_q \right] \\
& + \sum_{p\in\Lambda_+^*} \left[N^3\lambda_\ell\widehat{\chi}_\ell(p) + N^2\lambda_\ell \sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q - \frac{\kappa}{2N}\widehat{V}(p/N)\eta_0\right](b_p b_{-p} + b_p^* b_{-p}^*)
\end{aligned} \tag{2.152}$$

With the definition (2.54) and with the estimate (2.56) we find that

$$\left| \frac{(N-1)}{2}\kappa\widehat{V}(0) + \frac{\kappa}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\eta_p - \frac{N\kappa}{2}\int N^3 V(Nx) f_\ell(Nx)dx\right| \leq C\kappa$$

With the approximate identity (2.52), we conclude that

$$\left| \frac{(N-1)}{2}\kappa\widehat{V}(0) + \frac{\kappa}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\eta_p - 4\pi\mathfrak{a}_0 N\right| \leq C\kappa\,.$$

As for the terms on the second line on the r.h.s. of (2.152), they are all at most of order one. The first term can be estimated with (2.56) by

$$\Big|\frac{\kappa}{N}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\eta_p\eta_0\Big| \leq \frac{C\kappa^3}{N}\sum_{p\in\Lambda_+^*}\frac{\widehat{V}(p/N)}{p^2} \leq C\kappa^3$$

similarly to (2.108). The second term can be controlled using Lemma 2.3.1, part i), which implies that $\lambda_\ell N^3 \leq C\kappa$. We find

$$N^3\lambda_\ell\sum_{p\in\Lambda_+^*}\widehat{\chi}_\ell(p)\eta_p \leq C\kappa\|\chi_\ell\|\|\eta\| \leq C\kappa^2$$

As for the third term, we use again the bound $N^3\lambda_\ell \leq C\kappa$ to estimate

$$\Big|\lambda_\ell N^2\sum_{p\in\Lambda_+^*,q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q\eta_p\Big| \leq CN^{-1}\kappa\|\eta\|^2 \leq CN^{-1}\kappa^3$$

Next, we bound the expectation of the operator on the last line on the r.h.s. of (2.152). The first contribution can be estimated by

$$\begin{aligned}\Big|N^3\lambda_\ell\sum_{p\in\Lambda_+^*}\widehat{\chi}_\ell(p)\langle\xi, b_p b_{-p}\xi\rangle\Big| &\leq C\kappa\|(\mathcal{N}_++1)^{1/2}\xi\|\sum_{p\in\Lambda_+^*}|\widehat{\chi}_\ell(p)|\|a_{-p}\xi\| \\ &\leq C\kappa\|\chi_\ell\|\|(\mathcal{N}_++1)^{1/2}\xi\|^2 \leq C\kappa\|(\mathcal{N}_++1)^{1/2}\xi\|^2\end{aligned}$$

Similarly,

$$\begin{aligned}\Big|N^2\lambda_\ell\sum_{p\in\Lambda_+^*,q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q\langle\xi, b_p b_{-p}\xi\rangle\Big| &\leq CN^{-1}\kappa\|\widehat{\chi}_\ell*\eta\|\|(\mathcal{N}_++1)^{1/2}\xi\|^2 \\ &\leq CN^{-1}\kappa\|(\mathcal{N}_++1)^{1/2}\xi\|^2\end{aligned}$$

Finally, to estimate the contribution of the last term on the last line on the r.h.s. of (2.152), we switch to position space. We find

$$\begin{aligned}\Big|\frac{\kappa}{2N}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\eta_0\langle\xi, b_p b_{-p}\xi\rangle\Big| &\leq C\kappa\int dxdy\, N^2V(N(x-y))\|\check{a}_x\check{a}_y\xi\|\|\xi\| \\ &\leq CN^{-1/2}\kappa^{3/2}\|\mathcal{V}_N^{1/2}\xi\|\|\xi\|\end{aligned}$$

We conclude that

$$\mathcal{G}_N = 4\pi\mathfrak{a}_0 N + \mathcal{K} + \mathcal{V}_N + \widetilde{\mathcal{E}}_N$$

where the error term $\widetilde{\mathcal{E}}_N$ is such that, for all $\delta > 0$ there exists a constant $C > 0$ such that

$$\pm\,\widetilde{\mathcal{E}}_N \leq \delta(\mathcal{K}+\mathcal{V}_N) + C\kappa(\mathcal{N}_++1) \tag{2.153}$$

The statement of Prop. 2.3.2 now follows by the remark that, on $\mathcal{F}_+^{\leq N}$, $\mathcal{N}_+ \leq (2\pi)^{-2}\mathcal{K}$ (i.e. the kinetic energy operator on $\mathcal{F}_+^{\leq N}$ is gapped). Taking for example $\delta = 1$ in (2.153), we find

$$\mathcal{G}_N \leq 4\pi\mathfrak{a}_0 N + 2(\mathcal{K} + \mathcal{V}_N) + C(\mathcal{N}_+ + 1) \leq 4\pi\mathfrak{a}_0 N + C(\mathcal{K} + \mathcal{V}_N + 1)$$

Taking instead $\delta = 1/3$, we find the lower bound

$$\mathcal{G}_N \geq 4\pi\mathfrak{a}_0 N + \frac{2}{3}(\mathcal{K} + \mathcal{V}_N) - C\kappa(\mathcal{N}_+ + 1) \geq 4\pi\mathfrak{a}_0 N + \left[\frac{2}{3} - \frac{C\kappa}{(2\pi)^2}\right](\mathcal{K} + \mathcal{V}_N) - C$$

Now, if $\kappa \geq 0$ is small enough, we obtain that

$$\mathcal{G}_N \geq 4\pi\mathfrak{a}_0 N + \frac{1}{2}(\mathcal{K} + \mathcal{V}_N) - C \geq 4\pi\mathfrak{a}_0 N + 2\pi^2\mathcal{N}_+ - C$$

which completes the proof of Prop. 2.3.2.

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Chapter 3

Gross-Pitaevskii Dynamics for Bose-Einstein Condensates

In this chapter, we give the details for the proofs of Theorems 1.5.1 and 1.5.2. As discussed in Section 1.5, these results show that, in the Gross-Pitaevskii regime, Bose-Einstein condensation is dynamically stable and that the evolved condensate wavefunction is described by the solution of the time-dependent Gross-Pitaevskii equation. Our main results are proved in [20].

The following manuscript is a slightly modified version of the paper [20]. Section 3.1 is a partly rephrased and shortened verion of the introduction [20, Section 1]. Second, Section 3.2 is a shortened version of [20, Sections 2 and 3], since we already introduced the Fock space setting in which we work and related standard results in Section 1.2. Apart from these changes and up to the notational modifications already mentioned in Section 1.A, the following sections appear as in the paper [20].

3.1 Main Results

Let us recall from Section 1.5 that, in this chapter, we consider trapped gases of N bosons in $\Lambda = \mathbb{R}^3$ in the Gross-Pitaevskii regime, described by the Hamilton operator

$$H_N^{\text{trap}} = \sum_{j=1}^{N} \left[-\Delta_{x_j} + V_{\text{ext}}(x_j) \right] + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) \tag{3.1}$$

V_{ext} is a confining external potential and we assume the interaction potential V to be non-negative, spherically symmetric and compactly supported (but our results could be easily extended to potentials decaying sufficiently fast at infinity).

Characteristically for the Gross-Pitaevskii regime, the interaction $N^2V(N.)$ appearing in (3.9) scales with N so that its scattering length is of the order N^{-1}. The scattering length $\mathfrak{a}_0$ of the unscaled potential V is defined by the condition that the solution of the zero-energy scattering equation

$$\left[-\Delta + \frac{1}{2}V(x)\right] f(x) = 0, \tag{3.2}$$

with the boundary condition $f(x) \to 1$ for $|x| \to \infty$, has the form

$$f(x) = 1 - \frac{\mathfrak{a}_0}{|x|} \tag{3.3}$$

outside the support of V. Equivalently, $\mathfrak{a}_0$ is determined by

$$8\pi\mathfrak{a}_0 = \int V(x)f(x)dx \tag{3.4}$$

By scaling, the scattering equation (3.2) also implies that the rescaled potential $N^2V(N.)$ in (3.9) has scattering length $\mathfrak{a}_0/N$.

It has been shown in [71] (and more recently in [79]) that the ground state energy E_N of the Hamilton operator (3.1) is such that

$$\lim_{N\to\infty} \frac{E_N}{N} = \min_{\substack{\varphi\in L^2(\mathbb{R}^3):\\ \|\varphi\|_2=1}} \mathcal{E}_{\text{GP}}^{\text{trap}}(\varphi) \tag{3.5}$$

with the Gross-Pitaevskii energy functional

$$\mathcal{E}_{\text{GP}}^{\text{trap}}(\varphi) = \int \left[|\nabla\varphi(x)|^2 + V_{\text{ext}}(x)|\varphi(x)|^2 + 4\pi\mathfrak{a}_0|\varphi(x)|^4\right] dx \tag{3.6}$$

Furthermore, the results of [66, 79] imply Bose-Einstein condensation in the ground state of (3.1). More precisely, if $\gamma_N^{(1)} = \text{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$ denotes the one-particle reduced density associated with the ground state of (3.1), then

$$\gamma_N^{(1)} \to |\phi_{\text{GP}}\rangle\langle\phi_{\text{GP}}| \tag{3.7}$$

where $\phi_{\text{GP}} \in L^2(\mathbb{R}^3)$ is the unique non-negative minimizer of (3.6), among all $\varphi \in L^2(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$. The interpretation of (3.7) is straightforward: in the ground state of (3.1), all particles, up to a fraction vanishing in the limit of large N, are in the same one-particle state ϕ_{GP}.

In typical experiments, one observes the time-evolution of trapped Bose gases prepared in or close to their ground state, resulting from a change of the external fields. As an example, consider the situation in which the trapping potential is switched off at time $t = 0$. In this case, the dynamics is described, on the microscopic level, by the many-body Schrödinger equation

$$i\partial_t\psi_{N,t} = H_N\psi_{N,t} \tag{3.8}$$

with the translation invariant Hamilton operator

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) \tag{3.9}$$

and with the ground state of (3.1) as initial data. The next theorem shows how the solution of (3.8) can be described in terms of the time-dependent Gross-Pitaevskii equation.

Theorem 3.1.1. *Let $V_{ext} : \mathbb{R}^3 \to \mathbb{R}$ be locally bounded with $V_{ext}(x) \to \infty$ as $|x| \to \infty$. Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric. Let ψ_N be a sequence in $L^2_s(\mathbb{R}^{3N})$, with one-particle reduced density $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$. We assume that, as $N \to \infty$,*

$$\begin{aligned} a_N &= 1 - \langle \phi_{GP}, \gamma_N^{(1)} \phi_{GP}\rangle \to 0 \qquad \textit{and} \\ b_N &= \left| N^{-1}\langle \psi_N, H_N^{trap} \psi_N\rangle - \mathcal{E}_{GP}^{trap}(\phi_{GP})\right| \to 0 \end{aligned} \tag{3.10}$$

where $\phi_{GP} \in H^4(\mathbb{R}^3)$ is the unique non-negative minimizer of the Gross-Pitaevskii energy functional (3.6). Let $\psi_{N,t} = e^{-iH_N t}\psi_N$ be the solution of (3.8) with initial data ψ_N and let $\gamma_{N,t}^{(1)}$ be the one-particle reduced density associated with $\psi_{N,t}$. Then there are constants $C, c > 0$ such that

$$1 - \langle \varphi_t, \gamma_{N,t}^{(1)} \varphi_t\rangle \le C\left[a_N + b_N + N^{-1}\right] \exp\left(c \exp\left(c|t|\right)\right) \tag{3.11}$$

for all $t \in \mathbb{R}$. Here φ_t is the solution of the time-dependent Gross-Pitaevskii equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi\mathfrak{a}_0 |\varphi_t|^2 \varphi_t \tag{3.12}$$

with the initial data $\varphi_{t=0} = \phi_{GP}$.

Remarks:

1) The condition $a_N = 1 - \langle \phi_{\mathrm{GP}}, \gamma_N^{(1)} \phi_{\mathrm{GP}}\rangle \to 0$ is equivalent with $\gamma_N^{(1)} \to |\phi_{\mathrm{GP}}\rangle\langle\phi_{\mathrm{GP}}|$. Similarly, the bound (3.11) implies that $\gamma_{N,t}^{(1)} \to |\varphi_t\rangle\langle\varphi_t|$. More precisely, using the fact that $|\varphi_t\rangle\langle\varphi_t|$ is a rank-one projection, it follows from (3.11) that

$$\begin{aligned} \mathrm{tr}\left|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right| &\le 2\left\|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right\|_{\mathrm{HS}} \\ &\le 2^{3/2}\big[1 - \langle \varphi_t, \gamma_{N,t}^{(1)}\varphi_t\rangle\big]^{1/2} \\ &\le C\big[a_N + b_N + N^{-1}\big]^{1/2} \exp(c\exp(c|t|))\,. \end{aligned}$$

Hence, (3.11) is a statement about the stability of Bose-Einstein condensation with respect to the many-body Schrödinger equation (3.8).

2) To keep the notation as simple as possible, we consider the time evolution (3.8) generated by the translation invariant Hamiltonian (3.9). With the same techniques we use to prove Theorem 3.1.1, we could also have included in (3.9) an external potential W_{ext} (at least if the difference $W_{\text{ext}} - V_{\text{ext}}$ is bounded below). Under this assumption, the convergence (3.11) remains true, of course provided we introduce the external potential W_{ext} also in the time-dependent Gross-Pitaevskii equation (3.12). Physically, this would describe experiments where the system prepared at equilibrium (in the ground state) is perturbed by a change of the external potential, rather than by switching it off (we could also consider the situation where the external potential depends on time).

Theorem 3.1.1 is meant to describe the time-evolution of data prepared in the ground state of the trapped Hamilton operator (3.1). From the mathematical point of view, one may also ask whether it is possible to show that the evolution of an initial data exhibiting Bose-Einstein condensate in an arbitrary one-particle wave function $\varphi \in H^1(\mathbb{R}^3)$ which does not necessarily minimize $\mathcal{E}_{\text{GP}}^{\text{trap}}$. This is the content of our next theorem.

Theorem 3.1.2. *Assume that $V \in L^3(\mathbb{R}^3)$ is non-negative, compactly supported and spherically symmetric. Let ψ_N be a sequence in $L^2_s(\mathbb{R}^{3N})$, with one-particle reduced density $\gamma_N^{(1)} = \text{tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|$. Assume that, for a $\varphi \in H^4(\mathbb{R}^3)$,*

$$\begin{aligned} \widetilde{a}_N &= \text{tr}\left|\gamma_N^{(1)} - |\varphi\rangle\langle\varphi|\right| \to 0 \qquad \textit{and} \\ \widetilde{b}_N &= \left|N^{-1}\langle\psi_N, H_N\psi_N\rangle - \mathcal{E}_{GP}(\varphi)\right| \to 0 \end{aligned} \tag{3.13}$$

as $N \to \infty$. Here $\mathcal{E}_{GP}$ is the translation invariant Gross-Pitaevskii functional

$$\mathcal{E}_{GP}(\varphi) = \int \left[|\nabla\varphi|^2 + 4\pi\mathfrak{a}_0|\varphi|^4\right] dx \tag{3.14}$$

Let $\psi_{N,t} = e^{-iH_N t}\psi_N$ be the solution of the Schrödinger equation (3.8) with initial data ψ_N and let $\gamma_{N,t}^{(1)}$ denote the one-particle reduced density associated with $\psi_{N,t}$. Then

$$1 - \langle\varphi_t, \gamma_{N,t}^{(1)}\varphi_t\rangle \leq C\left[\widetilde{a}_N + \widetilde{b}_N + N^{-1}\right] \exp\left(c\exp\left(c|t|\right)\right) \tag{3.15}$$

where φ_t denotes the solution of the time-dependent Gross-Pitaevskii equation (3.12).

As mentioned in Section 1.5, the first proof of the convergence of the reduced density associated with the solution of the Schrödinger equation (3.8) towards the orthogonal projection onto the solution of the time-dependent Gross-Pitaevskii equation (3.12) was obtained in [36, 37, 40, 39] (partly simplified in [25], using also ideas from [56]). A different proof was later given in [85] whose methods were also used in the related results in [74, 83, 53, 54]. More recently, convergence with a rate similar to (3.11), (3.15) has been proven to hold in [11], for a class of Fock space initial data. The novelty of (3.11), (3.15) is the fact that convergence is shown with an explicit and (at least in (3.11)) optimal rate determined by the properties of the N-particle initial data.

More results are available about quantum dynamics in the mean-field regime. In this case, the evolution of the Bose gas is generated by an Hamilton operator of the form

$$H_N^{\mathrm{mf}} = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^{N} V(x_i - x_j) \tag{3.16}$$

In the limit $N \to \infty$, the solution of the Schrödinger equation $\psi_{N,t} = e^{-iH_N^{\mathrm{mf}}t}\psi_N$, for initial data ψ_N exhibiting Bose-Einstein condensation in a one-particle wave function $\varphi \in L^2(\mathbb{R}^3)$, can be approximated by products of the solution of the nonlinear Hartree equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + (V * |\varphi_t|^2)\varphi_t \tag{3.17}$$

Convergence towards Hartree dynamics has been established in different settings and using different methods in several works, including [2, 3, 4, 6, 5, 9, 26, 35, 41, 43, 42, 52, 57, 95, 98]. In the mean-field regime, it is also possible to find a norm approximation of the many-body evolution by taking into account fluctuations around the Hartree dynamics (3.17); see, for example, [10, 24, 47, 48, 55, 63, 75].

It is also interesting to consider the many-body evolution in scaling limits interpolating between the mean-field regime described by the Hamilton operator (3.16) and the Gross-Pitaevskii regime described by (3.9). In such regimes the interaction is of the form $N^{3\beta-1}V(N^\beta .)$ for $\beta \in (0;1)$. A norm-approximation of the time-evolution in these intermediate regimes was obtained for classes of Fock space initial data in [49, 58, 50], for $\beta \in \left(0; \frac{2}{3}\right)$, and in [16], valid for all interpolating regimes where $\beta \in (0;1)$. Analogous results for classes of N-particle initial data were provided in [76, 77] for scaling regimes where $\beta \in \left(0; \frac{1}{2}\right)$. In Chapter 5, we will come back to this question and present our main result in this direction which combines the methods of [16, 76, 77] to derive a norm-approximation of the many-body evolution for N-particle initial data, valid for all interpolating regimes where $\beta \in (0;1)$.

To prove Theorem 3.1.1 and Theorem 3.1.2. we will combine the strategies used in [11] and [63]. Let us briefly recall the main ideas of these works. In [11], the Bose gas was described on the Fock space $\mathcal{F} = \bigoplus_{n\geq 0} L^2_s(\mathbb{R}^{3n})$ by the Hamilton operator

$$\mathcal{H}_N = \int \nabla_x a_x^* \nabla_x a_x dx + \frac{1}{2}\int N^2 V(N(x-y)) a_x^* a_y^* a_y a_x \, dxdy$$

where a_x^*, a_x are the usual operator valued distributions, creating and, respectively, annihilating a particle at the point $x \in \mathbb{R}^3$. Notice that $\mathcal{H}_N$ commutes with the number of particle operator $\mathcal{N} = \int a_x^* a_x \, dx$, and that its restriction to the sector of $\mathcal{F}$ with exactly N particles coincides with (3.9).

On the Fock space $\mathcal{F}$, a Bose-Einstein condensate can be described by a coherent state of the form $W(\sqrt{N}\varphi)\Omega$, where $\Omega = \{1, 0, 0, \dots\}$ is the vacuum vector, $\varphi \in L^2(\mathbb{R}^3)$ is a normalized one-particle orbital, and where, for every $f \in L^2(\mathbb{R}^3)$,

$$W(f) = \exp(a^*(f) - a(f))$$

is a Weyl operator with wave function f. Here, we denoted by

$$a^*(f) = \int f(x) a_x^* \, dx \qquad \text{and } a(f) = \int \bar{f}(x) a_x$$

the usual creation and annihilation operators on $\mathcal{F}$, creating and annihilating a particle with wave function f. A simple computation shows that

$$W(\sqrt{N}\varphi)\Omega = e^{-N/2} \left\{ 1, N^{1/2}\varphi, \dots, \frac{N^{n/2}\varphi^{\otimes n}}{\sqrt{n!}}, \dots \right\}$$

In the coherent state $W(\sqrt{N}\varphi)\Omega$, the number of particles is Poisson distributed, with mean and variance equal to N.

On the Fock space $\mathcal{F}$, it is interesting to study the dynamics of approximately coherent initial states. In the Gross-Pitaevskii regime, however (in contrast with the mean field limit), we cannot expect the evolution of approximately coherent initial data to remain approximately coherent. On every sector of $\mathcal{F}$ with a fixed number of particles, the coherent states $W(\sqrt{N}\varphi)\Omega$ is factorized; it describes therefore uncorrelated particles. On the other hand, already from [37, 34] and more recently also from [27], we know that, in the Gross-Pitaevskii regime, particles develop important correlations. To provide a better approximation of the many-body dynamics, Weyl operators were combined in [11] with appropriate Bogoliubov transformations, leading to so-called squeezed coherent states. To be more precise, let f denote the solution of the zero-energy scattering equation (3.2) and $w = 1 - f$ (keep in mind that, for $|x| \gg 1$, $w(x) = \mathfrak{a}_0/|x|$). Using w, we define

$$k_{N,t}(x;y) = -Nw(N(x-y))\varphi_t(x)\varphi_t(y) \tag{3.18}$$

where φ_t is the solution of the time-dependent Gross-Pitaevskii equation (3.12). In fact, in [11] and also later in the present paper, it is more convenient to replace φ_t with the solution of the slightly modified Gross-Pitaevskii equation (3.68); to simplify the presentation, we neglect these technical details in this introduction. With (3.18), it is easy to check that $k_{N,t} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, with $\|k_{N,t}\|_2$ bounded, uniformly in N and in t. This implies that (3.18) is the integral kernel of a Hilbert-Schmidt operator; this remark allows us to define, on $\mathcal{F}$, the unitary Bogoliubov transformations

$$T_t = \exp\left[\frac{1}{2}\int dxdy \left(k_{N,t}(x;y)a_x^* a_y^* - \text{h.c.}\right)\right] \tag{3.19}$$

Notice that the action of T_t on creation and annihilation operators is explicitly given by

$$T_t^* a^*(g) T_t = a^*(\cosh_{k_{N,t}}(g)) + a(\sinh_{k_{N,t}}(\bar{g})) \tag{3.20}$$

where $g \in L^2(\mathbb{R}^3)$ and $\cosh_{k_{N,t}}$ and $\sinh_{k_{N,t}}$ are the bounded operators ($\sinh_{k_{N,t}}$ is even Hilbert-Schmidt) defined by the convergent series

$$\cosh_{k_{N,t}} = \sum_{n=0}^{\infty} \frac{(k_{N,t}\bar{k}_{N,t})^n}{(2n)!}, \qquad \text{and} \qquad \sinh_{k_{N,t}} = \sum_{n=0}^{\infty} \frac{(k_{N,t}\bar{k}_{N,t})^n k_{N,t}}{(2n+1)!}$$

Using the Bogoliubov transformation T_t to generate correlations at time t, it makes sense to study the time evolution of initial data close to the squeezed coherent state $W(\sqrt{N}\varphi)T_0\Omega$, and to approximate it with a Fock space vector of the same form. More precisely, for $\xi_N \in \mathcal{F}$ close to the vacuum (in a sense to be made precise later), we may consider the time evolution

$$e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T_0 \xi_N = W(\sqrt{N}\varphi_t) T_t \xi_{N,t} \tag{3.21}$$

where we defined $\xi_{N,t} = \mathcal{U}_N(t)\xi_N$ and the fluctuation dynamics

$$\mathcal{U}_N(t) = T_t^* W^*(\sqrt{N}\varphi_t) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi_0) T_0 \tag{3.22}$$

In order to show that the one-particle reduced density $\gamma_{N,t}^{(1)}$ associated with the l.h.s. of (3.21) is close to the orthogonal projection onto the solution of the Gross-Pitaevskii equation (3.68), it is enough to prove that $\xi_{N,t}$ is close to the vacuum, in an appropriate sense. In fact, it is enough to show that the expectation of the number of particles in $\xi_{N,t}$ is small, compared with the total number of particles N, assuming the same is true for the initial $\xi_N \in \mathcal{F}$. In other words, the problem of proving convergence towards the Gross-Pitaevskii dynamics reduces to the problem of showing that the expectation of the number of particles remains approximately preserved by the fluctuation dynamics (3.22). In [11], this strategy was used to show that the one-particle reduced density $\gamma_{N,t}^{(1)}$ associated with $\Psi_{N,t} = e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T_0 \xi_N$ is such that

$$\|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\|_{\mathrm{HS}} \leq C N^{-1/2} \exp(c \exp(c|t|))$$

for any $\xi_N \in \mathcal{F}$ with $\|\xi_N\| = 1$ and such that

$$\left\langle \xi_N, \left[\mathcal{N} + \mathcal{N}^2/N + \mathcal{H}_N\right] \xi_N \right\rangle \leq C$$

uniformly in N.

While the method of [11] works well to show convergence towards the Gross-Pitaevskii dynamics for the evolution of Fock space initial data of the form $W(\sqrt{N}\varphi)T_0\xi_N$, it is difficult to apply them to N-particle initial data in $L_s^2(\mathbb{R}^{3N})$ (a special class of N-particle states for which this is indeed possible is discussed in Appendix C of [11]). An alternative approach, tailored on N-particle initial data, was proposed in [63] for bosons in the mean field limit. From Section 1.2, we recall the important observation of [64], used also in [63], that, for a fixed normalized $\varphi \in L^2(\mathbb{R}^3)$, every $\psi_N \in L_s^2(\mathbb{R}^{3N})$ can be uniquely represented as

$$\psi_N = \sum_{n=0}^{N} \psi_N^{(n)} \otimes_s \varphi^{\otimes(N-n)} \tag{3.23}$$

for a sequence $\{\psi_N^{(n)}\}_{n=0}^N$ with $\psi_N^{(n)} \in L^2_{\perp\varphi}(\mathbb{R}^3)^{\otimes_s n}$, the symmetric tensor product of n copies of the orthogonal complement of φ in $L^2(\mathbb{R}^3)$.

This remark allows us to define the unitary map $U_N(\varphi) = U(\varphi)$, introduced already in (1.15), by

$$U(\varphi) : L^2_s(\mathbb{R}^{3N}) \to \mathcal{F}^{\leq N}_{\perp\varphi} \quad \text{through} \quad U(\varphi)\psi_N = \{\psi_N^{(0)}, \psi_N^{(1)}, \ldots, \psi_N^{(N)}\}. \tag{3.24}$$

Notice that a similar decomposition of states $\psi_N \in L^2_s(\mathbb{R}^{3N})$ (but with no second quantization) is also used in the approach of [85, 75] to identify excitations of the condensate. To prove convergence towards the Hartree evolution for N-particle systems in the mean field regime described by (3.16), we define a fluctuation dynamics

$$\mathcal{W}^{\text{mf}}_{N,t} = U(\varphi_t)e^{-iH_N^{\text{mf}}t}U^*(\varphi) : \mathcal{F}^{\leq N}_{\perp\varphi} \to \mathcal{F}^{\leq N}_{\perp\varphi_t} \tag{3.25}$$

where φ_t denotes the solution of the time-dependent Hartree equation (3.17). Similarly as above, the problem of proving convergence of the reduced density to the Hartree dynamics reduces to the problem of showing that the expectation of the number of particles operator is approximately preserved by the fluctuation dynamics $\mathcal{W}^{\text{mf}}_{N,t}$. In the mean field regime, one can even go one step further, constructing a quadratic evolution on $\mathcal{F}$ (i.e. a unitary group with a generator quadratic in creation and annihilation operators) that approximates $\mathcal{W}^{\text{mf}}_{N,t}$; in [63], this procedure was shown to provide a norm approximation for the many-body evolution.

In contrast to that, in the Gross-Pitaevskii regime, we need to modify our ansatz in order to take into account correlations developed by the many-body evolution. Similarly as above, where we argued that coherent states are not a good ansatz to describe the evolution of Fock space initial data, the reason is that we cannot expect here that factorized N-particles states of the form $U^*_{\varphi_t}\Omega = \varphi_t^{\otimes N}$ provide a good approximation for the solution of the Schrödinger equation (3.8) in the Gross-Pitaevskii regime. The tool we use for this incorporating correlations are, in analogy to the methods of [11], *generalized Bogoliubov transformations* of the form

$$S_t = \exp\left[\frac{1}{2}\int dxdy\,(\eta_t(x;y)b_x^*b_y^* - \text{h.c.})\right] \tag{3.26}$$

for a kernel $\eta_t \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, orthogonal to φ_t in both its variables. Compared with the standard Bogoliubov transformations in (3.19), (3.26) has an important advantage: it maps $\mathcal{F}^{\leq N}_{\perp\varphi_t}$ back into itself.

For this reason, with (3.26) we can define the modified fluctuation dynamics $\mathcal{W}_{N,t} = S_t^*U(\varphi_t)e^{-iH_Nt}U^*(\varphi_0)S_0 : \mathcal{F}^{\leq N}_{+} \to \mathcal{F}^{\leq N}_{+}$, which will play in our analysis a similar role as (3.22) played in [11]. To prove Theorem 3.1.1 and Theorem 3.1.2 it will then be enough to show a bound for the growth of the expectation of the number of particles with respect to $\mathcal{W}_{N,t}$. To achieve this goal, we will establish several properties of its generator. Technically, the main challenge we will have to face is the fact that, in contrast with (3.20), there is no explicit formula for the action of the generalized Bogoliubov transformation (3.26) on creation and annihilation operators. For this reason, we will have to expand expressions like $S_t^*b(g)S_t$ in absolutely convergent infinite series, and we will have to control the contribution of several different terms. The main tool to control these expansions is Lemma 3.2.3 below.

3.2 Generalized Bogoliubov Transformations for Inhomogeneous Systems

As explained in the previous section, we will have to analyse the action of generalized Bogoliubov transformations of the form (3.26) on the creation and annihilation operators. Similar to Section 2.2, we carefully introduce here the operators (3.26) and study their properties. In particular, we need to introduce two types of monomials in creation and annihilation operators that will play an important role in our analysis. We define

$$\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n) = \int b^{\flat_0}_{x_1} a^{\sharp_1}_{y_1} a^{\flat_1}_{x_2} a^{\sharp_2}_{y_2} a^{\flat_2}_{x_3} \dots a^{\sharp_{n-1}}_{y_{n-1}} a^{\flat_{n-1}}_{x_n} b^{\sharp_n}_{y_n} \prod_{\ell=1}^{n} j_\ell(x_\ell;y_\ell)\, dx_\ell dy_\ell \tag{3.27}$$

where $j_k \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ for $k=1,\dots,n$ and where $\sharp = (\sharp_1,\dots,\sharp_n), \flat = (\flat_0,\dots,\flat_{n-1}) \in \{\cdot,*\}^n$. In other words, for every index $j\in\{1,\dots,n\}$, we have either $\sharp_j = \cdot$ (meaning that $a^{\sharp_j} = a$ or $b^{\sharp_j} = b$) or $\sharp_j = *$ (meaning that $a^{\sharp_j} = a^*$ or $b^{\sharp_j} = b^*$) and analogously for $\flat_j$, if $j\in\{0,\dots,n-1\}$. Furthermore, for $j=1,\dots,n-1$, we impose the condition that either $\sharp_\ell = \cdot$ and $\flat_\ell = *$ or $\sharp_\ell = *$ and $\flat_\ell = \cdot$ (so that the product $a^{\sharp_\ell}_{y_\ell} a^{\flat_\ell}_{x_{\ell+1}}$ always preserves the number of particles, for all $\ell = 1,\dots,n-1$). If $\flat_{i-1} = \cdot$ and $\sharp_i = *$ (i.e. if the product $a^{\flat_{i-1}}_{x_i} a^{\sharp_i}_{y_i}$ for $i=2,\dots,n$, or the product $b^{\flat_0}_{x_1} a^{\sharp_1}_{y_1}$ for $i=1$, is not normally ordered) we require additionally $x \to j_i(x;x)$ to be integrable. An operator of the form (3.27), with all the properties listed above, will be called a $\Pi^{(2)}$-operator of order n.

Next, we define

$$\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f) = \int b^{\flat_0}_{x_1} a^{\sharp_1}_{y_1} a^{\flat_1}_{x_2} a^{\sharp_2}_{y_2} a^{\flat_2}_{x_3} \dots a^{\sharp_{n-1}}_{y_{n-1}} a^{\flat_{n-1}}_{x_n} a^{\sharp_n}_{y_n} a^{\flat_n}(f) \prod_{\ell=1}^{n} j_\ell(x_\ell;y_\ell)\, dx_\ell dy_\ell \tag{3.28}$$

where $f\in L^2(\mathbb{R}^3)$ and , as above $j_k \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ for all $k=1,\dots,n$, $\sharp = (\sharp_1,\dots,\sharp_n)\in \{\cdot,*\}^n$, $\flat = (\flat_0,\dots,\flat_n)\in\{\cdot,*\}^{n+1}$ with the condition that, for all $\ell=1,\dots,n$, we either have $\sharp_\ell = \cdot$ and $\flat_\ell = *$ or $\sharp_\ell = *$ and $\flat_\ell = \cdot$. Additionally, we assume that $x\to j_i(x;x)$ is integrable, if $\flat_{i-1} = \cdot$ and $\sharp_i = *$ for an $i=1,\dots,n$. An operator of the form (3.28) will be called a $\Pi^{(1)}$-operator of order n. Operators of the form $b(f)$, $b^*(f)$, for a $f\in L^2(\mathbb{R}^3)$, will be called $\Pi^{(1)}$-operators of order zero. It will also be useful to consider

$$\widetilde{\Pi}^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f) = \int a^{\flat_0}(f) a^{\sharp_0}_{x_1} a^{\flat_1}_{y_1} a^{\sharp_1}_{x_2} a^{\flat_2}_{y_2} a^{\sharp_2}_{x_3} \dots a^{\flat_{n-1}}_{y_{n-1}} a^{\sharp_{n-1}}_{x_n} b^{\flat_n}_{y_n} \prod_{\ell=1}^{n} j_\ell(x_\ell;y_\ell) dx_\ell dy_\ell \tag{3.29}$$

where $f\in L^2(\mathbb{R}^3)$, $j_k\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ for all $k=1,\dots,n$, $\sharp = (\sharp_0,\dots,\sharp_{n-1})\in\{\cdot,*\}^n$, $\flat = (\flat_0,\dots,\flat_n)\in\{\cdot,*\}^{n+1}$ with the condition that, for every $\ell\in\{0,\dots,n-1\}$, either $\sharp_\ell = \cdot$ and $\flat_\ell = *$ or $\sharp_\ell = *$ and $\flat_\ell = \cdot$. As above, we also assume that that $x\to j_i(x;x)$ is integrable, if $\flat_{i-1} = \cdot$ and $\sharp_i = *$ for $i=1,\dots,n$. Observe that

$$\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)^* = \widetilde{\Pi}^{(1)}_{\sharp',\flat'}(j_n,\dots,j_1;f)$$

with $\flat' = (\bar{\flat}_n,\dots,\bar{\flat}_0)$, $\sharp' = (\bar{\sharp}_n,\dots,\bar{\sharp}_1)$, where $\bar{\flat} = \cdot$ if $\flat = *$ and $\bar{\flat} = *$ if $\flat = \cdot$ (and similarly for $\bar{\sharp}$).

Notice that $\Pi^{(2)}$-operators involve two b operators and therefore may create or annihilate up to two excitations of the condensate (depending on the choice of $\flat_0$ and $\sharp_n$, it may excite two particles, it may annihilate two excitations or it may create one and annihilate another excitation, leaving the total number of excitations invariant). $\Pi^{(1)}$- and $\widetilde{\Pi}^{(1)}$-operators, on the other hand, create or annihilate exactly one excitation. The conditions on the number of creation and annihilation operators guarantee that $\Pi^{(2)}$-, $\Pi^{(1)}$- and $\widetilde{\Pi}^{(1)}$-operators always map $\mathcal{F}^{\leq N}$ back into itself. In the next lemma we collect bounds that we are going to use to control these operators.

Lemma 3.2.1. *Let $n \in \mathbb{N}$, $f \in L^2(\mathbb{R}^3)$, $j_1, \dots, j_n \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. We assume the operators $\Pi^{(2)}_{\sharp,\flat}(j_1, \dots, j_n)$ and $\Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_n; f)$ are defined as in (3.27), (3.28). Then we have the bounds*

$$\begin{aligned} \left\| \Pi^{(2)}_{\sharp,\flat}(j_1, \dots, j_n)\xi \right\| &\leq 6^n \prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell} \left\| (\mathcal{N}+1)^n \left(1 - \frac{\mathcal{N}-2}{N}\right) \xi \right\| \\ \left\| \Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_n; f)\xi \right\| &\leq 6^n \|f\| \prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell} \left\| (\mathcal{N}+1)^{n+1/2} \left(1 - \frac{\mathcal{N}-2}{N}\right)^{1/2} \xi \right\| \end{aligned} \tag{3.30}$$

where

$$K_\ell^{\flat_{\ell-1},\sharp_\ell} = \begin{cases} \|j_\ell\|_2 + \int |j_\ell(x;x)|\,dx & \text{if } \flat_{\ell-1} = \cdot \text{ and } \sharp_\ell = * \\ \|j_\ell\|_2 & \text{otherwise} \end{cases}$$

Since $\mathcal{N} \leq N$ on $\mathcal{F}^{\leq N}$, it follows that $\Pi^{(2)}_{\sharp,\flat}(j_1, \dots, j_n), \Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_n; f)$ are bounded operators on $\mathcal{F}^{\leq N}$, with

$$\begin{aligned} \left\| \Pi^{(2)}_{\sharp,\flat}(j_1, \dots, j_n) \right\| &\leq (12N)^n \prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell} \\ \left\| \Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_n; f) \right\| &\leq (12N)^n \sqrt{N} \|f\|_2 \prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell} \end{aligned}$$

Remark: if $j_i \in (q_{\varphi^{\bar{\flat}_{i-1}}} \otimes q_{\varphi^{\bar{\sharp}_i}}) L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ for all $i = 1, \dots, n$ and if $f \in L^2_\perp(\mathbb{R}^3)$, then $\Pi^{(2)}_{\sharp,\flat}(j_1, \dots, j_n)$ and $\Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_n; f)$ map $\mathcal{F}^{\leq N}_{\perp\varphi}$ into itself.

Proof. We consider operators of the form (3.27). Let us assume, for example, that $\flat_0 = \cdot$ and $\sharp_n = \cdot$. Then we have, writing $b_{x_1} = a_{x_1}(1 - \mathcal{N}/N)^{1/2}$ and $b_{y_n} = a_{y_n}(1 - \mathcal{N}/N)^{1/2}$

and using the pull-through formula $g(\mathcal{N})a_x = a_x g(\mathcal{N}-1)$,

$$\begin{aligned}
&\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)\\
&\quad = \int a_{x_1}\left(\frac{N-\mathcal{N}}{N}\right)^{1/2} a^{\sharp_1}_{y_1}\dots a^{\sharp_{n-1}}_{y_{n-1}} a^{\flat_{n-1}}_{x_n} a_{y_n}\left(\frac{N-\mathcal{N}}{N}\right)^{1/2}\prod_{\ell=1}^n j_\ell(x_\ell;y_\ell)dx_\ell dy_\ell\\
&\quad = \int a_{x_1} a^{\sharp_1}_{y_1}\dots a^{\sharp_{n-1}}_{y_{n-1}} a^{\flat_{n-1}}_{x_n} a_{y_n}\left(\frac{N-\mathcal{N}+1}{N}\right)^{1/2}\left(\frac{N-\mathcal{N}}{N}\right)^{1/2}\prod_{\ell=1}^n j_\ell(x_\ell;y_\ell)dx_\ell dy_\ell\\
&\quad = \prod_{\ell=1}^n A^{\flat_{\ell-1},\sharp_\ell}(j_\ell)\left(\frac{N-\mathcal{N}+1}{N}\right)^{1/2}\left(\frac{N-\mathcal{N}}{N}\right)^{1/2}
\end{aligned}$$

where we used the definition (1.13). The first bound in (3.30) follows therefore from Lemma 1.2.1. The other estimates can be shown similarly. □

For a kernel $\eta \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ with $\eta(x;y)=\eta(y;x)$, we define

$$B(\eta) = \frac{1}{2}\int \left[\eta(x;y)b^*_x b^*_y - \bar{\eta}(x;y)b_x b_y\right] dxdy \tag{3.31}$$

Observe that, with the notation introduced in (1.24),

$$B(\eta) = \frac{1}{2}\left[B_{*,*}(\eta) - B^*_{*,*}(\eta)\right] = -\frac{1}{2}\left[B_{\cdot,\cdot}(\eta) - B^*_{\cdot,\cdot}(\eta)\right] .$$

We will consider unitary operators of the form

$$e^{B(\eta)} = \exp\left[\frac{1}{2}\int (\eta(x;y)b^*_x b^*_y - \bar{\eta}(x;y)b_x b_y)\right]$$

which we are going to call generalized Bogoliubov transformations. It is clear that $B(\eta), e^{B(\eta)} : \mathcal{F}^{\leq N} \to \mathcal{F}^{\leq N}$. Furthermore, if $\eta \in (q_\varphi\otimes q_\varphi)L^2(\mathbb{R}^3\times\mathbb{R}^3)$ then we have $B(\eta), e^{B(\eta)} : \mathcal{F}^{\leq N}_{\perp\varphi} \to \mathcal{F}^{\leq N}_{\perp\varphi}$ for any normalized $\varphi \in L^2(\mathbb{R}^3)$ (as above, $q_\varphi = 1-|\varphi\rangle\langle\varphi|$ is the projection into the orthogonal complement of φ). It may be helpful to observe that, with the unitary operator $U(\varphi)$ defined in (3.24), we can write, according to (1.18),

$$B(\eta) = \frac{1}{2}U(\varphi)\int dxdy\left[\eta(x;y)a^*_x a^*_y \frac{a(\varphi)a(\varphi)}{N} - \bar{\eta}(x;y)\frac{a^*(\varphi)a^*(\varphi)}{N}a_x a_y\right]U^*(\varphi) \tag{3.32}$$

In other words, after transforming with $U(\varphi)$, the operator $B(\eta)$ creates and annihilates pair excitations of the condensate (moving two particles from the condensate to its orthogonal complement or, viceversa, moving two excitations orthogonal to the condensate back into the condensate). Notice that, while $U^*(\varphi)B(\eta)U(\varphi)$ preserves the number of particles, $B(\eta)$ does not (because it does not preserve the number of excitations, obviously). When acting on many-body states exhibiting complete condensation, the operators $a(\varphi)^2/N$ and $a^*(\varphi)^2/N$ appearing in (3.32) can be approximated, in leading order, by 1. With this replacement, the operator $B(\eta)$ is approximated by

$$\widetilde{B}(\eta) = \frac{1}{2}\int dxdy \left(\eta(x;y)a_x^* a_y^* - \bar{\eta}(x;y)a_x a_y\right)$$

which is quadratic in the fields a, a^*. Exponentiating $\widetilde{B}(\eta)$ we find a Bogoliubov transformation, whose action on creation and annihilation operators is explicitly given by

$$e^{-\widetilde{B}(\eta)}a(f)e^{\widetilde{B}(\eta)} = a(\cosh_\eta(f)) + a^*(\sinh_\eta(\bar{f})) \tag{3.33}$$

where $\cosh_\eta$ and $\sinh_\eta$ denote the operators defined by the absolutely convergent series

$$\cosh_\eta = \sum_{j=0}^{\infty} \frac{(\eta\bar{\eta})^n}{(2n)!}, \qquad \text{and} \qquad \sinh_\eta = \sum_{j=0}^{\infty} \frac{(\eta\bar{\eta})^n \eta}{(2n+1)!} \tag{3.34}$$

Here we think of η as the Hilbert-Schmidt operator with integral kernel given by $\eta(x;y)$ (and $\bar{\eta}$ is the operator with integral kernel given by $\bar{\eta}(x;y)$). Bogoliubov transformations of the form $\exp(\widetilde{B}(\eta))$ have been used in [11] to model correlations among particles in the Gross-Pitaevskii regime, for approximately coherent initial data on the Fock space. In the setting of the present paper, however, the Bogoliubov transformation $\exp(\widetilde{B}(\eta))$ has the annoying problem that it does not map $\mathcal{F}_{\perp\varphi}^{\leq N}$ into itself (it maps $\mathcal{F}_{\perp\varphi}$ into itself, if $\eta \in (q_\varphi \otimes q_\varphi)L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, but since it does not preserve the number of particles, it will not preserve the condition that $\mathcal{N} \leq N$). For this reason, in this paper, we are going to use generalized Bogoliubov transformations of the form $\exp(B(\eta))$. The price that we have to pay is the fact that, in contrast to (3.33), the action of $\exp(B(\eta))$ on creation and annihilation operators is not explicit. Let us remark here that generalized Bogoliubov transformations of the form $\exp(B(\eta))$ have already been used in [96, 46] to study the excitation spectrum in the mean field regime. Here we will need more detailed information on the action of these operators; the rest of this section is therefore devoted to the study of the properties of generalized Bogoliubov transformations.

First of all, we need the following generalization of Lemma 4.3 of [11] (a similar result has also been proven in [96]).

Lemma 3.2.2. *Let $\eta \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Let $B(\eta)$ be the antisymmetric operator defined in (3.31). For every $n_1, n_2 \in \mathbb{Z}$ there exists a constant $C = C(n_1, n_2)$ such that*

$$e^{-B(\eta)}(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2}\, e^{B(\varphi)} \leq C(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2}$$

as operator inequality on $\mathcal{F}^{\leq N}$.

Proof. We use Gronwall's inequality. For a fixed $\xi \in \mathcal{F}^{\leq N}$ and $s \in [0;1]$, let

$$f(s) = \left\langle \xi, e^{-sB(\eta)}(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2} e^{sB(\eta)}\xi \right\rangle$$

We compute

$$
\begin{aligned}
f'(s) &= \langle \xi, e^{-sB(\eta)}\,[(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2}, B(\eta)]\,e^{sB(\eta)}\xi\rangle \\
&= \Big\langle e^{sB(\eta)}\xi, \{(\mathcal{N}+1)^{n_1}[(N+1-\mathcal{N})^{n_2}, B(\eta)] \\
&\qquad\qquad +[(\mathcal{N}+1)^{n_1}, B(\eta)](N+1-\mathcal{N})^{n_2}\}\,e^{sB(\eta)}\xi\Big\rangle
\end{aligned}
\tag{3.35}
$$

From the pull-through formula $\mathcal{N}b^* = b^*(\mathcal{N}+1)$, we conclude that

$$
\begin{aligned}
[(N+1-\mathcal{N})^{n_2}, B(\eta)] &= \frac{1}{2}B_{*,*}(\eta)\ [(N-1-\mathcal{N})^{n_2} - (N+1-\mathcal{N})^{n_2}] + \text{h.c.} \\
[(\mathcal{N}+1)^{n_1}, B(\eta)] &= \frac{1}{2}B_{*,*}(\eta)\ [(\mathcal{N}+3)^{n_1} - (\mathcal{N}+1)^{n_1}] + \text{h.c.}
\end{aligned}
$$

By the mean value theorem, we can find functions $\theta_1, \theta_2 : \mathbb{N} \to (0;2)$ (depending also on N, n_1, n_2) such that

$$
\begin{aligned}
(N-j+1)^{n_2} - (N-j-1)^{n_2} &= 2n_2(N+1-j-\theta_1(j))^{n_2-1} \\
(j+3)^{n_1} - (j+1)^{n_1} &= 2n_1(j+1+\theta_2(j))
\end{aligned}
$$

Hence, the first term on the r.h.s. of (3.35) can be written as

$$
\begin{aligned}
&\langle e^{sB(\eta)}\xi, (\mathcal{N}+1)^{n_1}[(N+1-\mathcal{N})^{n_2}, B(\eta)]e^{sB(\eta)}\xi\rangle \\
&\quad= \frac{1}{2}\langle (\mathcal{N}+1)^{n_1}e^{sB(\eta)}\xi, \big(B_{*,*}(\eta)(N+1-\mathcal{N}-\theta_1(\mathcal{N}))^{n_2-1} + h.c.\big)\,e^{sB(\eta)}\xi\rangle \\
&\quad= \frac{1}{2}\langle (\mathcal{N}+1)^{n_1/2}(N+3-\mathcal{N}-\theta_1(\mathcal{N}-2))^{n_2/2}e^{sB(\eta)}\xi, \\
&\qquad\qquad B_{*,*}(\eta)(\mathcal{N}+3)^{n_1/2}(N+1-\mathcal{N}-\theta_1(\mathcal{N}))^{n_2/2-1}e^{sB(\eta)}\xi\rangle \\
&\qquad + \frac{1}{2}\langle (\mathcal{N}+1)^{n_1/2}(N+1-\mathcal{N}-\theta_1(\mathcal{N}))^{n_2/2}e^{sB(\eta)}\xi, \\
&\qquad\qquad B_{\cdot,\cdot}(\eta)(\mathcal{N}-1)^{n_1/2}(N+3-\mathcal{N}-\theta_1(\mathcal{N}-2))^{n_2/2-1}e^{sB(\eta)}\xi\rangle
\end{aligned}
$$

The Cauchy-Schwarz inequality implies with Lemma 1.2.4

$$
\begin{aligned}
&\Big|\langle e^{sB(\eta)}\xi, (\mathcal{N}+1)^{n_1}[(N+1-\mathcal{N})^{n_2}, B(\eta)]e^{sB(\eta)}\xi\rangle\Big| \\
&\quad\le C\,\Big\|(\mathcal{N}+1)^{n_1/2}(N+3-\mathcal{N}-\theta_1(\mathcal{N}-2))^{n_2/2}e^{sB(\eta)}\xi\Big\| \\
&\qquad\qquad \times \Big\|(\mathcal{N}+3)^{n_1/2+1}(N+1-\mathcal{N}-\theta_1(\mathcal{N}))^{n_2}N^{-1}e^{sB(\eta)}\xi\Big\|
\end{aligned}
$$

Since on $\mathcal{F}^{\le N}$ we have $\mathcal{N} \le N$ and since $0 \le \theta_1(n) \le 2$ for all $n \in \mathbb{N}$, we conclude that

$$
\Big|\langle e^{sB(\eta)}\xi, (\mathcal{N}+1)^{n_1}[(N+1-\mathcal{N})^{n_2}, B(\eta)]e^{sB(\eta)}\xi\rangle\Big| \le Cf(s)
$$

The second term on the r.h.s. of (3.35) can be bounded similarly. We infer that $f'(s) \le Cf(s)$. Gronwall's inequality implies that $f(s) \le e^{Cs}f(0)$. Hence, taking $s = 1$, and renaming the constant C, we obtain

$$
\Big\langle \xi, e^{-B(\eta)}(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2}e^{B(\eta)}\xi\Big\rangle \le C\,\langle \xi, (\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2}\xi\rangle
$$

which concludes the proof of the lemma. $\square$

We will need to express the action of the generalized Bogoliubov transformation $e^{B(\eta)}$ on the b-fields by means of a convergent series of nested commutators. To this end, we start by noticing that, for $f \in L^2(\mathbb{R}^3)$,

$$
\begin{aligned}
e^{-B(\eta)} b(f) e^{B(\eta)} &= b(f) + \int_0^1 ds \, \frac{d}{ds} e^{-sB(\eta)} b(f) e^{sB(\eta)} \\
&= b(f) - \int_0^1 ds \, e^{-sB(\eta)} [B(\eta), b(f)] e^{sB(\eta)} \\
&= b(f) - [B(\eta), b(f)] + \int_0^1 ds_1 \int_0^{s_1} ds_2 \, e^{-s_2 B(\eta)} [B(\eta), [B(\eta), b(f)] e^{s_2 B(\eta)}
\end{aligned}
$$

Iterating m times, we obtain

$$
\begin{aligned}
e^{-B(\eta)} b(f) e^{B(\eta)} = &\sum_{n=1}^{m-1} (-1)^n \frac{\mathrm{ad}^{(n)}_{B(\eta)}(b(f))}{n!} \\
&+ \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m \, e^{-s_m B(\eta)} \mathrm{ad}^{(m)}_{B(\eta)}(b(f)) e^{s_m B(\eta)}
\end{aligned}
\tag{3.36}
$$

where we introduced the notation $\mathrm{ad}^{(n)}_{B(\eta)}(A)$ defined recursively by

$$
\mathrm{ad}^{(0)}_{B(\eta)}(A) = A \quad \text{and} \quad \mathrm{ad}^{(n)}_{B(\eta)}(A) = [B(\eta), \mathrm{ad}^{(n-1)}_{B(\eta)}(A)]
$$

We will show later that, under suitable assumptions on η, the error term on the r.h.s. of (3.36) is negligible in the limit $m \to \infty$. This means that the action of the generalized Bogoliubov transformation $B(\eta)$ on $b(f)$ and similarly on $b^*(f)$ can be described in terms of the nested commutators $\mathrm{ad}_{B(\eta)}(A)$, for $A = b(f)$ or $A = b^*(f)$. In the next lemma, we give a detailed analysis of these terms.

For a kernel $\eta \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, we will use the notation

$$
\eta^{(n)} = \begin{cases} 1, & \text{for } n = 0 \\ (\eta\bar{\eta})^\ell, & \text{if } n = 2\ell, \ell \in \mathbb{N}\backslash\{0\} \\ (\eta\bar{\eta})^\ell \eta & \text{if } n = 2\ell + 1, \ell \in \mathbb{N} \end{cases}
\tag{3.37}
$$

Here we, identify $\eta \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ with the Hilbert-Schmidt operator acting on $L^2(\mathbb{R}^3)$, with the integral kernel given by η. To avoid keeping track of complex conjugations of η-kernels, we also introduce the following notation. For $\natural \in \{\cdot, *\}$ we write $\eta_\natural = \eta$, if $\natural = \cdot$, and $\eta_\natural = \bar{\eta}$ if $\natural = *$. More generally, for $n \in \mathbb{N}$, and $(\natural_1, \ldots, \natural_n) \in \{\cdot, *\}^n$, we will use the notation $\eta^{(n)}_\natural = \eta_{\natural_1} \eta_{\natural_2} \ldots \eta_{\natural_n}$, in the sense of products of operators. Also for a function $f \in L^2(\mathbb{R}^3)$, we use the notation $f_\natural = f$ if $\natural = \cdot$ and $f_\natural = \bar{f}$ if $\natural = *$.

Lemma 3.2.3. *Let $\eta \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ be such that $\eta(x; y) = \eta(y; x)$ for all $x, y \in \mathbb{R}^3$. Let $B(\eta)$ be defined as in (3.31). Let $n \in \mathbb{N}$ and $f \in L^2(\mathbb{R}^3)$. Then the nested commutators $ad^{(n)}_{B(\eta)}(b(f))$ can be written as the sum of exactly $2^n n!$ terms, with the following properties.*

i) Possibly up to a sign, each term has the form

$$\Lambda_1\Lambda_2\dots\Lambda_i\frac{1}{N^k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond)) \tag{3.38}$$

for some $i,k,s\in\mathbb{N}$, $j_1,\dots,j_k\in\mathbb{N}\backslash\{0\}$, $\Diamond\in\{\cdot,*\}$, $\sharp\in\{\cdot,*\}^k$, $\flat\in\{\cdot,*\}^{k+1}$, $\natural_v\in\{\cdot,*\}^{j_v}$ *for all* $v=1,\dots,k$ *and* $\natural\in\{\cdot,*\}^s$. *In (3.38), each operator* $\Lambda_w:\mathcal{F}^{\leq N}\to\mathcal{F}^{\leq N}$ *is either a factor* $(N-\mathcal{N})/N$*, a factor* $(N+1-\mathcal{N})/N$ *or an operator of the form*

$$\frac{1}{N^p}\Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1},\eta^{(m_2)}_{\natural_2},\dots,\eta^{(m_p)}_{\natural_p}) \tag{3.39}$$

for some $p,m_1,\dots,m_p\in\mathbb{N}\backslash\{0\}$, $\sharp,\flat\in\{\cdot,*\}^p$, $\natural_v\in\{\cdot,*\}^{m_v}$ *for all* $v=1,\dots,p$.

ii) If a term of the form (3.38) contains $m\in\mathbb{N}$ *factors* $(N-\mathcal{N})/N$ *or* $(N+1-\mathcal{N})/N$ *and* $j\in\mathbb{N}$ *factors of the form (3.39) with* $\Pi^{(2)}$*-operators of order* $p_1,\dots,p_j\in\mathbb{N}\backslash\{0\}$*, then we have*

$$m+(p_1+1)+\dots+(p_j+1)+(k+1)=n+1 \tag{3.40}$$

iii) If a term of the form (3.38) contains (considering all Λ*-operators and the* $\Pi^{(1)}$*-operator) the kernels* $\eta^{(i_1)}_{\natural_1},\dots,\eta^{(i_m)}_{\natural_m}$ *and the wave function* $\eta^{(s)}_{\natural}(f_\Diamond)$ *for some* $m,s\in\mathbb{N}$, $i_1,\dots,i_m\in\mathbb{N}\backslash\{0\}$, $\natural_r\in\{\cdot,*\}^{i_r}$ *for all* $r=1,\dots,m$, $\natural\in\{\cdot,*\}^s$ *then*

$$i_1+\dots+i_m+s=n.$$

iv) There is exactly one term having the form

$$\left(\frac{N-\mathcal{N}}{N}\right)^{n/2}\left(\frac{N+1-\mathcal{N}}{N}\right)^{n/2}b(\eta^{(n)}(f)) \tag{3.41}$$

if n *is even, and*

$$-\left(\frac{N-\mathcal{N}}{N}\right)^{(n+1)/2}\left(\frac{N-\mathcal{N}+1}{N}\right)^{(n-1)/2}b^*(\eta^{(n)}(\bar{f})) \tag{3.42}$$

if n *is odd.*

v) If the $\Pi^{(1)}$*-operator in (3.38) is of order* $k\in\mathbb{N}\backslash\{0\}$*, it has either the form*

$$\int b^{\flat_0}_{x_1}\prod_{i=1}^{k-1}a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}a^*_{y_k}a(\eta^{(2r)}(f))\prod_{i=1}^{k}\eta^{(j_i)}_{\natural_i}(x_i;y_i)dx_idy_i$$

or the form

$$\int b^{\flat_0}_{x_1}\prod_{i=1}^{k-1}a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}a_{y_k}a^*(\eta^{(2r+1)}(\bar{f}))\prod_{i=1}^{k}\eta^{(j_i)}_{\natural_i}(x_i;y_i)dx_idy_i$$

for some $r\in\mathbb{N}$, $j_1,\dots,j_k\in\mathbb{N}\backslash\{0\}$. *If it is of order* $k=0$*, then it is either given by* $b(\eta^{(2r)}_{\natural}(f_\Diamond))$ *or by* $b^*(\eta^{(2r+1)}_{\natural}(f_\Diamond))$*, for some* $r\in\mathbb{N}$.

vi) For every non-normally ordered term of the form

$$\int dxdy\,\eta_\natural^{(i)}(x;y)a_x a_y^*,\quad \int dxdy\,\eta_\natural^{(i)}(x;y)b_x a_y^*$$
$$\int dxdy\,\eta_\natural^{(i)}(x;y)a_x b_y^*,\quad or\quad \int dxdy\,\eta_\natural^{(i)}(x;y)b_x b_y^*$$

appearing either in the Λ-operators or in the $\Pi^{(1)}$-operator in (3.38), we have $i \geq 2$.

Remark: Similarly, the nested commutator $\mathrm{ad}^{(n)}(b^*(f))$ can be written as the sum of $2^n n!$ terms of the form

$$\frac{1}{N^k}\widetilde{\Pi}^{(1)}_{\sharp,\flat}(\eta_{\natural_1}^{(j_1)},\dots,\eta_{\natural_k}^{(j_k)};\eta_{\natural_{k+1}}^{(\ell)}(f_\Diamond))\Lambda_1\Lambda_2\dots\Lambda_i$$

satisfying properties analogous to those listed in i)-vi).

Proof. We prove the lemma by induction. For $n = 0$ all claims are trivially satisfied. For the induction step from n to $n+1$ we first compute, using (1.20) and (1.21) the commutators

$$\begin{aligned}
[B(\eta), b_z] &= -\frac{N-\mathcal{N}}{N}b^*(\eta_z) + \frac{1}{N}\int dxdy\,\eta(x;y)b_x^* a_y^* a_z \\
&= -b^*(\eta_z)\frac{N+1-\mathcal{N}}{N} + \frac{1}{N}\int dxdy\,\eta(x;y)a_z a_y^* b_x^*, \\
[B(\eta), b_z^*] &= -b(\eta_z)\frac{N-\mathcal{N}}{N} + \frac{1}{N}\int dxdy\,\bar\eta(x;y)a_z^* a_y b_x \\
&= -\frac{N+1-\mathcal{N}}{N}b(\eta_z) + \frac{1}{N}\int dxdy\,\bar\eta(x;y)b_x a_y a_z^*, \\
[B(\eta), a_z^* a_w] &= [B(\eta), a_w a_z^*] = -b_z^* b^*(\eta_w) - b(\eta_z)b_w, \\
[B(\eta), N-\mathcal{N}] &= [B(\eta), N+1-\mathcal{N}] = \int dxdy\,(\eta(x,y)b_x^* b_y^* + \bar\eta(x;y)b_x b_y).
\end{aligned} \tag{3.43}$$

From $\mathrm{ad}^{(n+1)}_{B(\eta)}(b(f)) = [B(\eta), \mathrm{ad}^{(n)}_{B(\eta)}(b(f))]$ and by linearity, it is enough to analyze

$$\Big[B(\eta), \Lambda_1\Lambda_2\dots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta_{\natural_1}^{(j_1)},\dots,\eta_{\natural_k}^{(j_k)};\eta_{\natural_{k+1}}^{(\ell)}(f_\Diamond))\Big] \tag{3.44}$$

with the operator $\Lambda_1\Lambda_2\dots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta_{\natural_1}^{(j_1)},\dots,\eta_{\natural_k}^{(j_k)};\eta_\natural^{(s)}(f_\Diamond))$ satisfying properties (i) to (vi). Applying Leibniz rule $[A, BC] = [A,B]C + B[A,C]$, the commutator (3.44) is given by a sum of terms, where $B(\eta)$ is either commuted with a Λ-operator, or with the $\Pi^{(1)}$-operator.

Let's consider first the case that $B(\eta)$ is commuted with a Λ-operator, assuming further that Λ is either the operator $(N-\mathcal{N})/N$ or the operator $(N+1-\mathcal{N})/N$. The last line in (3.43) implies that such an operator Λ is replaced, after commutation with $B(\eta)$, by the sum

$$N^{-1}\Pi^{(2)}_{*,*}(\eta) + N^{-1}\Pi^{(2)}_{\cdot,\cdot}(\bar\eta). \tag{3.45}$$

With this replacement, we generate two terms contributing to $\mathrm{ad}^{(n+1)}_{B(\eta)}(b(f))$. Let us check that these new terms satisfy the properties (i)-(vi) (of course, with n replaced by $(n+1)$). (i) is obviously true. Also (ii) remains valid, because replacing a factor $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$ by one of the two summands in (3.45), the index m will decrease by one, but there will be an additional factor of 2 because we added a $\Pi^{(2)}$-operator of the order one. Since exactly one additional kernel $\eta_\natural$ is inserted, also (iii) continues to hold true. The factor $\Pi^{(1)}$ is not affected by the replacement, hence the new terms will continue to satisfy (v). Furthermore, since both terms in (3.45) are normally ordered, also (vi) remains valid, by the induction assumption. We observe, finally, that the two terms we generated here do not have the form appearing in (iv).

Next, we consider the commutator of $B(\eta)$ with a Λ-operator of the form $\Lambda = N^{-p}\Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_p)}_{\natural_p})$ for a $p\in\mathbb{N}$ ($p\leq n$ by (ii)). By definition

$$\Lambda = N^{-p}\int b^{\flat_0}_{x_1}\prod_{i=1}^{p-1} a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}b^{\sharp_p}_{y_p}\prod_{i=1}^{p}\eta^{(m_i)}_{\natural_i}(x_i;y_i)dx_idy_i \tag{3.46}$$

If $[B(\eta),\cdot]$ hits $b^{\flat_0}_{x_1}$, the first two relations in (3.43) imply that Λ is replaced by a sum of two operators, the first one being either

$$\begin{aligned}&-\frac{N-\mathcal{N}}{N}N^{-p}\Pi^{(2)}_{\sharp,\widetilde{\flat}}(\eta^{(m_1+1)}_{\natural_1},\eta^{(m_2)}_{\natural_2},\dots,\eta^{(m_p)}_{\natural_p}) \quad \text{or}\\ &\qquad\qquad -\frac{N+1-\mathcal{N}}{N}N^{-p}\Pi^{(2)}_{\sharp,\widetilde{\flat}}(\eta^{(m_1+1)}_{\natural_1},\eta^{(m_2)}_{\natural_2},\dots,\eta^{(m_p)}_{\natural_p})\end{aligned} \tag{3.47}$$

depending on whether $\flat_0=\cdot$ or $\flat_0=*$ (here $\widetilde{\flat}=(\bar{\flat}_0,\flat_1,\dots,\flat_{p-1})$ with $\bar{\flat}_0=\cdot$ if $\flat_0=*$ and $\bar{\flat}_0=*$ if $\flat_0=\cdot$). The second operator emerging when $[B(\eta),\cdot]$ hits $b^{\flat_0}_{x_1}$ is a $\Pi^{(2)}$-operator of order $(p+1)$, given by

$$N^{-(p+1)}\Pi^{(2)}_{\widetilde{\sharp},\widetilde{\flat}}(\eta_{\natural_0},\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_p)}_{\natural_p}) \tag{3.48}$$

where $\widetilde{\sharp}=(\bar{\flat}_0,\sharp_1,\dots,\sharp_p)$, $\widetilde{\flat}=(\bar{\flat}_0,\flat_0,\dots,\flat_{p-1})$ and $\natural_0=\flat_0$.

For both terms (3.47) and (3.48), (i) is clearly correct and also (ii) remains true (when we replace (3.46) with (3.47), the number of $(N-\mathcal{N})/N$ or $(N-\mathcal{N}+1)/N$-operators increases by one, while everything else remains unchanged; similarly, when we replace (3.46) with (3.48), the order of the $\Pi^{(2)}$-operator increases by one, while the rest remains unchanged). (iii) remains true as well, since, in (3.47), the power m_1+1 of the first η-kernel is increased by one unit and, in (3.48), there is one additional factor η, compared with (3.46). (v) remains valid, since the $\Pi^{(1)}$-operator on the right is not affected by this commutator. (vi) remains true in (3.47), because $m_1+1\geq 2$. It remains true also in (3.48). In fact, according to (3.43), when switching from (3.46) to (3.48), we are effectively replacing $b\to b^*a^*a$ or $b^*\to baa^*$. Hence, the first pair of operators in (3.48) is always normally ordered. As for the second pair of creation and annihilation operators (the one associated with the kernel $\eta^{(m_1)}_{\natural_1}$ in (3.48)), the first field is of the same type as the original b-field appearing in (3.46); hence non-normally ordered pairs cannot be created. Finally, we remark that the terms we generated here are certainly not of the form in (iv) (because for terms as in (iv) all Λ-factors must be either $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$, and this is not the case, for terms containing (3.47) or (3.48)).

The same arguments can be applied if $B(\eta)$ hits the factor $b^{\sharp_p}_{y_p}$ on the right of (3.46) (in this case, we use the identities for the first two commutators in (3.43) having the b-field to the left of the factors $(N+1-\mathcal{N})/N$ and $(N-\mathcal{N})/N$ and to the right of the $a_z a_y^*$ and $a_z^* a_y$ operators).

If now $B(\eta)$ hits a term $a^*_{y_r} a_{x_{r+1}}$ or $a_{y_r} a^*_{x_{r+1}}$ in (3.46), for an $r = 1, \dots, p-1$, then (3.43) implies that $\Lambda = N^{-p} \Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1}, \dots, \eta^{(m_p)}_{\natural_p})$ is replaced by the sum of the two terms, given by

$$- \left[N^{-r} \Pi^{(2)}_{\sharp',\flat'}(\eta^{(m_1)}_{\natural_1}, \dots, \eta^{(m_r+1)}_{\natural'_r}) \right] \left[N^{-(p-r)} \Pi^{(2)}_{\sharp'',\flat''}(\eta^{(m_{r+1})}_{\natural_{r+1}}, \dots, \eta^{(m_p)}_{\natural_p}) \right] \tag{3.49}$$

and by

$$- \left[N^{-r} \Pi^{(2)}_{\sharp''',\flat'}(\eta^{(m_1)}_{\natural_1}, \dots, \eta^{(m_r)}_{\natural'_r}) \right] \left[N^{-(p-r)} \Pi^{(2)}_{\sharp'',\flat'''}(\eta^{(m_{r+1}+1)}_{\natural'_{r+1}}, \dots, \eta^{(m_p)}_{\natural_p}) \right] \tag{3.50}$$

with $\flat' = (\flat_0, \dots, \flat_{r-1})$, $\flat'' = (\flat_r, \dots, \flat_{p-1})$, $\flat''' = (\bar{\flat}_r, \flat_{r+1}, \dots, \flat_{p-1})$ and with $\sharp' = (\sharp_1, \dots, \sharp_{r-1}, \bar{\sharp}_r)$, $\sharp'' = (\sharp_{r+1}, \dots, \sharp_p)$, $\sharp''' = (\sharp_1, \dots, \sharp_r)$ (here, we denote $\bar{\sharp}_r = *$ if $\sharp_r = \cdot$ and $\bar{\sharp}_r = \cdot$ if $\sharp_r = *$, and similarly for $\bar{\flat}_{r-1}$). The precise form of $\natural'_r$ and $\natural'_{r+1}$ does not play an important role (they are given by $\natural'_r = (\natural_r, \sharp_r)$ and $\natural'_{r+1} = (\natural_{r+1}, \flat_r)$). The new terms containing (3.49) and (3.50) clearly satisfy (i). Furthermore, (ii) remains true because the contribution of the original Λ to the sum in (3.40), which was given by $(p+1)$ is now replaced by $(r+1) + (p-r+1) = p+2$. Clearly, (iii) remains true as well, since, for both terms (3.49) and (3.50), the total powers of the η-kernels is increased exactly by one. As before, the terms we generated do not have the form (iv). (v) continues to hold true, because the $\Pi^{(1)}$ term is unaffected. As for (vi), we observe that non-normally ordered pairs can only be created where $\sharp_r$ is changed to $\bar{\sharp}_r$ (in the term where $\sharp'$ appears) or where $\flat_r$ is changed to $\bar{\flat}_r$ (in the term where $\flat'''$ appears). In both cases, however, the change $\sharp_r \to \bar{\sharp}_r$ and $\flat_r \to \bar{\flat}_r$ comes together with an increase in the power of the corresponding η-kernel (i.e. $\eta^{(m_r)}_{\natural_r}$ is changed to $\eta^{(m_r+1)}_{\natural'_r}$ in the first case, while $\eta^{(m_{r+1})}_{\natural_{r+1}}$ is changed to $\eta^{(m_{r+1}+1)}_{\natural'_{r+1}}$ in the second case). Since $m_r + 1, m_{r+1} + 1 \geq 2$, even if non-normally ordered terms are created, they still satisfy (vi).

Next, let us consider the terms arising from commuting $B(\eta)$ with the operator

$$\begin{aligned} N^{-k} \Pi^{(1)}_{\sharp,\flat}&(\eta^{(j_1)}_{\natural_1}, \dots, \eta^{(j_k)}_{\natural_k}; \eta^{(s)}_{\natural}(f_\Diamond)) \\ &= N^{-k} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k-1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} a^{\sharp_k}_{y_k} a^{\flat_k} (\eta^{(s)}_{\natural}(f_\Diamond)) \prod_{i=1}^{k} \eta^{(j_i)}_{\natural_i}(x_i; y_i) dx_i dy_i \end{aligned} \tag{3.51}$$

We argue similarly to the case in which $B(\eta)$ hits a $\Pi^{(2)}$-operator like (3.46). In particular, if $B(\eta)$ hits the operator $b^{\flat_0}_{x_1}$, the operator (3.51) is replaced by the sum of two terms, the first one being

$$-\frac{N-\mathcal{N}}{N}N^{-p}\Pi^{(1)}_{\sharp,\widetilde{\flat}}(\eta^{(m_1+1)}_{\natural'_1},\eta^{(m_2)}_{\natural_2},\dots,\eta^{(m_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond)) \qquad \text{or} \qquad -\frac{N+1-\mathcal{N}}{N}N^{-p}\Pi^{(1)}_{\sharp,\widetilde{\flat}}(\eta^{(m_1+1)}_{\natural'_1},\eta^{(m_2)}_{\natural_2},\dots,\eta^{(m_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond))$$

depending on whether $\flat_0=\cdot$ or $\flat_0=*$ (with $\widetilde{\flat}=(\bar{\flat}_0,\flat_1,\dots,\flat_{k-1})$) and the second one being

$$N^{-(k+1)}\Pi^{(1)}_{\widetilde{\sharp},\widetilde{\flat}}(\eta,\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_k)}_{\natural_k},\eta^{(s)}_{\natural}(f_\Diamond))$$

with $\widetilde{\sharp}=(\bar{\flat}_0,\sharp_1,\dots,\sharp_k)$ and $\widetilde{\flat}=(\bar{\flat}_0,\flat_1,\dots,\flat_k)$. As we did in the analysis of (3.47) and (3.48), one can show that both these terms satisfy all properties (i), (ii), (iii), (v), (vi) (we will discuss the properties (iv) below).

If instead $B(\eta)$ hits one of the factors $a^{\sharp_r}_{y_r}a^{\flat_r}_{x_{r+1}}$ for an $r=1,\dots,k-1$, the resulting two terms will have the form

$$-\left[N^{-r}\Pi^{(2)}_{\sharp',\flat'}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_r+1)}_{\natural'_r})\right]\left[N^{-(k-r)}\Pi^{(1)}_{\sharp'',\flat''}(\eta^{(m_{r+1})}_{\natural_{r+1}},\dots,\eta^{(m_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond))\right] \tag{3.52}$$

and by

$$-\left[N^{-r}\Pi^{(2)}_{\sharp''',\flat'}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_r)}_{\natural'_r})\right]\left[N^{-(k-r)}\Pi^{(1)}_{\sharp'',\flat'''}(\eta^{(m_{r+1}+1)}_{\natural'_{r+1}},\dots,\eta^{(m_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond))\right] \tag{3.53}$$

with $\sharp',\sharp'',\sharp'''$ and $\flat',\flat'',\flat'''$ as defined after (3.50). Proceeding similarly as we did in (3.50), we can show that these terms satisfy (i),(ii),(iii),(v),(vi).

Let us now consider the case that (3.51) is commuted with the last pair of operators appearing in (3.51). From the induction assumption, we know that this pair can only be $a^*_{y_k}a(\eta^{(2r)}(f))$ or $a_{y_k}a^*(\eta^{(2r+1)}(\bar{f}))$. In the first case, (3.51) is replaced by

$$-\Pi^{(2)}_{\sharp,\flat'}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k})\,b^*(\eta^{(2r+1)}(\bar{f}))-\Pi^{(2)}_{\sharp',\flat'}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_{k-1})}_{\natural_{k-1}},\eta^{(j_k+1)}_{\natural'_k})\,b(\eta^{(2r)}(f)) \tag{3.54}$$

In the second case, it is replaced by

$$-\Pi^{(2)}_{\sharp',\flat'}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_{k-1})}_{\natural_{k-1}},\eta^{(j_k+1)}_{\natural'_k})b^*(\eta^{(2r+1)}(\bar{f}))-\Pi^{(2)}_{\sharp',\flat'}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k})b(\eta^{(2r+2)}(f)) \tag{3.55}$$

In (3.54), (3.55), we used the notation $\flat'=(\flat_0,\dots,\flat_{k-1})$, $\sharp'=(\sharp_1,\dots,\bar{\sharp}_k)$ (as usual, the precise form of $\natural'_k$ is not important). From the expression (3.54), (3.55), we see that also in this case, (i), (ii), (iii), (v), (vi) are satisfied.

As for (iv), from the induction assumption we know that there is exactly one term, in the expansion for $\mathrm{ad}^{(n)}_{B(\eta)}(b(f))$, given by (3.41) if n is even and by (3.42) if n is odd. Let us take, for example, (3.41). If we commute the zero-order $\Pi^{(1)}$-operator $b(\eta^{(n)}(f))$ in (3.41) with $B(\eta)$, we obtain exactly the term in (3.42), with n replaced by $(n+1)$ (together with a second term, containing a $\Pi^{(1)}$-operator of order one). Similarly, if we take (3.42) and we commute the $\Pi^{(1)}$-operator $b^*(\eta^{(n)}(\bar{f})$ with $B(\eta)$, we get (3.41), with n replaced by $(n+1)$. Clearly, looking at the terms above, it is clear that there can be only one term with this form. This shows that also in the expansion for $\text{ad}^{(n+1)}_{B(\eta)}(b(f))$, there is exactly one term of the form given in (iv).

Finally, let us count the number of terms in the expansion for $\text{ad}^{(n+1)}_{B(\eta)}(b(f))$. By the inductive assumption, the expansion for $\text{ad}^{(n)}_{B(\eta)}(b(f))$ contains exactly $2^n n!$ terms. By (ii), each of these terms is a product of exactly $(n+1)$ operators, each of them being either $(N-\mathcal{N})$, $(N+1-\mathcal{N})$, a field operator $b^\sharp_x$ or a quadratic factor $a^\sharp_y a^\flat_x$ commuting with the number of particles operator. By (3.43), the commutator of $B(\eta)$ with each such factor gives a sum of two terms. Therefore, by the product rule, $\text{ad}^{(n+1)}_{B(\eta)}(b(f))$ contains $2^n(n!) \times 2(n+1) = 2^{(n+1)}(n+1)!$ summands. This concludes the proof of the lemma. □

From Lemma 3.2.3, we immediately obtain a convergent series expansion for the conjugation of the fields $b(f)$ and $b^*(f)$ with the unitary operator $\exp(B(\eta))$.

Lemma 3.2.4. *Let $\eta \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ be symmetric, with $\|\eta\|_2$ sufficiently small. Then we have*

$$\begin{aligned} e^{-B(\eta)} b(f) e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} ad^{(n)}_{B(\eta)}(b(f)) \\ e^{-B(\eta)} b^*(f) e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} ad^{(n)}_{B(\eta)}(b^*(f)) \end{aligned} \tag{3.56}$$

where the series on the r.h.s. are absolutely convergent.

Proof. We start from the expression (3.36).

$$\begin{aligned} e^{-B(\eta)} b(f) e^{B(\eta)} = &\sum_{n=1}^{m-1} (-1)^n \frac{\text{ad}^{(n)}_{B(\eta)}(b(f))}{n!} \\ &+ \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m \, e^{-s_m B(\eta)} \text{ad}^{(m)}_{B(\eta)}(b(f)) e^{s_m B(\eta)} \end{aligned} \tag{3.57}$$

To prove (3.56), we show that the norm of the error term converges to zero, as $m \to \infty$. By Lemma (3.2.3), $\text{ad}^{(n)}_{B(\eta)}(b(f)$ is given by a sum of $2^n n!$ terms of the form

$$\Lambda_1 \dots \Lambda_i \, \frac{1}{N^k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1}, \dots, \eta^{(j_k)}_{\natural_k}; \eta^{(\ell)}(f_\Diamond)) \tag{3.58}$$

with $i, k, \ell \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\backslash\{0\}$ and where each Λ_r is either $(N-\mathcal{N})/N$, $(N+1-\mathcal{N})/N$ or an operator of the form

$$\frac{1}{N^p} \Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1}, \dots, \eta^{(m_p)}_{\natural_p})$$

On $\mathcal{F}^{\leq N}$, we have the bounds $\|(N-\mathcal{N})/N\| \leq 1$ and $\|(N+1-\mathcal{N})/N\| \leq 2$. Lemma 3.2.1 implies that

$$N^{-p}\left\|\Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_p)}_{\natural_p})\right\| \leq (12)^p(2\|\eta\|_2)^{m_1+\dots+m_p}$$

and that

$$N^{-k}\left\|\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k};\eta^{(\ell)}(f_\lozenge))\right\| \leq (12)^k\sqrt{N}\|f\|_2(2\|\eta\|_2)^{\ell+j_1+\dots+j_k}$$

Here we used the fact that, if a kernel $\eta^{(j)}$ is associated with a normally ordered pairs of creation and annihilation operators, then $\|\eta^{(j)}\|_{\mathrm{HS}} \leq \|\eta\|^j_{\mathrm{HS}}$. If instead $\eta^{(j)}$ is associated with a non-normally ordered pair, then point (vi) in Lemma 3.2.3 implies that $j \geq 2$. Hence,

$$\begin{aligned}\int\left|\eta^{(j)}(x;x)\right|dx &= \int\left|\int\eta(x;y)\eta^{(j-1)}(y;x)dy\right|dx \\ &\leq \left(\int|\eta(x;y)|^2dxdy\right)^{1/2}\left(\int|\eta^{(j-1)}(x;y)|^2dxdy\right)^{1/2} \\ &\leq \|\eta\|_2\|\eta^{(j-1)}\|_2 \leq \|\eta\|_2^j\end{aligned}$$

Therefore, if the term (3.58) contains $\Pi^{(2)}$-operators of order $p_1,\dots,p_j \in \mathbb{N}\backslash\{0\}$, we can bound

$$\begin{aligned}&\left\|\Lambda_1\cdot\dots\cdot\Lambda_i\,\frac{1}{N^k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k};\eta^{(\ell)}(f_\lozenge))\right\| \\ &\qquad\leq 12^{p_1+\dots+p_j+k}\sqrt{N}(2\|\eta\|_2)^m \leq \sqrt{N}\|f\|_2C^m\|\eta\|^m\end{aligned}$$

and therefore, since $\mathrm{ad}^{(m)}_{B(\eta)}(b(f))$ is the sum of $2^m m!$ terms,

$$\|\mathrm{ad}^{(m)}_{B(\eta)}(b(f))\| \leq \sqrt{N}\|f\|_2(2C\|\eta\|_2)^m m! \tag{3.59}$$

This proves, first of all, that the series on the r.h.s. of (3.56) converges absolutely, if $\|\eta\|_2 \leq (4C)^{-1}$. Under this condition, (3.59) also implies that the error term on the r.h.s. of (3.57) converges to zero, as $m\to\infty$, since

$$\left\|\int_0^1 ds_1\cdots\int_0^{s_{m-1}} ds_m e^{-s_mB(\eta)}\mathrm{ad}_{B(\eta)}(b(f))e^{s_mB(\eta)}\right\| \leq \sqrt{N}\|f\|_2(2C\|\eta\|)^m$$

□

It follows from the proof of Lemma 3.2.4 that the norm of $e^{-B(\eta)}b(f)e^{B(\eta)}$ is bounded by a constant proportional to $\sqrt{N}$. For states with bounded number of particles, this is quite a pessimistic bound. It follows from Lemma 3.2.2, that $e^{-B(\eta)}b(f)e^{B(\eta)}$, like the original field operator $b(f)$, can be bounded by the operator $\mathcal{N}^{1/2}(1-\mathcal{N}/N)^{1/2}$ (see Lemma 1.2.4). In fact, most terms in the expansion (3.56) are small, on states with bounded number of particles. The only "large" contributions arise from the terms described in point (iv) of Lemma 3.2.3, replacing the factors $(1-\mathcal{N}/N)$ and $(1-(\mathcal{N}-1)/N)$ by 1 (which is a good approximation, on states in $\mathcal{F}^{\leq N}$ having a bounded number of particles). Summing over all $n \in \mathbb{N}$, we conclude that

$$e^{-B(\eta)} b(f) e^{B(\eta)} = b(\cosh_\eta(f)) + b^*(\sinh_\eta(\bar{f})) + \varepsilon_\eta(f) \tag{3.60}$$

where the operators $\cosh_\eta$ and $\sinh_\eta$ are defined in (3.34) and where $\varepsilon_\eta(f)$ is a bounded operator on $\mathcal{F}^{\leq N}$ (with norm of order $\sqrt{N}$) which is small on states with bounded number of particles. In fact, we will prove later that $\varepsilon_\eta(f)$ can be bounded by the operator $\mathcal{N}^{3/2}/N$. Eq. (3.60) explains why we refer to the unitary operators $e^{-B(\eta)}$ as generalized Bogoliubov transformations. The difference with respect to the action (3.33) of a standard Bogoliubov transformation is just the appearance of the small (in appropriate sense) error $\varepsilon_\eta(f)$.

3.3 Fluctuation Dynamics

In this section, we are going to define the fluctuation dynamics describing the evolution of orthogonal excitations of the Bose-Einstein condensate.

Instead of comparing the solution of the many-body Schrödinger equation (3.8) directly with the solution of the Gross-Pitaevskii equation (3.12), it is convenient to introduce a modified, N-dependent, Gross-Pitaevskii equation. To this end, we consider the ground state f_ℓ of the Neumann problem

$$\left(-\Delta + \frac{1}{2} V\right) f_\ell = \lambda_\ell f_\ell \tag{3.61}$$

on the ball $|x| \leq N\ell$ (we omit the N-dependence in the notation for f_ℓ and for λ_ℓ; notice that λ_ℓ scales as N^{-3}), with the normalization $f_\ell(x) = 1$ if $|x| = N\ell$. We extend f_ℓ to $\mathbb{R}^3$ by setting $f_\ell(x) = 1$ for all $|x| > N\ell$. It is also useful to set $w_\ell = 1 - f_\ell$ (so that $w_\ell(x) = 0$ if $|x| > N\ell$). By scaling, we observe that $f_\ell(N.)$ satisfies the equation

$$\left(-\Delta + \frac{N^2}{2} V(N.)\right) f_\ell(N.) = N^2 \lambda_\ell f_\ell(N.) \tag{3.62}$$

on the ball $|x| \leq \ell$. We will consider $\ell > 0$ of order one, independent of N. With this choice, we expect that f_ℓ will be close, in the limit of large N, to the solution of the zero-energy scattering equation (3.2). This is confirmed by the next lemma, where we collect some important properties of f_ℓ. Most of the these results are taken from Lemma A.1 of [36].

Lemma 3.3.1. *Let $V \in L^3(\mathbb{R}^3)$ be a non-negative, spherically symmetric potential with $V(x) = 0$ for all $|x| > R$. Fix $\ell > 0$ and let f_ℓ denote the solution of* (3.61)*.*

i) We have

$$\lambda_\ell = \frac{3\mathfrak{a}_0}{N^3 \ell^3} \left(1 + \mathcal{O}(\mathfrak{a}_0/N\ell)\right)$$

ii) We have $0 \leq f_\ell, w_\ell \leq 1$ and

$$\int dx\, V(x) f_\ell(x) = 8\pi \mathfrak{a}_0 + \mathcal{O}(N^{-1}). \tag{3.63}$$

iii) There exists a constant $C > 0$, depending on the potential V, such that

$$w_\ell(x) \leq \frac{C}{|x|+1} \qquad \text{and} \qquad |\nabla w_\ell(x)| \leq \frac{C}{|x|^2+1}. \tag{3.64}$$

for all $|x| \leq N\ell$.

Proof. Statement (i), the fact that $0 \leq f_\ell, w_\ell \leq 1$, and statement (iii) follow from Lemma A.1 in [36]. We have to show (3.63). To this end, we adapt the proof of Lemma 5.1 (iv) of [40]. With $r = |x|$, we may write $m(r) = r f_\ell(r)$. We find that, for all $r \in (R, N\ell]$,

$$m(r) = \lambda_\ell^{-\frac{1}{2}} \sin(\lambda_\ell^{\frac{1}{2}}(r - N\ell)) + N\ell \cos(\lambda_\ell^{\frac{1}{2}}(r - N\ell)). \tag{3.65}$$

By expanding up to the order $\mathcal{O}(\lambda_\ell^2)$ we obtain

$$m(r) = r - \mathfrak{a}_0 + \mathcal{O}(N^{-1}), \quad m'(r) = 1 + \mathcal{O}(N^{-1}). \tag{3.66}$$

Hence

$$\begin{aligned} \int dx\, V(x) f_\ell(x) &= 4\pi \int_0^R dr\, r V(r) m(r) \\ &= 8\pi \int_0^R dr\, (r m''(r) + \lambda_\ell r^2 f_\ell(r)) \\ &= 8\pi \int_0^R dr\, r m''(r) + O(N^{-3}) \\ &= 8\pi (R m'(R) - m(R)) + \mathcal{O}(N^{-1}) = 8\pi \mathfrak{a}_0 + \mathcal{O}(N^{-1}). \end{aligned} \tag{3.67}$$

□

Next, we introduce next the modified Gross-Pitaevskii equation

$$i\partial_t \widetilde{\varphi_{\xi_t}} = -\Delta \widetilde{\varphi_{\xi_t}} + \left(N^3 V(N.) f_\ell(N.) * |\widetilde{\varphi_{\xi_t}}|^2\right) \widetilde{\varphi_{\xi_t}} \tag{3.68}$$

with initial data $\widetilde{\varphi_{\xi_{t=0}}} = \varphi$ describing the Bose-Einstein condensate at time $t = 0$. While in Theorem 3.1.2 the notation φ is already used to indicate the initial condensate wave function, in the proof of Theorem 3.1.1 we will choose $\varphi = \phi_{\mathrm{GP}}$ to be the minimizer of the Gross-Pitaevskii functional (3.6). In both cases, we assume that $\varphi \in H^4(\mathbb{R}^3)$.

Notice that, in contrast with the initial data φ, the solution $\widetilde{\varphi_{\xi_t}}$ depends on N. With (3.63), one can show that $\widetilde{\varphi_{\xi_t}}$ converges towards the solution of the original Gross-Pitaevskii equation (3.12), as $N \to \infty$. This fact and some other important properties of the solutions of (3.12) and (3.68) are listed in the next proposition, whose proof can be found in Theorem 3.1 of [11], with the only difference that, in [11], the modified Gross-Pitaevskii equation was defined through the solution f of the zero energy scattering equation, while here we work with the Neumann ground state f_ℓ. The only relevant consequence is the fact that, here, the integral of f_ℓ against V is not exactly equal to $8\pi\mathfrak{a}_0$; the error, however, is of order N^{-1} by (3.63).

Proposition 3.3.2. *Let $V \in L^3(\mathbb{R}^3)$ be a non-negative, spherically symmetric, compactly supported potential. Let $\varphi \in H^1(\mathbb{R})$ with $\|\varphi\|_2 = 1$.*

i) Well-Posedness. For any $\varphi \in H^1(\mathbb{R}^3)$, with $\|\varphi\|_2 = 1$, there exist unique global solutions $t \to \varphi_t$ and $t \to \widetilde{\varphi_{\xi}}_t$ in $C(\mathbb{R}, H^1(\mathbb{R}^3))$ of the Gross-Pitaevskii equation (3.12) and, respectively, of the modified Gross-Pitaevskii equation (3.68) with initial datum φ. We have $\|\varphi_t\|_2 = \|\widetilde{\varphi_{\xi}}_t\|_2 = 1$ for all $t \in \mathbb{R}$. Furthermore, there exists a constant $C > 0$ such that

$$\|\varphi_t\|_{H^1}, \|\widetilde{\varphi_{\xi}}_t\|_{H^1} \leq C$$

ii) Propagation of higher regularity. If $\varphi \in H^m(\mathbb{R})$ for some $m \geq 2$, then $\varphi_t, \widetilde{\varphi_{\xi}}_t \in H^m(\mathbb{R})$ for every $t \in \mathbb{R}$. Moreover, there exist constants $C > 0$ depending on m and on $\|\varphi\|_{H^m}$, and $c > 0$, depending on $\|\varphi\|_{H^1}$ and m, such that, for all $t \in \mathbb{R}$,

$$\|\varphi_t\|_{H^m}, \|\widetilde{\varphi_{\xi}}\|_{H^m} \leq Ce^{c|t|}. \tag{3.69}$$

iii) Regularity of time derivatives. Suppose $\varphi \in H^4(\mathbb{R})$. Then there exist $C > 0$, depending on $\|\varphi\|_{H^4}$, and $c > 0$, depending on $\|\varphi\|_{H^1}$, such that, for all $t \in \mathbb{R}$,

$$\|\dot{\widetilde{\varphi_{\xi}}}\|_{H^2}, \|\ddot{\widetilde{\varphi_{\xi}}}\|_{H^2} \leq Ce^{c|t|}.$$

iv) Comparison of Dynamics. Suppose $\varphi \in H^2(\mathbb{R})$. Then there exists a constant $c > 0$, depending on $\|\varphi\|_{H^2}$, such that for all $t \in \mathbb{R}$,

$$\|\varphi_t - \widetilde{\varphi_{\xi}}_t\|_2 \leq CN^{-1} \exp(c \exp(c|t|)). \tag{3.70}$$

To compare the many-body evolution $\psi_{N,t}$ with products of the solution $\widetilde{\varphi_{\xi}}_t$ of the modified Gross-Pitaevskii equation (3.12), we are going to use a unitary map (already discussed in the introduction, after (3.23)) that was first introduced in [64] to analyze the excitation spectrum of mean field bosonic systems and then in [63] to study fluctuations around Hartree dynamics, again in the mean field regime. To define this map, we remark that every $\psi_N \in L^2_s(\mathbb{R}^{3N})$ has a unique representation of the form

$$\psi_N = \sum_{n=0}^{N} \psi_N^{(n)} \otimes_s \widetilde{\varphi_{\xi}}_t^{\otimes(N-n)} \tag{3.71}$$

where $\psi_N^{(n)} \in L^2_{\perp\widetilde{\varphi}_{\xi_t}}(\mathbb{R}^3)^{\otimes_s n}$ is symmetric with respect to permutations and orthogonal to $\widetilde{\varphi}_{\xi_t}$, in each of its coordinate, and where, for $\psi_N^{(n)} \in L^2_\perp(\mathbb{R}^3)^{\otimes_s n}$ and $\psi_N^{(k)} \in L^2_\perp(\mathbb{R}^3)^{\otimes_s k}$, we denote by $\psi_N^{(n)} \otimes_s \psi_N^{(k)}$ the symmetrized product defined by

$$\begin{aligned}\psi_N^{(k)} \otimes_s \psi_N^{(n)} &(x_1, \dots, x_{k+n}) \\ &= \frac{1}{\sqrt{k!n!(k+n)!}} \sum_{\sigma \in S_{k+n}} \psi_N^{(k)}(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \psi_N^{(n)}(x_{\sigma(k+1)}, \dots, x_{\sigma(k+n)}).\end{aligned} \tag{3.72}$$

Using the representation (3.71), we define the map $U_{N,t} : L^2_s(\mathbb{R}^{3N}) \to \mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$ by setting

$$U_{N,t}\psi_N = \{\psi_N^{(0)}, \psi_N^{(1)}, \dots, \psi_N^{(N)}, 0, 0, \dots\}. \tag{3.73}$$

In terms of creation and annihilation operators, the map $U_{N,t}$ is given by

$$U_{N,t}\psi_N = \bigoplus_{n=0}^{N} (1 - |\widetilde{\varphi}_{\xi_t}\rangle\langle\widetilde{\varphi}_{\xi_t}|)^{\otimes n} \frac{a(\widetilde{\varphi}_{\xi_t})^{N-n}}{\sqrt{(N-n)!}} \psi_N \, .$$

Here, and frequently in the sequel, we identify $\psi_N \in L^2_s(\mathbb{R}^{3N})$ with the Fock space vector $\{0, \dots, 0, \psi_N, 0, \dots\} \in \mathcal{F}$. $U_{N,t}$ is clearly an isometry. A simple computation shows that $U_{N,t}$ (defined as in (3.73), as a map from $L^2_s(\mathbb{R}^{3N})$ into $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_\xi}$) is a unitary operator, with inverse given by

$$U^*_{N,t}\{\psi_N^{(0}, \psi_N^{(1)}, \dots, \psi_N^{(N)}, 0, \dots\} = \sum_{n=0}^{N} \frac{a^*(\widetilde{\varphi}_{\xi_t})^{N-n}}{\sqrt{(N-n)!}} \psi_N^{(n)}$$

The action of $U_{N,t}$ on creation and annihilation operators is determined by the following rules (see [64, 63]):

$$\begin{aligned} U_{N,t} a^*(\widetilde{\varphi}_{\xi_t}) a(\widetilde{\varphi}_{\xi_t}) U^*_{N,t} &= N - \mathcal{N} \\ U_{N,t} a^*(f) a(\widetilde{\varphi}_{\xi_t}) U^*_{N,t} &= a^*(f)\sqrt{N - \mathcal{N}} \\ U_{N,t} a^*(\widetilde{\varphi}_{\xi_t}) a(g) U^*_{N,t} &= \sqrt{N - \mathcal{N}} a(g) \\ U_{N,t} a^*(f) a(g) U^*_{N,t} &= a^*(f) a(g) \end{aligned} \tag{3.74}$$

for all $f, g \in L^2_{\perp\widetilde{\varphi}_{\xi_t}}(\mathbb{R}^3)$.

The unitary map $U_{N,t}$ acts on $\psi_N \in L^2_s(\mathbb{R}^{3N})$ by factoring out the condensate, consisting of all particles in the state $\widetilde{\varphi}_{\xi_t}$, and mapping ψ_N to the excitations orthogonal to the condensate. However, measuring excitations with respect to the condensate wave function is not a very good idea in the Gross-Pitaevskii regime considered in this paper. The many-body dynamics is known to develop correlations (see, for example, [40, 34]); $\psi_{N,t}$ is quite far from being factorized. To take into account correlations, we are going to use a generalized Bogoliubov transformation, as introduced in Section 3.2. We define

$$k_t(x; y) = -N w_\ell(N(x-y)) \widetilde{\varphi}_{\xi_t}(x) \widetilde{\varphi}_{\xi_t}(y) \tag{3.75}$$

From Lemma 4.3.1, it follows that $k_t \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, with L^2-norm bounded uniformly in N. Hence, k_t is the integral kernel of a Hilbert-Schmidt operator on $L^2(\mathbb{R}^3)$, which we denote again with k_t. We define a new Hilbert-Schmidt operator setting

$$\eta_t = (1 - |\widetilde{\varphi_{\xi_t}}\rangle\langle\widetilde{\varphi_{\xi_t}}|)\, k_t\, (1 - |\bar{\widetilde{\varphi_{\xi_t}}}\rangle\langle\bar{\widetilde{\varphi_{\xi_t}}}|) \tag{3.76}$$

Also in this case, we will denote by η_t both the Hilbert-Schmidt operator defined in (3.76) and its integral kernel. Note that $\eta_t \in (q_{\widetilde{\varphi_{\xi_t}}} \otimes q_{\widetilde{\varphi_{\xi_t}}})L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, where $q_{\widetilde{\varphi_{\xi_t}}} = 1 - |\widetilde{\varphi_{\xi_t}}\rangle\langle\widetilde{\varphi_{\xi_t}}|$. Let us write $\eta_t = k_t + \mu_t$, with the Hilbert-Schmidt operator

$$\mu_t = |\widetilde{\varphi_{\xi_t}}\rangle\langle\widetilde{\varphi_{\xi_t}}|\, k_t\, |\bar{\widetilde{\varphi_{\xi_t}}}\rangle\langle\bar{\widetilde{\varphi_{\xi_t}}}| - |\widetilde{\varphi_{\xi_t}}\rangle\langle\widetilde{\varphi_{\xi_t}}|\, k_t - k_t|\bar{\widetilde{\varphi_{\xi_t}}}\rangle\langle\bar{\widetilde{\varphi_{\xi_t}}}| \tag{3.77}$$

In the next lemma we collect some important properties of the operators η_t, k_t, μ_t. The proof is a simple generalization of the proof of Lemma 3.3 and Lemma 3.4 in [11]; we omit the details.

Lemma 3.3.3. *Let $\widetilde{\varphi_{\xi_t}}$ be the solution of* (3.68) *with initial datum $\varphi \in H^4(\mathbb{R})$. Let $w_\ell = 1 - f_\ell$ with f_ℓ the ground state solution of the Neumann problem* (3.61). *Let k_t, η_t, μ_t be defined as in (3.75), (3.76), (3.77). Then there exist constants $C, c > 0$ depending only on $\|\varphi\|_{H^4}$ (in many cases, these constants actually depend only on lower Sobolev norms of φ) and on V such that the following bounds hold true, for all $t \in \mathbb{R}$.*

i) We have

$$\|\eta_t\|_2 \le C, \quad \|\eta_t^{(n)}\|_2 \le \|\eta_t\|_2^n \le C^n \quad \text{and} \quad \lim_{\ell\to 0} \sup_{t\in\mathbb{R}} \|\eta_t\|_2 = 0 \tag{3.78}$$

and also

$$\|\nabla_j \eta_t\|_2 \le C\sqrt{N}, \ \|\nabla_j \mu_t\|_2 \le C, \ \|\nabla_j \eta_t^{(n)}\|_2 \le C\|\eta_t\|_2^{n-2}, \ \|\Delta_j \eta_t^{(n)}\|_2 \le C\|\eta_t\|_2^{n-2}$$

for $j = 1, 2$ and for all $n \ge 2$. Here $\nabla_1\eta_t$ and $\nabla_2\eta_t$ denote the kernels $\nabla_x\eta_t(x;y)$ and $\nabla_y\eta_t(x;y)$ ($\Delta_1\eta_t$ and $\Delta_2\eta_t$ are defined similarly). Decomposing $\cosh_{\eta_t} = 1 + p_{\eta_t}$ and $\sinh_{\eta_t} = \eta_t + r_{\eta_t}$, we obtain

$$\|sinh_{\eta_t}\|_2, \|p_{\eta_t}\|_2, \|r_{\eta_t}\|_2, \|\nabla_j p_{\eta_t}\|_2, \|\nabla_j r_{\eta_t}\|_2 \le C \tag{3.79}$$

ii) For a.e. $x, y \in \mathbb{R}^3$ and $n \in \mathbb{N}$, $n \ge 2$, we have the pointwise bounds

$$\begin{aligned} |\eta_t(x;y)| &\le \frac{C}{|x-y| + N^{-1}} |\widetilde{\varphi_{\xi_t}}(x)||\widetilde{\varphi_{\xi_t}}(y)| \\ |\eta_t^{(n)}(x;y)| &\le C\|\eta_t\|_2^{n-2} |\widetilde{\varphi_{\xi_t}}(x)||\widetilde{\varphi_{\xi_t}}(y)| \\ |\mu_t(x;y)|, |p_{\eta_t}(x;y)|, |r_{\eta_t}(x;y)| &\le C|\widetilde{\varphi_{\xi_t}}(x)||\widetilde{\varphi_{\xi_t}}(y)| \end{aligned} \tag{3.80}$$

iii) We have

$$\sup_x \int |\eta_t(x;y)|^2 dy, \ \sup_x \int |k_t(x;y)|^2 dy, \ \sup_x \int |\mu_t(x;y)|^2 dy \le C\|\widetilde{\varphi_{\xi_t}}\|_{H^2} \le Ce^{c|t|}$$

and

$$\sup_x \int |\eta_t^{(n)}(x;y)|^2 dy \le C\|\eta_t\|_2^{n-2}\|\widetilde{\varphi}_{\xi_t}\|_{H^2} \le C\|\eta_t\|_2^{n-2}e^{c|t|}$$

for all $n \ge 2$. Therefore

$$\sup_x \int |p_{\eta_t}(x;y)|^2 dy, \sup_x \int |r_{\eta_t}(x;y)|^2 dy, \sup_x \int |sinh_{\eta_t}(x;y)|^2 dy \le Ce^{c|t|}$$

iv) For $j = 1, 2$ and $n \ge 2$, we have

$$\|\partial_t \eta_t\|_2, \|\partial_t^2 \eta_t\|_2 \le Ce^{c|t|}, \quad \|\partial_t \eta_t^{(n)}\|_2 \le Cne^{c|t|}\|\eta_t\|_2^{n-1}$$

and also

$$\|\partial_t \nabla_j \eta_t\|_2 \le C\sqrt{N}e^{c|t|}, \quad \|\partial_t \nabla_j \mu_t\|_2 \le Ce^{c|t|}, \quad \|\partial_t \nabla_j \eta_t^{(n)}\|_2 \le Cn\|\eta_t\|^{n-2}e^{c|t|}$$

Therefore

$$\|\partial_t p_{\eta_t}\|_2, \|\partial_t r_{\eta_t}\|_2, \|\partial_t sinh_{\eta_t}\|_2, \|\nabla_j \partial_t p_{\eta_t}\|_2, \|\nabla_j \partial_t r_{\eta_t}\|_2 \le Ce^{c|t|}$$

v) For a.e. $x \in \mathbb{R}^3$, we have the pointwise bounds

$$\begin{aligned}|\partial_t \eta_t(x;y)| \le C &\left[1 + \frac{1}{|x-y| + N^{-1}}\right] \\ &\times \left[|\dot{\widetilde{\varphi}}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)| + |\widetilde{\varphi}_{\xi_t}(x)||\dot{\widetilde{\varphi}}_{\xi_t}(y)| + |\widetilde{\varphi}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)|\right]\end{aligned}$$

Moreover, for $n \ge 2$, we have

$$|\partial_t \eta_t^{(n)}(x;y)| \le Cne^{c|t|}\|\eta_t\|_2^{n-2}\left[|\dot{\widetilde{\varphi}}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)| + |\widetilde{\varphi}_{\xi_t}(x)||\dot{\widetilde{\varphi}}_{\xi_t}(y)| + |\widetilde{\varphi}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)|\right]$$

Therefore

$$\begin{aligned}&|\partial_t \mu_t(x;y)|, |\partial_t r_{\eta_t}(x;y)|, |\partial_t p_{\eta_t}(x;y)| \\ &\qquad \le Ce^{c|t|}\left[|\dot{\widetilde{\varphi}}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)| + |\widetilde{\varphi}_{\xi_t}(x)||\dot{\widetilde{\varphi}}_{\xi_t}(y)| + |\widetilde{\varphi}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)|\right]\end{aligned}$$

vi) Finally, we find

$$\sup_x \int |\partial_t \eta_t(x;y)|^2 dy, \ \sup_x \int |\partial_t k_t(x;y)|^2 dy, \ \sup_x \int |\partial \mu_t(x;y)|^2 dy \le Ce^{c|t|}$$

Furthermore, for all $n \ge 2$,

$$\sup_x \int |\partial_t \eta_t^{(n)}(x;y)| dy \le Cne^{c|t|}\|\eta_t\|_2^{n-2}$$

and therefore

$$\sup_x \int |\partial_t p_{\eta_t}(x;y)|^2 dy, \ \sup_x \int |\partial_t r_{\eta_t}(x;y)|^2 dy, \ \sup_x \int |\partial_t \mathrm{sinh}_{\eta_t}(x;y)|^2 dy \le Ce^{c|t|}$$

We model correlations in the solution $\psi_{N,t}$ of the many-body Schrödinger equation (3.8) by means of the generalized Bogoliubov transformation $\exp(B(\eta_t)) : \mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}} \to \mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$ with the integral kernel $\eta_t \in (q_{\widetilde{\varphi}_{\xi_t}} \otimes q_{\widetilde{\varphi}_{\xi_t}})L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ defined in (3.76). We define therefore the fluctuation dynamics

$$\mathcal{W}_{N,t} = e^{-B(\eta_t)}\, U_{N,t}\, e^{-iH_N t}\, U^*_{N,0}\, e^{B(\eta_t \mathfrak{a}_0)} \tag{3.81}$$

Then $\mathcal{W}_{N,t} : \mathcal{F}^{\leq N}_{\perp\varphi} \to \mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$ is a unitary operator. Clearly, $\mathcal{W}_{N,t}$ depends on the length parameter ℓ (the radius of the ball in (3.61)), through the modified Gross-Pitaevskii equation (3.68) and also through the kernel η_t defined in (3.75), (3.76). While $\mathcal{W}_{N,t}$ is well-defined for any value of $\ell > 0$, we will have to choose $\ell > 0$ small, to make sure that $\|\eta_t\|_2$ is sufficiently small; this will allow us to expand the action of the generalized Bogoliubov transformation $\exp(B(\eta_t))$ appearing in (3.81) using the series expansion (3.56) (because, by (3.78), smallness of ℓ implies that $\|\eta_t\|_2$ is small, uniformly in t).

For $\xi \in \mathcal{F}^{\leq N}_{\perp\varphi}$, the operator $\mathcal{W}_{N,t}$ is defined so that

$$e^{-iH_N t}\, U^*_{N,0}\, e^{B(\eta_t \mathfrak{a}_0)}\xi = U^*_{N,t}\, e^{B(\eta_t)}\, \mathcal{W}_{N,t}\xi\,.$$

It allows us to describe the many-body evolution of initial data of the form

$$\psi_N = U^*_{N,0} e^{B(\eta_t \mathfrak{a}_0)}\xi, \tag{3.82}$$

and to express the evolved state again in the form

$$\psi_{N,t} = e^{-iH_N t}\psi_N = U^*_{N,t} e^{B(\eta_t)}\xi_t, \tag{3.83}$$

where we defined $\xi_t = \mathcal{W}_{N,t}\,\xi$. As we will see below, a vector of the form (3.82) exhibits Bose-Einstein condensation in the one-particle state φ if and only if the expectation of the number of particles operator $\langle \xi, \mathcal{N}\xi\rangle$ is small, compared with the total number of particles N. Hence, to prove Theorems 3.1.1 and 3.1.2, we will have to show first that every initial $\psi_N \in L^2_s(\mathbb{R}^{3N})$ satisfying (3.10) can be written in the form (3.82) for a $\xi \in \mathcal{F}^{\leq N}_{\perp\varphi}$ with $\langle \xi, \mathcal{N}\xi\rangle \ll N$ and then, secondly, that the bound on the expectation of the number of particles is approximately preserved by $\mathcal{W}_{N,t}$. In fact, it turns out that to control the growth of the expectation of $\mathcal{N}$ along the fluctuation dynamics, it is not enough to have a bound on $\langle \xi, \mathcal{N}\xi\rangle$; instead, we will also need a bound on the energy of ξ (this is why we need to assume $b_N \to 0$, in (3.10)).

To control the growth of the number of particles with respect to the fluctuation dynamics it is important to compute the generator of $\mathcal{W}_{N,t}$. A simple computation shows that

$$i\partial_t \mathcal{W}_{N,t} = \mathcal{G}_{N,t}\mathcal{W}_{N,t}$$

with the time-dependent generator

$$\mathcal{G}_{N,t} = (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} + e^{-B(\eta_t)}\left[(i\partial_t U_{N,t})U^*_{N,t} + U_{N,t} H_N U^*_{N,t}\right] e^{B(\eta_t)} \tag{3.84}$$

Notice, that $\mathcal{G}_{N,t}$ maps $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$ into $\mathcal{F}^{\leq N}$, but not into $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$. This is due to the fact that the space $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$ depends on time (and thus $\mathcal{G}_{N,t}$ must have a component which allows $\mathcal{W}_{N,t}$ to move to different spaces). We will mostly be interested in the expectation of $\mathcal{G}_{N,t}$ for states in $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$, but at some point (when we will be interested in the variation of the expectation of $\mathcal{G}_{N,t}$) it will be important to remember the component of $\mathcal{G}_{N,t}$ mapping out of $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$.

In the next proposition, we collect important properties of the generator $\mathcal{G}_{N,t}$.

Theorem 3.3.4. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, spherically symmetric and compactly supported. Let $\mathcal{W}_{N,t}$ be defined as in (3.81) with the length parameter $\ell > 0$ sufficiently small and using the solution of the modified Gross-Pitaevskii equation (3.68), with an initial data $\varphi \in H^4(\mathbb{R}^3)$. Let*

$$\begin{aligned} C_{N,t} &= \frac{1}{2}\left\langle \widetilde{\varphi}_{\xi_t}, \left([N^3V(N.)(N-1-2Nf_\ell(N.))] * |\widetilde{\varphi}_{\xi_t}|^2\right)\widetilde{\varphi}_{\xi_t}\right\rangle \\ &\quad + \int dxdy\, |\nabla_x k_t(x;y)|^2 + \frac{1}{2}\int dxdy\, N^2V(N(x-y))|k_t(x;y)|^2 \\ &\quad + \mathrm{Re}\int dxdy\, N^3V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)k_t(x;y). \end{aligned} \tag{3.85}$$

Then there exist constants $C, c > 0$ such that, in the sense of quadratic forms on $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$,

$$\begin{aligned} \frac{1}{2}\mathcal{H}_N - Ce^{c|t|}(\mathcal{N}+1) \leq (\mathcal{G}_{N,t} - C_{N,t}) &\leq 2\mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1) \\ \pm i\,[\mathcal{N}, \mathcal{G}_{N,t}] &\leq \mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1), \\ \pm\partial_t(\mathcal{G}_{N,t} - C_{N,t}) &\leq \mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1), \\ \pm\,\mathrm{Re}[a^*(\partial_t\widetilde{\varphi}_{\xi_t})a(\widetilde{\varphi}_{\xi_t}), \mathcal{G}_{N,t}] &\leq \mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1). \end{aligned} \tag{3.86}$$

where $\mathcal{H}_N$ is the Fock space Hamiltonian

$$\mathcal{H}_N = \int dx\, \nabla_x a_x^* \nabla_x a_x + \frac{1}{2}\int dxdy\, N^2V(N(x-y))a_x^* a_y^* a_y a_x \tag{3.87}$$

Note that, on $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$, we have $[a^*(\partial_t\widetilde{\varphi}_{\xi_t})a(\widetilde{\varphi}_{\xi_t}), \mathcal{G}_{N,t}] = a^*(\partial_t\widetilde{\varphi}_{\xi_t})a(\widetilde{\varphi}_{\xi_t})\mathcal{G}_{N,t}$.

The proof of Theorem 3.3.4 is given in the next section. From the technical point of view, it represents the main part of our paper. In Section 3.5, we show then how to use the properties of $\mathcal{G}_{N,t}$ established in Theorem 3.3.4 to complete the proof of Theorems 3.1.1 and 3.1.2.

3.4 Analysis of the Generator of Fluctuation Dynamics

In this section we study the properties of the generator

$$\mathcal{G}_{N,t} = (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} + e^{-B(\eta_t)}\left[(i\partial_t U_{N,t})U_{N,t}^* + U_{N,t}H_N U_{N,t}^*\right]e^{B(\eta_t)} \tag{3.88}$$

of the fluctuation dynamics (3.81); the goal is to prove Theorem 3.3.4.

As forms on $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}} \times \mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$, we find (see Lemma 6 in [63])

$$(i\partial_t U_{N,t})U^*_{N,t} = -\langle i\partial_t\widetilde{\varphi}_{\xi_t}, \widetilde{\varphi}_{\xi_t}\rangle(N-\mathcal{N}) - \sqrt{N}\left[b(i\partial_t\widetilde{\varphi}_{\xi_t}) + b^*(i\partial_t\widetilde{\varphi}_{\xi_t})\right] \tag{3.89}$$

Using (3.74) to compute $U_{N,t}H_N U^*_{N,t}$ a lengthy but straightforward computation (see Appendix B of [63]) shows then that

$$(i\partial_t U_{N,t})U^*_{N,t} + U_{N,t}H_N U^*_{N,t} = \sum_{j=0}^{4} \mathcal{L}^{(j)}_{N,t}$$

where

$$\begin{aligned}
\mathcal{L}^{(0)}_{N,t} =& \frac{1}{2}\left\langle \widetilde{\varphi}_{\xi_t}, [N^3V(N.)(1-2f_\ell(N.)) * |\widetilde{\varphi}_{\xi_t}|^2]\widetilde{\varphi}_{\xi_t}\right\rangle (N-\mathcal{N}) \\
& - \frac{1}{2}\left\langle \widetilde{\varphi}_{\xi_t}, [N^3V(N.) * |\widetilde{\varphi}_{\xi_t}|^2]\widetilde{\varphi}_{\xi_t}\right\rangle (\mathcal{N}+1)\frac{(N-\mathcal{N})}{N} \\
\mathcal{L}^{(1)}_{N,t} =& \sqrt{N}\, b([N^3V(N.)w_\ell(N.) * |\widetilde{\varphi}_{\xi_t}|^2]\widetilde{\varphi}_{\xi_t}) - \frac{\mathcal{N}+1}{\sqrt{N}} b([N^3V(N.) * |\widetilde{\varphi}_{\xi_t}|^2]\widetilde{\varphi}_{\xi_t}) + h.c. \\
\mathcal{L}^{(2)}_{N,t} =& \int dx\, \nabla_x a^*_x \nabla_x a_x \\
& + \int dxdy\, N^3V(N(x-y))|\widetilde{\varphi}_{\xi_t}(y)|^2 \left(b^*_x b_x - \frac{1}{N}a^*_x a_x\right) \\
& + \int dxdy\, N^3V(N(x-y))\widetilde{\varphi}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y) \left(b^*_x b_y - \frac{1}{N}a^*_x a_y\right) \\
& + \frac{1}{2}\left[\int dxdy\, N^3V(N(x-y))\widetilde{\varphi}_{\xi_t}(x)\widetilde{\varphi}_{\xi_t}(y) b^*_x b^*_y + h.c.\right] \\
\mathcal{L}^{(3)}_{N,t} =& \int dxdy\, N^{5/2}V(N(x-y))\widetilde{\varphi}_{\xi_t}(y) b^*_x a^*_y a_x + \text{h.c.} \\
\mathcal{L}^{(4)}_{N,t} =& \frac{1}{2}\int dxdy\, N^2V(N(x-y))a^*_x a^*_y a_y a_x
\end{aligned} \tag{3.90}$$

The generator (3.88) of the fluctuation dynamics is therefore given by

$$\mathcal{G}_{N,t} = (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} + \sum_{j=0}^{4} e^{-B(\eta_t)}\mathcal{L}^{(j)}_{N,t} e^{B(\eta_t)}$$

In the next subsections, we will study separately the six terms contributing to $\mathcal{G}_{N,t}$. Before doing so, however, we collect some preliminary results, which will be useful for our analysis.

Notation and Conventions. For the rest of this section we use the short-hand notation η_x k_x, μ_x for the wave functions $\eta_x(y) = \eta_t(x;y)$, $k_x(y) = k_t(x;y)$ and $\mu_x(y) = \mu_t(x;y)$.

We will always assume that $\sup_{t\in\mathbb{R}} \|\eta_t\|_2$ is sufficiently small, so that we can use the expansions obtained in Lemma 3.2.4. Finally, by C and c we denote generic constants which only depend on fixed parameters, but not on N or t, and which may vary from one line to the next.

3.4.1 Preliminary results

In this subsection we show some simple but important auxiliary results which will be used throughout the rest of Section 3.4. Recall the operators

$$\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n) = \int b^{\flat_0}_{x_1} \prod_{i=1}^{n-1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} b^{\sharp_n}_{y_n} \prod_{i=1}^{n} j_i(x_i;y_i)dx_i dy_i$$
$$\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f) = \int b^{\flat_0}_{x_1} \prod_{i=1}^{n-1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} a^{\sharp_n}_{y_n} a^{\flat_n}(f) \prod_{i=1}^{n} j_i(x_i;y_i)dx_i dy_i$$

introduced in Section 3.2. For each $i \in \{1,\dots,n\}$, we recall in particular the condition that either $\sharp_i = *$ and $\flat_i = \cdot$ or $\sharp_i = \cdot$ and $\flat_i = *$.

In the next lemma, we consider commutators of these operators with the number of particles operator $\mathcal{N}$ and with operators of the form $a^*(g_1)a(g_2)$.

Lemma 3.4.1. *Let $n \in \mathbb{N}$, $f, g_1, g_2 \in L^2(\mathbb{R}^3)$, $j_1,\dots,j_n \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$.*

i) We have

$$\left[\mathcal{N}, \Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)\right] = \kappa_{\flat_0,\sharp_n} \Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n) \qquad \text{for all} \quad \sharp,\flat \in \{\cdot,*\}^n$$
$$\left[\mathcal{N}, \Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)\right] = \nu_{\flat_0} \Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f) \quad \text{for all} \quad \sharp \in \{\cdot,*\}^n, \flat \in \{\cdot,*\}^{n+1}.$$

Here $\kappa_{\flat_0,\sharp_n} = 2$, if $\flat_0 = \sharp_n = *$, $\kappa_{\flat_0,\sharp_n} = -2$ if $\flat_0 = \sharp_n = \cdot$, and $\kappa_{\flat_0,\sharp_n} = 0$ otherwise, while $\nu_{\flat_0} = 1$ if $\flat_0 = *$ and $\nu_{\flat_0} = -1$ if $\flat_0 = \cdot$.

ii) The commutator

$$\left[a^*(g_1)a(g_2), \Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)\right]$$

can be written as the sum of $2n$ terms, all having the form

$$\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_{i-1},h_i,j_{i+1},\dots,j_n)$$

for some $i \in \{1,\dots,n\}$. Here $h_i \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ has (up to a possible sign) one of the following forms:

$$h_i(x;y) = g_1(x) j_i(\bar{g}_2)(y), \qquad h_i(x;y) = g_1(y) j_i(\bar{g}_2)(y) \tag{3.91}$$

or the same, but with g_1 and $\bar{g}_2$ exchanged. Here $j_i(g)(x) = \int j_i(x;z)g(z)dz$. Notice that

$$\|h_i\|_2 \le \|g_1\|_2 \|g_2\|_2 \|j_i\|_2 \tag{3.92}$$

and

$$|h_i(x;y)| \leq \max \Big\{ |g_1(x)| \|j_i(.;y)\|_2 \|g_2\|_2, |g_1(y)| \|j_i(x;.)\|_2 \|g_2\|_2, \\ |g_2(x)| \|j_i(.;y)\|_2 \|g_1\|_2, |g_2(y)| \|j_i(x;.)\|_2 \|g_1\|_2 \Big\} \tag{3.93}$$

iii) The commutator

$$\left[a^*(g_1) a(g_2), \Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_n; f) \right] \tag{3.94}$$

can be written as the sum of $2n+1$ terms. $2n$ of them have the form

$$\Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_{i-1}, h_i, j_{i+1}, \dots, j_n; f)$$

where h_i is (up to a possible sign) one of the kernels appearing in (3.91) (or the same with g_1 and $\bar{g}_2$ exchanged), and satisfying the bounds in (3.92), (3.93). The remaining term in the expansion for (3.94) has the form

$$\Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_n; k) \tag{3.95}$$

where $k \in L^2(\mathbb{R}^3)$ is (up to a possible sign) one of the functions

$$k(x) = \langle g_1, f \rangle\, g_2(x), \quad k(x) = \langle g_2, f \rangle\, g_1(x) \tag{3.96}$$

or one of their complex conjugated functions. In any event, we have

$$\|k\|_2 \leq \|g_1\|_2 \|g_2\|_2 \|f\|_2$$

and

$$|k(x)| \leq \|f\|_2 \max\{ \|g_1\|_2 |g_2(x)|, \|g_2\|_2 |g_1(x)| \}$$

iv) If $f \in L^2(\mathbb{R}^3)$ and/or $j_1, \dots, j_n \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ depend on time $t \in \mathbb{R}$, we have

$$\begin{aligned} \partial_t \Pi^{(2)}_{\sharp,\flat}(j_1, \dots, j_n) &= \sum_{i=1}^n \Pi^{(2)}_{\sharp,\flat}(j_1, \dots, j_{i-1}, \partial_t j_i, j_{i+1}, \dots, j_n) \\ \partial_t \Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_n; f) &= \Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_n; \partial_t f) \\ &\quad + \sum_{i=1}^n \Pi^{(1)}_{\sharp,\flat}(j_1, \dots, j_{i-1}, \partial_t j_i, j_{i+1}, \dots, j_n; f). \end{aligned}$$

Proof. Part (i) follows from $(\mathcal{N}+1) b_x = b_x \mathcal{N}$ and $\mathcal{N} b_x^* = b_x^* (\mathcal{N}+1)$. Part (iv) follows easily from the Leibniz rule. To prove part (ii), we apply Leibniz rule:

$$\begin{aligned}
\Big[a^*(g_1)a(g_2),&\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)\Big] \\
&= \int [a^*(g_1)a(g_2), b^{\flat_0}_{x_1}] \prod_{i=1}^n a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} b^{\sharp_n}_{y_n} \prod_{i=1}^n j_i(x_i;y_i)dx_i dy_i \\
&\quad + \sum_{m=1}^{n-1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{m-1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} \left[a^*(g_1)a(g_2), a^{\sharp_m}_{y_m} a^{\flat_m}_{x_{m+1}}\right] \\
&\qquad\qquad \times \prod_{i=m+1}^{n-1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} b^{\sharp_n}_{y_n} \prod_{i=1}^n j_i(x_i;y_i)dx_i dy_i \\
&\quad + \int b^{\flat_0}_{x_1} \prod_{i=1}^n a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} [a^*(g_1)a(g_2), b^{\sharp_n}_{y_n}] \prod_{i=1}^n j_i(x_i;y_i)dx_i dy_i
\end{aligned} \tag{3.97}$$

Using the commutation relations

$$\begin{aligned}
[a^*(g_1)a(g_2), b_x] &= -g_1(x)b(g_2), \\
[a^*(g_1)a(g_2), b^*_x] &= \bar{g}_2(x)b^*(g_1) \\
[a^*(g_1)a(g_2), a^*_x a_y] &= [a^*(g_1)a(g_2), a_y a^*_x] = \bar{g}_2(x)a^*(g_1)a_y - g_1(y)a^*_x a(g_2)
\end{aligned} \tag{3.98}$$

we conclude that on the r.h.s. of (3.97) we have $2n$ terms, each of them being a $\Pi^{(2)}$-operator (with the same indices $\sharp,\flat$ as the $\Pi^{(2)}$ operator on the l.h.s. of (3.97)). Furthermore, from (3.98) it is clear that for each $\Pi^{(2)}$ operator on the r.h.s. of (3.97), only one j-kernel will differ from the j-kernels of the $\Pi^{(2)}$ operator on the l.h.s. of (3.97). In the first term on the r.h.s. of (3.97), we only have to replace the j_1 kernel (either with $g_1(x_1)j_1(\bar{g}_2)(y_1)$ or with $\bar{g}_2(x_1)j_1(g_1)(y_1)$, depending on $\flat_0 \in \{\cdot,*\}$). Similarly, in the last term on the r.h.s. of (3.97), only the j_n kernel has to be changed. In the m-th term in the sum, on the other hand, the commutator leads to the sum of two $\Pi^{(2)}$-operators, one where the kernel j_m is changed and one where the kernel j_{m+1} is replaced. From (3.98), it is easy to check that the new kernel can only have one of the forms listed in (3.91). The bounds (3.92), (3.93) follow easily from the explicit formula in (3.91). Part (iii) can be shown similarly; the only difference is that, in this case, the commutator can hit the last pair $a^{\sharp_n}_{y_n} a^{\flat_n}(f)$ instead of the $b^{\sharp_n}_{y_n}$ appearing in the $\Pi^{(2)}$-operator. $\square$

It follows from Lemma 3.4.1 that

$$\begin{aligned}
[\mathcal{N}, e^{-B(\eta)}b(f)e^{B(\eta)}] &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} [\mathcal{N}, \mathrm{ad}^{(n)}_{B(\eta)}(b(f))] \\
[a^*(g_1)a(g_2), e^{-B(\eta)}b(f)e^{B(\eta)}] &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} [a^*(g_1)a(g_2), \mathrm{ad}^{(n)}_{B(\eta)}(b(f))] \\
\partial_t(e^{-B(\eta)}b(f)e^{B(\eta)}) &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \partial_t \mathrm{ad}^{(n)}_{B(\eta)}(b(f))
\end{aligned} \tag{3.99}$$

where the series on the r.h.s. are absolutely convergent.

In the next subsections we are going to study what happens to the operators $\mathcal{L}_{N,t}^{(j)}$ defined in (3.90), when they are conjugated with the generalized Bogoliubov transformation $e^{B(\eta_t)}$. The general strategy is to expand $e^{-B(\eta_t)}\mathcal{L}_{N,t}^{(j)}e^{B(\eta_t)}$ using Lemma 3.56, and then use Lemma 3.2.3 to express every nested commutator. Therefore, we will have to bound expectations of operators of the form

$$\Lambda_1 \dots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(s)}(g))$$

or of products of such operators. To this end, the next lemma will be frequently used.

Lemma 3.4.2. *Let $g \in L^2(\mathbb{R}^3)$, $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$ and $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \in \mathbb{N}\backslash\{0\}$. Suppose that, for $s = 1, \dots, i_1$, $s' = 1, \dots, i_2$, Λ_s, $\Lambda'_{s'}$ is either a factor $(N-\mathcal{N})/N$, a factor $(N-\mathcal{N}+1)/N$ or an operator of the form*

$$N^{-p}\, \Pi^{(2)}_{\sharp,\flat}(\eta^{(q_1)}_{t,\natural_1}, \dots, \eta^{(q_p)}_{t,\natural_p}) \tag{3.100}$$

i) Assume that the operator

$$\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(\ell_1)}_{t,\diamondsuit}(g))$$

appears in the expansion of $ad^{(n)}_{B(\eta_t)}(b(g))$ for some $n \in \mathbb{N}$ (as discussed in Lemma 3.2.3). Then

$$\left\| (\mathcal{N}+1)^{-1/2} \Lambda_1 \dots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi \right\| \leq C^n \|\eta_t\|^n \|g\| \|\xi\|$$

If moreover, at least one of the Λ_s operators has the form (3.100) or if $k \geq 1$, we also have

$$\begin{aligned} &\left\| (\mathcal{N}+1)^{-1/2} \Lambda_1 \dots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi \right\| \\ &\qquad\qquad \leq C^n N^{-1/2} \|\eta_t\|^n \|g\| \|(\mathcal{N}+1)^{1/2}\xi\| \end{aligned} \tag{3.101}$$

ii) Let $r : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be a bounded linear operator. We use the notation $(\eta^{(s)}r)_x(y) := (\eta^{(s)}r)(x;y)$ (if $s = 0$, $(\eta^{(s)}r)_x(y) = r_x(y) = r(x;y)$, as a distribution). Assume that the operator

$$\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1)}_{t,\diamondsuit} r)_x)$$

appears in the expansion of $ad^{(n)}_{B(\eta_t)}(b(r_x))$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned} &\left\| \Lambda_1 \dots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; (\eta^{(\ell_1)}_{t,\diamondsuit} r)_x)\xi \right\| \\ &\qquad\qquad \leq \begin{cases} C^n \|\eta_t\|^{n-1} \|(\eta_t r)_x\| \|(\mathcal{N}+1)^{1/2}\xi\| & \text{if } \ell_1 \geq 1 \\ C^n \|\eta_t\|^n \|a(r_x)\xi\| & \text{if } \ell_1 = 0 \end{cases} \end{aligned} \tag{3.102}$$

iii) Suppose that the operators

$$\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1+1)}_{t,\diamondsuit} r)_x),$$
$$\Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})$$

appear in the expansion of $ad^{(n)}_{B(\eta_t)}(b((\eta_t r)_x))$ *and respectively of* $ad^{(k)}_{B(\eta_t)}(b_x)$ *for some* $n, k \in \mathbb{N}$. *Then*

$$\begin{aligned} &\Big\|(\mathcal{N}+1)^{-1/2} \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1+1)}_{t,\diamondsuit} r)_x) \\ &\qquad \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})\xi\Big\| \\ &\qquad \leq \begin{cases} C^{n+k}\|\eta_t\|^{n+k-1}\|(\eta_t r)_x\|\|\eta_x\|\|(\mathcal{N}+1)^{1/2}\xi\| & \textit{if } \ell_2 > 0 \\ C^{n+k}\|\eta_t\|^{n+k}\|(\eta_t r)_x\|\|a_x\xi\| & \textit{if } \ell_2 = 0 \end{cases} \end{aligned} \tag{3.103}$$

Similarly, if the operators

$$\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1)}_{t,\diamondsuit} \partial_t \eta_{t,\widetilde{\diamondsuit}})_x),$$
$$\Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})$$

appear in the expansion of $ad^{(n)}_{B(\eta_t)}(b(\partial_t\eta_t))$ *and respectively of* $ad^{(k)}_{B(\eta_t)}(b_x)$ *for some* $n, k \in \mathbb{N}$, *we have*

$$\begin{aligned} &\Big\|(\mathcal{N}+1)^{-1/2} \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1)}_{t,\diamondsuit} \partial_t \eta_{t,\widetilde{\diamondsuit}})_x) \\ &\qquad \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})\xi\Big\| \\ &\qquad \leq \begin{cases} C^{n+k}\|\eta_t\|^{n+k-1}\|(\partial_t\eta_t)_x\|\|\eta_x\|\|(\mathcal{N}+1)^{1/2}\xi\| & \textit{if } \ell_2 > 0 \\ C^{n+k}\|\eta_t\|^{n+k}\|(\partial_t\eta_t)_x\|\|a_x\xi\| & \textit{if } \ell_2 = 0 \end{cases} \end{aligned} \tag{3.104}$$

iv) Suppose that the operators

$$\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \eta^{(\ell_1)}_{y,\diamondsuit}),$$
$$\Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})$$

appear in the expansion of $ad^{(k)}_{B(\eta_t)}(b_y)$ *and respectively of* $ad^{(n)}_{B(\eta_t)}(b_x)$ *for some* $n, k \in \mathbb{N}$. *For* $\alpha \in \mathbb{N}$, *let*

$$\begin{aligned} D = \Big\|(\mathcal{N}+1)^{(\alpha-1)/2} \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \eta^{(\ell_1)}_{y,\diamondsuit}) \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_1}}; \eta^{(\ell_2)}_{x,\diamondsuit'})\xi\Big\| \end{aligned}$$

Then, if $\ell_1 > 0$, we have, for every $\alpha \in \mathbb{N}$,

$$D \leq \begin{cases} C^{n+k}\|\eta\|^{n+k-2}\|\eta_x\|\|\eta_y\|\|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\| & \text{if } \ell_2 \geq 1 \\ C^{n+k}\|\eta\|^{n+k-1}\|\eta_y\|\|a_x(\mathcal{N}+1)^{\alpha/2}\xi\| & \text{if } \ell_2 = 0 \end{cases} \tag{3.105}$$

If instead $\ell_1 = 0$, we distinguish three cases. For $\ell_2 > 1$, we obtain

$$\begin{aligned} D \leq C^{n+k}\|\eta_t\|^{n+k-2}\Big\{&\|\eta_y\|\|\eta_x\|(\|(\mathcal{N}+1)^{(\alpha-1)/2}\xi\| + n/N\|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\|) \\ &+ \|\eta_t\|\|\eta_x\|\|a_y(\mathcal{N}+1)^{\alpha/2}\xi\|\Big\} \end{aligned} \tag{3.106}$$

If $\ell_1 = 0$ and $\ell_2 = 1$, we find

$$\begin{aligned} D \leq C^{n+k}\|\eta_t\|^{n+k-2}\Big\{&\,[n\|\eta_x\|\|\eta_y\| + \|\eta_t\||\eta_t(x;y)|]\,\|(\mathcal{N}+1)^{(\alpha-1)/2}\xi\| \\ &+ \|\eta_t\|\|\eta_x\|\|a_y(\mathcal{N}+1)^{\alpha/2}\xi\|\Big\} \end{aligned} \tag{3.107}$$

If $\ell_1 = 0$ and $\ell_2 = 1$ and we additionally assume that that $k+n \geq 2$ (since $\ell_1 \leq k$, $\ell_2 \leq n$ from Lemma 3.2.3, this assumption only excludes the case $k = \ell_1 = 0$, $n = \ell_2 = 1$), we find the improved estimate

$$\begin{aligned} D \leq C^{n+k}\|\eta_t\|^{n+k-2}\Big\{&N^{-1}\,[n\|\eta_x\|\|\eta_y\| + \|\eta_t\||\eta_t(x;y)|]\,\|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\| \\ &+ \|\eta_t\|\|\eta_x\|\|a_y(\mathcal{N}+1)^{\alpha/2}\xi\|\Big\} \end{aligned} \tag{3.108}$$

Finally, let $\ell_1 = \ell_2 = 0$. Then

$$\begin{aligned} D \leq C^{n+k}\|\eta_t\|^{n+k-1}\Big\{&nN^{-1}\|\eta_y\|\|a_x(\mathcal{N}+1)^{\alpha/2}\xi\| \\ &+ \|\eta_t\|\|a_xa_y(\mathcal{N}+1)^{(\alpha-1)/2}\xi\|\Big\} \end{aligned} \tag{3.109}$$

If, however, $\ell_1 = \ell_2 = 0$ and, additionally, $k+n \geq 1$ (excluding the case $n = \ell_1 = k = \ell_2 = 0$), we find the improved bound

$$D \leq C^{n+k}\|\eta_t\|^{n+k-1}\left\{nN^{-1}\|\eta_y\|\|a_x\xi\| + N^{-1/2}\|\eta_t\|\|a_xa_y(\mathcal{N}+1)^{\alpha/2}\xi\|\right\} \tag{3.110}$$

Proof. Let us start with part i). If Λ_1 is either the operator $(N-\mathcal{N})/N$ or $(N-\mathcal{N}+1)/N$, then, on $\mathcal{F}^{\leq N}$,

$$\begin{aligned} &\left\|(\mathcal{N}+1)^{-1/2}\Lambda_1\ldots\Lambda_iN^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\right\| \\ &\qquad\leq 2\left\|(\mathcal{N}+1)^{-1/2}\Lambda_2\ldots\Lambda_iN^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\right\| \end{aligned} \tag{3.111}$$

If instead Λ_1 has the form (3.140) for a $p \geq 1$, we apply Lemma 3.2.1 and we find (using part vi) in Lemma 3.2.3)

$$\begin{aligned}\Big\|(\mathcal{N}+1)^{-1/2}\Lambda_1\dots\Lambda_{i_1}N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\Big\| \\ \leq C^p\|\eta_t\|^{\bar{p}}\|(\mathcal{N}+1)^{-1/2}\Lambda_2\dots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\|\end{aligned} \tag{3.112}$$

where we used the notation $\bar{p} = q_1 + \dots + q_p$ for the total number of η_t-kernels appearing in (3.100). Iterating the bounds (3.111) and (3.112), we conclude that

$$\begin{aligned}&\|(\mathcal{N}+1)^{-1/2}\Lambda_1\dots\Lambda_{i_1}N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| \\ &\leq C^{r+p_1+\dots+p_s}\|\eta_t\|^{\bar{p}_1+\dots+\bar{p}_s}\|(\mathcal{N}+1)^{1/2}N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\|\end{aligned} \tag{3.113}$$

if r of the operators $\Lambda_1,\dots,\Lambda_{i_1}$ have either the form $(N-\mathcal{N})/N$ or the form $(N-\mathcal{N}+1)/N$, and the other $s = i_1 - r$ are $\Pi^{(2)}$-operators of the form (3.128) of order $p_1,\dots,p_s$, containing $\bar{p}_1,\dots\bar{p}_s$ η_t-kernels. Again with Lemma 3.2.1, we obtain

$$\begin{aligned}\|(\mathcal{N}+1)^{-1/2}\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| \\ \leq C^{r+p_1+\dots+p_s+j_1+\dots+j_{k_1}+l_1}\|\eta_t\|^{\bar{p}_1+\dots+\bar{p}_s+j_1+\dots+j_{k_1}+l_1}\|g\|\|\xi\| \\ \leq C^n\|\eta_t\|^n\|g\|\|\xi\|\,.\end{aligned} \tag{3.114}$$

This shows the first bound in part i). Now, assume that at least one of the Λ_m operators, for $m \in \{1,\dots,i_1\}$, has the form (3.100). Since, for $\Psi \in \mathcal{F}^{\leq N}$,

$$\begin{aligned}\|(\mathcal{N}+1)^{-1/2}N^{-p}\Pi^{(2)}_{\sharp,\flat}(\eta^{(q_1)}_{t,\natural_1},\dots,\eta^{(q_p)}_{t,\natural_p})\Psi\| \\ \leq C^p\|\eta_t\|^{q_1+\dots+q_p}N^{-p}\|(\mathcal{N}+1)^{p-1/2}\Psi\| \\ \leq C^p\|\eta_t\|^{q_1+\dots+q_p}N^{-1/2}\|\Psi\|\end{aligned}$$

for any $p \geq 1$, in this case we can improve (3.114) to

$$\begin{aligned}\|(\mathcal{N}+1)^{-1/2}\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| \\ \leq C^n N^{-1/2}\|\eta_t\|^n\|g\|\|(\mathcal{N}+1)^{1/2}\xi\|\,.\end{aligned}$$

Similarly, if $k_1 \geq 1$, we have by Lemma 3.2.1,

$$\begin{aligned}N^{-k_1}\Big\|(\mathcal{N}+1)^{-1/2}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_k};\eta^{(\ell_1)}_{t,\natural_{k+1}}(g))\xi\Big\| \\ \leq N^{-k_1}C^{k_1}\|\eta_t\|^{j_1+\dots+j_{k_1}+\ell_1}\|g\|\|(N+1)^{k_1-1/2}\xi\| \\ \leq C^k N^{-1/2}\|\eta_t\|^{j_1+\dots+j_{k_1}+\ell_1}\|g\|\|(N+1)^{1/2}\xi\|\end{aligned}$$

Hence, also in this case, the bound (3.101) holds true.

If $\ell_1 \geq 1$, part ii) can be proven similarly to part i), noticing that

$$\|(\eta^{(\ell_1)}_{t,\diamondsuit} r)_x\| \leq \|\eta_t\|^{\ell_1-1}\|(\eta_t r)_x\|\,.$$

If instead $\ell_1 = 0$, it follows from Lemma 3.2.3, part v), that the field operator associated with $(\eta^{(\ell_1)}_{t,\diamondsuit} r)_x = r_x$ (the one appearing on the right of $\Pi^{(1)}$) is an annihilation operator (acting directly on ξ). Hence, (3.102) holds true also in this case.

Let us now consider part iii). We can bound, first of all

$$\left\|(\mathcal{N}+1)^{-1/2}\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1+1)}_t r)_x)\Psi\right\| \leq C^n\|\eta_t\|^n\|(\eta_t r)_x\|\|\Psi\|$$

and

$$\begin{aligned}\Big\|(\mathcal{N}+1)^{-1/2}\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1)}_{t,\diamondsuit}\partial_t\eta_{t,\widetilde{\diamondsuit}})_x)\Psi\Big\|\\ \leq C^n\|\eta_t\|^n\|(\partial_t\eta_t)_x\|\|\Psi\|\end{aligned}$$

Choosing now

$$\Psi = \Lambda'_1\dots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\diamondsuit'})\xi,$$

and proceeding as in part ii), distinguishing the cases $\ell_2 \geq 1$ and $\ell_2 = 0$, we obtain (3.103) and (3.104).

Finally, we consider part iv). If $\ell_1 > 0$, we can proceed as in part iii) to show (3.105). So, let us focus on the case $\ell_1 = 0$. In this case, the field operator on the right of the first $\Pi^{(1)}$-operator (the one on the left) is an annihilation operator, a_y. To estimate D, we need to commute a_y to the right, until it hits ξ. To commute a_y through factors of $\mathcal{N}$, we just use the pull-through formula $a_y\mathcal{N} = (\mathcal{N}+1)a_y$. When we commute a_y through a pair of creation and/or annihilation operators associated with a kernel $\eta^{(j)}_t$ for a $j \geq 1$ (as the ones appearing in the $\Pi^{(2)}$-operators of the form (3.100) or in the operator $\Pi^{(1)}$-operator), we generate a creation or an annihilation operator with argument $\eta^{(j)}_y$, whose L^2-norm is uniformly bounded. At the same time, we spare a factor N^{-1}. For example, we have

$$\left[a_y, \int a^*_{x_i}a_{y_i}\eta^{(j)}(x_i;y_i)dx_idy_i\right] = a(\bar{\eta}^{(j)}_y)$$

At the end, we have to commute a_y through the field operator with argument $\eta^{(\ell_2)}_{x,\diamondsuit'}$. The commutator is trivial if ℓ_2 is even (because then the corresponding field operator is an annihilation operator; see Lemma 3.2.3, part v)). It is given by

$$[a_y, a^*(\eta^{(\ell_2)}_{x,\diamondsuit'})] = \eta^{(\ell_2)}_{t,\diamondsuit'}(x;y) \tag{3.115}$$

if ℓ_2 is odd. If $\ell_2 \geq 2$, we can bound $|\eta^{(\ell_2)}_{t,\diamondsuit'}(x;y)| \leq \|\eta_t\|^{\ell_2-2}\|\eta_x\|\|\eta_y\|$ and we obtain (taking into account the fact that there are at most n pairs of fields with which a_y has to be commuted)

$$\begin{aligned}\mathrm{D} \leq & C^{k+n}\|\eta_t\|^{k+n-2}\Big\{nN^{-1}\|\eta_y\|\|\eta_x\|\|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\|\\ &+\|\eta_x\|\|\eta_y\|\|(\mathcal{N}+1)^{(\alpha-1)/2}\xi\| + \|\eta_t\|\|\eta_x\|\|a_y(\mathcal{N}+1)^{\alpha/2}\xi\|\Big\}\,.\end{aligned}$$

If instead $\ell_2 = 1$, the r.h.s. of (3.115) blows up, as $N \to \infty$. To make up for this singularity, we use the additional assumption $k + n \geq 2$. Combining this information with $\ell_1 = 0$, $\ell_2 = 1$, we conclude that either $k_1 > 0$ or $k_2 > 0$ or there exists $i \in \mathbb{N}$ such that either Λ_i or Λ_i' is a $\Pi^{(2)}$-operator of the form (3.100) with $p \geq 1$. This factor allows us to gain a factor $(\mathcal{N}+1)/N$ in the estimate for the term arising from the commutator (3.115). We conclude that, in this case,

$$\begin{aligned}\mathrm{D} \leq C^{k+n} \|\eta_t\|^{k+n-2} \Big\{ & nN^{-1} \|\eta_y\| \|\eta_x\| \|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\| \\ & + N^{-1} |\eta_t(x;y)| \|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\| + \|\eta_t\| \|\eta_x\| \|a_y (\mathcal{N}+1)^{\alpha/2}\xi\| \Big\}.\end{aligned}$$

Finally, let us consider the case $\ell_2 = 0$. Here we proceed as before, commuting a_y to the right. The commutator produces at most n factors, whose norm can be bounded similarly as before. We easily conclude that

$$\mathrm{D} \leq C^{k+n} \|\eta_t\|^{k+n-1} \left\{ nN^{-1} \|\eta_x\| \|a_y (\mathcal{N}+1)^{\alpha/2}\xi\| + \|\eta_t\| \|a_x a_y (\mathcal{N}+1)^{(\alpha-1)/2}\xi\| \right\}$$

If we impose the additional condition $k + n \geq 1$, we deduce that either $k_1 > 0$ or $k_2 > 0$ or there exists $i \in \mathbb{N}$ such that either Λ_i or Λ_i' is a $\Pi^{(2)}$-operator of the form (3.100) with $p \geq 1$. Similarly as we argued in the case $\ell_2 = 1$, when estimating the contribution with the two annihilation operators a_x, a_y acting on ξ, we can therefore extract an additional factor $(\mathcal{N}+1)/N$. Under this additional condition, we obtain

$$\mathrm{D} \leq C^{k+n} \|\eta_t\|^{k+n-1} \left\{ nN^{-1} \|\eta_x\| \|a_y \xi\| + N^{-1/2} \|\eta_t\| \|a_x a_y (\mathcal{N}+1)^{(\alpha-1)/2}\xi\| \right\}$$

which proves (3.110). □

3.4.2 Analysis of $e^{-B(\eta_t)} \mathcal{L}_{N,t}^{(0)} e^{B(\eta_t)}$

From the definition (3.90), we can write

$$\begin{aligned}\mathcal{L}_{N,t}^{(0)} = C_{N,t} & - \langle \widetilde{\varphi_\xi}_t, \left[N^3 V(N.) w_\ell(N.) * |\widetilde{\varphi_\xi}_t|^2\right] \widetilde{\varphi_\xi}_t \rangle \mathcal{N} \\ & + \frac{1}{2N} \langle \widetilde{\varphi_\xi}_t, \left[N^3 V(N.) * |\widetilde{\varphi_\xi}_t|^2\right] \widetilde{\varphi_\xi}_t \rangle \mathcal{N} + \frac{1}{2N} \langle \widetilde{\varphi_\xi}_t, \left[N^3 V(N.) * |\widetilde{\varphi_\xi}_t|^2\right] \widetilde{\varphi_\xi}_t \rangle \mathcal{N}^2\end{aligned}$$

with the t- and N-dependent number

$$C_{N,t} = \frac{N}{2} \langle \widetilde{\varphi_\xi}_t, \left[N^3 V(N.) w_\ell(N.) * |\widetilde{\varphi_\xi}_t|^2\right] \widetilde{\varphi_\xi}_t \rangle - \frac{1}{2} \langle \widetilde{\varphi_\xi}_t, \left[N^3 V(N.) * |\widetilde{\varphi_\xi}_t|^2\right] \widetilde{\varphi_\xi}_t \rangle$$

The properties of the other terms are described in the next proposition.

Proposition 3.4.3. *Under the same assumptions as in Theorem 3.3.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned}
\left| \left\langle \xi, e^{-B(\eta_t)} \left(\mathcal{L}^{(0)}_{N,t} - C_{N,t} \right) e^{B(\eta_t)} \xi \right\rangle \right| &\leq C \langle \xi, (\mathcal{N}+1)\xi \rangle \\
\left| \left\langle \xi, \left[\mathcal{N}, e^{-B(\eta_t)} \left(\mathcal{L}^{(0)}_{N,t} - C_{N,t} \right) e^{B(\eta_t)} \right] \xi \right\rangle \right| &\leq C \langle \xi, (\mathcal{N}+1)\xi \rangle \\
\left| \left\langle \xi, \left[a^*(g_1)a(g_2), e^{-B(\eta_t)} \left(\mathcal{L}^{(0)}_{N,t} - C_{N,t} \right) e^{B(\eta_t)} \right] \xi \right\rangle \right| &\leq C \|g_1\| \|g_2\| \langle \xi, (\mathcal{N}+1)\xi \rangle \\
\left| \partial_t \left\langle \xi, e^{-B(\eta_t)} \left(\mathcal{L}^{(0)}_{N,t} - C_{N,t} \right) e^{B(\eta_t)} \xi \right\rangle \right| &\leq C e^{c|t|} \langle \xi, (\mathcal{N}+1)\xi \rangle
\end{aligned} \tag{3.116}$$

for all $t \in \mathbb{R}$, $g_1, g_2 \in L^2(\mathbb{R}^3)$, $\xi \in \mathcal{F}^{\leq N}$.

In order to show Proposition 3.4.3, we need to conjugate the number of particles operator $\mathcal{N}$ with the generalized Bogoliubov transformation $e^{-B(\eta_t)}$. To this end, we make use of the following lemma, where, for later convenience, we consider conjugation of more general quadratic operators.

Lemma 3.4.4. *Let $r : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be a bounded linear operator. Consider the Fock-space operators*

$$R_1 = \int dxdy\, r(y;x)\, b^*_x b_y \qquad \text{and} \qquad R_2 = \int dxdy\, r(y;x)\, a^*_x a_y$$

mapping $\mathcal{F}^{\leq N}$ in itself. Then we have the bounds

$$\begin{aligned}
\left| \left\langle \xi_1, e^{-B(\eta_t)} R_i e^{B(\eta_t)} \xi_2 \right\rangle \right| &\leq C \|r\|_{op} \|(\mathcal{N}+1)^{1/2}\xi_1\| \|(\mathcal{N}+1)^{1/2}\xi_2\| \\
\left| \left\langle \xi_1, \left[\mathcal{N}, e^{-B(\eta_t)} R_i e^{B(\eta_t)} \right] \xi_2 \right\rangle \right| &\leq C \|r\|_{op} \|(\mathcal{N}+1)^{1/2}\xi_1\| \|(\mathcal{N}+1)^{1/2}\xi_2\| \\
\left| \left\langle \xi_1, \left[a^*(g_1)a(g_2), e^{-B(\eta_t)} R_i e^{B(\eta_t)} \right] \xi_2 \right\rangle \right| &\leq C \|r\|_{op} \|g_1\| \|g_2\| \\
&\quad \times \|(\mathcal{N}+1)^{1/2}\xi_1\| \|(\mathcal{N}+1)^{1/2}\xi_2\|
\end{aligned} \tag{3.117}$$

for $i = 1, 2$ and all $\xi_1, \xi_2 \in \mathcal{F}^{\leq N}$. Furthermore, if $r = r_t$ is differentiable in t, we find

$$\left| \partial_t \left\langle \xi_1, e^{-B(\eta_t)} R_i e^{B(\eta_t)} \xi_2 \right\rangle \right| \leq C e^{c|t|} (\|r\|_{op} + \|\dot{r}\|_{op}) \|(\mathcal{N}+1)^{1/2}\xi_1\| \|(\mathcal{N}+1)^{1/2}\xi_2\| \tag{3.118}$$

for $i = 1, 2$ and all $\xi_1, \xi_2 \in \mathcal{F}^{\leq N}$.

Proof. We consider first the operator R_1. By Lemma 3.2.4, we expand

$$\begin{aligned}
e^{-B(\eta_t)} R_1 e^{B(\eta_t)} &= \int dx\, e^{-B(\eta_t)} b^*(r_x) b_x e^{B(\eta_t)} \\
&= \sum_{k,n \geq 0} \frac{(-1)^{k+n}}{k!n!} \int dx\, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*(r_x)) \mathrm{ad}^{(k)}_{B(\eta_t)}(b_x)
\end{aligned} \tag{3.119}$$

with the notation $r_x(y) = r(x;y)$. According to Lemma 3.2.3 the operator

$$\int dx\, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*(r_x))\, \mathrm{ad}^{(k)}_{B(\eta_t)}(b_x)$$

is given by the sum of $2^{n+k}n!k!$ terms having the form

$$\begin{aligned}\mathrm{E} := \int dx\, N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1)}_{t,\diamond} r)_x)^*\Lambda^*_{i_1}\dots\Lambda^*_1 \\ \Lambda'_1\dots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\diamond'})\end{aligned} \tag{3.120}$$

where $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \geq 0$, $j_1,\dots,j_{k_1}, m_1,\dots,m_{k_2} \geq 1$, and where each operator Λ_i and Λ'_i is either a factor $(N-\mathcal{N})/N$, a factor $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1},\dots,\eta^{(q_p)}_{t,\underline{\natural}_p}) \tag{3.121}$$

for a $p \geq 1$ and powers $q_1,\dots,q_p \geq 1$. With Cauchy-Schwarz we find

$$\begin{aligned}|\langle \xi_1, \mathrm{E}\xi_2\rangle| \leq \int dx\, \Big\| \Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1)}_{t,\diamond} r)_x)\xi_1\Big\| \\ \times \Big\|\Lambda'_1\dots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\diamond'})\xi_2\Big\|\end{aligned} \tag{3.122}$$

for every $\xi_1, \xi_2 \in \mathcal{F}^{\leq N}$. With Lemma 3.4.2, part ii), we find that

$$|\langle \xi_1, \mathrm{E}\,\xi_2\rangle| \leq C^{k+n}\|r\|_{\mathrm{op}}\|\eta_t\|^{n+k}\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\| \tag{3.123}$$

where we used the fact that

$$\int dx\, \|a(r_x)\xi_1\|^2 = \langle \xi_1, d\Gamma(r^2)\xi_1\rangle \leq \|r^2\|_{\mathrm{op}}\|\mathcal{N}^{1/2}\xi_1\|^2 \leq \|r\|^2_{\mathrm{op}}\|\mathcal{N}^{1/2}\xi_1\|^2$$

From (3.119), we conclude that, if $\sup_t \|\eta_t\|$ is small enough,

$$\Big|\langle \xi_1, e^{-B(\eta_t)}R_1e^{B(\eta_t)}\xi_2\rangle\Big| \leq C\|r\|_{\mathrm{op}}\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\| \tag{3.124}$$

This proves the first bound in (3.117), if $i=1$. The other two bounds in (3.117) and the bound in (3.118) for $i=1$ can be proven similarly. To be more precise, we first expand the operator $e^{-B(\eta_t)}R_1e^{B(\eta_t)}$ as in (3.119), where the (n,k)-th term can be written as the sum of $2^{n+k}k!n!$ terms of the form (3.120). Then we use Lemma 3.4.1 to express the commutator of (3.120) with $\mathcal{N}$ or with $a^*(g_1)a(g_2)$ or its time-derivative as a sum of at most $2(k+n+1)$ terms having again the form (3.120), with just one of the η_t-kernels appropriately replaced. Finally, we proceed as above to show that the matrix elements of such a term can be bounded as in (3.123). We omit further details.

Let us now consider the operator R_2. We start by writing

$$
\begin{aligned}
e^{-B(\eta_t)} R_2 e^{B(\eta_t)} &= R_2 + \int_0^1 ds\, e^{-sB(\eta_t)} [R_2, B(\eta_t)] e^{sB(\eta_t)} \\
&= R_2 + \int_0^1 ds \int dx dy\, r(y;x) e^{-sB(\eta_t)} \left[a_x^* a_y, B(\eta_t)\right] e^{sB(\eta_t)} \\
&= R_2 + \int_0^1 ds \int dx e^{-sB(\eta_t)} \left[b((\eta_t r)_x) b_x + \text{h.c.}\right] e^{sB(\eta_t)}
\end{aligned}
$$

Expanding as in Lemma 3.2.4 and then integrating over s, we find

$$
\begin{aligned}
&e^{-B(\eta_t)} R_2 e^{B(\eta_t)} \\
&\quad = R_2 + \sum_{k,n\geq 0} \frac{(-1)^{k+n}}{k!n!(k+n+1)} \int dx \left[\text{ad}^{(n)}_{B(\eta_t)}(b((\eta_t r)_x)) \text{ad}^{(k)}_{B(\eta_t)}(b_x) + \text{h.c.}\right]
\end{aligned}
\tag{3.125}
$$

With Lemma 3.2.3, we can write the operator

$$
\int dx\, \text{ad}^{(n)}_{B(\eta_t)}(b((\eta_t r)_x)) \text{ad}^{(k)}_{B(\eta_t)}(b_x) \tag{3.126}
$$

as a sum of $2^{n+k} k! n!$ contributions of the form

$$
\begin{aligned}
\text{E} = \int dx\, \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1+1)}_{t,\diamondsuit} r)_x) \\
\times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})
\end{aligned}
\tag{3.127}
$$

where each Λ_i and Λ'_i is either $(N-\mathcal{N})/N$, $(N+1-\mathcal{N})/N$ or an operator of the form

$$
N^{-p}\, \Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1}, \dots \eta^{(q_p)}_{t,\underline{\natural}_p}) \tag{3.128}
$$

From Lemma 3.4.2, part iii), we obtain that

$$
\begin{aligned}
|\langle \xi_1, \text{E} \xi_2 \rangle| \leq\; & \|(\mathcal{N}+1)^{1/2} \xi_1\| \\
& \times \int dx \Big\| (\mathcal{N}+1)^{-1/2} \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1+1)}_{t,\diamondsuit} r)_x) \\
& \qquad \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'}) \xi_2 \Big\| \\
\leq\; & C^{n+k} \|r\|_{\text{op}} \|\eta_t\|^{k+n+1} \|(\mathcal{N}+1)^{1/2} \xi_1\| \|(\mathcal{N}+1)^{1/2} \xi_2\|
\end{aligned}
$$

This implies that, if $\sup_t \|\eta_t\|$ is small enough,

$$
\left| \langle \xi_1, e^{-B(\eta_t)} R_2 e^{B(\eta_t)} \xi_2 \rangle \right| \leq C \|r\|_{\text{op}} \|(\mathcal{N}+1)^{1/2} \xi_1\| \|(\mathcal{N}+1)^{1/2} \xi_2\|
$$

As in the analysis of R_1 above, also here one can show the other bounds in (3.117) for the commutators of $e^{-B(\eta_t)} R_1 e^{B(\eta_t)}$ with $\mathcal{N}$ and with $a^*(g_1)a(g_2)$ and for its time-derivative. □

Next, we use Lemma 3.4.4 to show Prop. 3.4.3.

Proof of Prop. 3.4.3. To control $\mathcal{L}^{(0)}_{N,t}$ we start by noticing that, with Young's inequality,

$$\begin{aligned} \big|\langle \widetilde{\varphi_{\xi_t}}, \big[N^3V(N.)*|\widetilde{\varphi_{\xi_t}}|^2\big]\,\widetilde{\varphi_{\xi_t}}\rangle\big| &\leq \int N^3V(N(x-y))|\widetilde{\varphi_{\xi_t}}(x)|^2|\widetilde{\varphi_{\xi_t}}(y)|^2dxdy \\ &\leq C\|\widetilde{\varphi_{\xi_t}}\|_4^4 \leq C\|\widetilde{\varphi_{\xi_t}}\|_{H^1}^4 \leq C \end{aligned} \tag{3.129}$$

and

$$\big|\partial_t\langle \widetilde{\varphi_{\xi_t}}, \big[N^3V(N.)*|\widetilde{\varphi_{\xi_t}}|^2\big]\,\widetilde{\varphi_{\xi_t}}\rangle\big| \leq C\|\widetilde{\varphi_{\xi_t}}\|_4^3\|\dot{\widetilde{\varphi_{\xi_t}}}\|_4 \leq C\|\widetilde{\varphi_{\xi_t}}\|_{H^1}^3\|\widetilde{\varphi_{\xi_t}}\|_{H^3} \leq Ce^{c|t|} \tag{3.130}$$

for constants $C,c>0$. Similarly, we also have

$$\begin{aligned} \big|\langle \widetilde{\varphi_{\xi_t}}, \big[N^3V(N.)w_\ell(N.)*|\widetilde{\varphi_{\xi_t}}|^2\big]\,\widetilde{\varphi_{\xi_t}}\rangle\big| &\leq C \\ \big|\partial_t\langle \widetilde{\varphi_{\xi_t}}, \big[N^3V(N.)w_\ell(N.)*|\widetilde{\varphi_{\xi_t}}|^2\big]\,\widetilde{\varphi_{\xi_t}}\rangle\big| &\leq Ce^{c|t|}\,. \end{aligned} \tag{3.131}$$

By (3.129), (3.130), (3.131), it is enough to show the four bound in (3.116) with $\mathcal{L}^{(0)}_{N,t}-C_{N,t}$ replaced by $\mathcal{N}$ and by $\mathcal{N}^2/N$. If we replace $\mathcal{L}^{(0)}_{N,t}-C_{N,t}$ with $\mathcal{N}$, the bounds in (3.116) follow from Lemma 3.4.4. To prove that these bounds also hold for $\mathcal{N}^2/N$, we use again Lemma 3.4.4. Setting $\xi_2 = e^{-B(\eta_t)}(\mathcal{N}/N)e^{B(\eta_t)}\xi$, we have

$$\Big|\langle \xi, e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)}\xi\rangle\Big| = \Big|\langle \xi, e^{-B(\eta_t)}\mathcal{N}e^{B(\eta_t)}\xi_2\rangle\Big| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi_2\|$$

Since, by Lemma 3.2.2,

$$\begin{aligned} \|(\mathcal{N}+1)^{1/2}\xi_2\|^2 &= N^{-2}\langle \xi, e^{-B(\eta_t)}\mathcal{N}e^{B(\eta_t)}(\mathcal{N}+1)e^{-B(\eta_t)}\mathcal{N}e^{B(\eta_t)}\xi\rangle \\ &\leq N^{-2}\langle \xi, (\mathcal{N}+1)^3\xi\rangle \leq C\langle \xi, (\mathcal{N}+1)\xi\rangle \end{aligned}$$

for all $\xi \in \mathcal{F}^{\leq N}$, we have

$$\Big|\langle \xi, e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)}\xi\rangle\Big| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

Using Lemma 3.4.4 and Leibniz rule, we also find

$$\begin{aligned} \Big|\langle \xi, \big[\mathcal{N}, e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)}\big]\xi\rangle\Big| &\leq C\|(\mathcal{N}+1)^{1/2}\xi\|^2 \\ \Big|\langle \xi, \big[a^*(g_1)a(g_2), e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)}\big]\xi\rangle\Big| &\leq C\|g_1\|\|g_2\|\|(\mathcal{N}+1)^{1/2}\xi\|^2 \\ \Big|\langle \xi, \partial_t\big(e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)}\big)\xi\rangle\Big| &\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|^2 \end{aligned}$$

This concludes the proof of the proposition. $\square$

3.4.3 Analysis of $e^{-B(\eta_t)}\mathcal{L}^{(1)}_{N,t}e^{B(\eta_t)}$

We recall that

$$\mathcal{L}^{(1)}_{N,t} = \sqrt{N}b(h_{N,t}) - \frac{\mathcal{N}+1}{\sqrt{N}}b(\widetilde{h}_{N,t}) + \text{h.c.}$$

where we used the notation $h_{N,t} = (N^3V(N.)w_\ell(N.) * |\widetilde{\varphi_{\xi_t}}|^2)\widetilde{\varphi_{\xi_t}}$ and $\widetilde{h}_{N,t} = (N^3V(N.) * |\widetilde{\varphi_{\xi_t}}|^2)\widetilde{\varphi_{\xi_t}}$. We write

$$e^{-B(\eta_t)}\mathcal{L}^{(1)}_{N,t}e^{B(\eta_t)} = \sqrt{N}\left[b(\cosh_{\eta_t}(h_{N,t})) + b^*(\sinh_{\eta_t}(\bar{h}_{N,t})) + \text{h.c.}\right] + \mathcal{E}^{(1)}_{N,t} \qquad (3.132)$$

In the next proposition we show that the operator $\mathcal{E}^{(1)}_{N,t}$, defined in (3.132), its commutator with $\mathcal{N}$ and its time-derivative can all be controlled by the number of particles operator $\mathcal{N}$ (while the first term on the r.h.s. of (3.132) will cancel with contributions arising from conjugation of $\mathcal{L}^{(3)}_{N,t}$).

Proposition 3.4.5. *Under the same assumptions as in Theorem 3.3.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned} \left|\langle \xi, \mathcal{E}^{(1)}_{N,t}\xi\rangle\right| &\leq C\langle \xi, (\mathcal{N}+1)\xi\rangle \\ \left|\langle \xi, \left[\mathcal{N}, \mathcal{E}^{(1)}_{N,t}\right]\xi\rangle\right| &\leq C\langle \xi, (\mathcal{N}+1)\xi\rangle \\ \left|\langle \xi, \left[a^*(g_1)a(g_2), \mathcal{E}^{(1)}_{N,t}\right]\xi\rangle\right| &\leq C\|g_1\|\|g_2\|\langle \xi, (\mathcal{N}+1)\xi\rangle \\ \left|\partial_t\langle \xi, \mathcal{E}^{(1)}_{N,t}\xi\rangle\right| &\leq Ce^{c|t|}\langle \xi, (\mathcal{N}+1)\xi\rangle \end{aligned} \qquad (3.133)$$

for all $\xi \in \mathcal{F}^{\leq N}$.

Proof. We start with the observation that

$$\begin{aligned} \|h_{N,t}\|, \|\widetilde{h}_{N,t}\| &\leq C\|\widetilde{\varphi_{\xi_t}}\|^3_{H^1} \leq C \\ \|\partial_t h_{N,t}\|, \|\partial_t\widetilde{h}_{N,t}\| &\leq \|\widetilde{\varphi_{\xi_t}}\|^2_{H^1}\|\widetilde{\varphi_{\xi_t}}\|_{H^3} \leq Ce^{c|t|} \end{aligned} \qquad (3.134)$$

uniformly in N and for all $t \in \mathbb{R}$. Recall that, by (3.132),

$$\begin{aligned} \mathcal{E}^{(1)}_{N,t} = &\left[e^{-B(\eta_t)}\mathcal{L}^{(1)}_{N,t}e^{B(\eta_t)} - \sqrt{N}\left(b(\cosh_{\eta_t}(h_{N,t}) + b^*(\sinh_{\eta_t}(h_{N,t}) + \text{h.c.}\right)\right] \\ = &\sqrt{N}\left[e^{-B(\eta_t)}b(h_{N,t})e^{B(\eta_t)} - (b(\cosh_{\eta_t}(h_{N,t}) + b^*(\sinh_{\eta_t}(h_{N,t}))\right] + \text{h.c.} \\ &+ N^{-1/2}e^{-B(\eta_t)}(\mathcal{N}+1)b(\widetilde{h}_{N,t})e^{B(\eta_t)} \end{aligned} \qquad (3.135)$$

Set

$$D(g) = e^{-B(\eta_t)}b(g)e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t}(g))$$

We observe that Proposition 3.4.5 follows if we prove that

$$\begin{aligned} |\langle \xi_1, D(g)\xi_2\rangle| &\leq CN^{-1/2}\|g\|\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\| \\ |\langle \xi_1, [\mathcal{N}, D(g)]\xi_2\rangle| &\leq CN^{-1/2}\|g\|\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\| \\ |\langle \xi_1, [a^*(g_1)a(g_2), D(g)]\xi_2\rangle| &\leq CN^{-1/2}\|g\|\|g_1\|\|g_2\|\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\| \\ |\langle \xi_1, \partial_t D(g)\xi_2\rangle| &\leq CN^{-1/2}(\|g\|+\|\dot{g}\|)\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\| \end{aligned} \tag{3.136}$$

for every, possibly time-dependent, $g \in L^2(\mathbb{R}^3)$. In fact, applying (3.136) with $g = h_{N,t}$, we obtain the desired bounds for the first line on the r.h.s. of (3.135). To bound the expectation of the operator on the second line on the r.h.s. of (3.135), on the other hand, we apply (3.136) with $g = \widetilde{h}_{N,t}$, $\xi_1 = \xi$ and $\xi_2 = e^{-B(\eta_t)}(\mathcal{N}+1)e^{B(\eta_t)}\xi$. We find

$$\begin{aligned} N^{-1/2} &\left| \langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)b(\widetilde{h}_{N,t})e^{B(\eta_t)}\xi\rangle \right| \\ &= N^{-1/2}\left|\langle \xi_2, e^{-B(\eta_t)}b(\widetilde{h}_{N,t})e^{B(\eta_t)}\xi\rangle\right| \\ &\leq N^{-1/2}\left|\langle \xi_2, \left[b(\cosh_{\eta_t}(\widetilde{h}_{N,t})) + b^*(\sinh_{\eta_t}(\widetilde{h}_{N,t}))\right]\xi\rangle\right| \\ &\quad + CN^{-1}\|\widetilde{h}_{N,t}\|\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi_2\| \\ &\leq CN^{-1/2}\|(\mathcal{N}+1)^{1/2}\xi\|\|\xi_2\| + CN^{-1}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi_2\| \end{aligned} \tag{3.137}$$

where we used Lemma 1.2.4, the fact that $\cosh_{\eta_t}, \sinh_{\eta_t}$ are bounded operators (uniformly in t and N), and (3.134). From Lemma 3.2.2, we obtain

$$\|\xi_2\|^2 = \langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)^2 e^{B(\eta_t)}\xi\rangle \leq C\langle \xi, (\mathcal{N}+1)^2\xi\rangle = C\|(\mathcal{N}+1)\xi\|^2$$

and, similarly,

$$\begin{aligned} \|(\mathcal{N}+1)^{1/2}\xi_2\|^2 &= \langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)e^{B(\eta_t)}(\mathcal{N}+1)e^{-B(\eta_t)}(\mathcal{N}+1)e^{B(\eta_t)}\xi\rangle \\ &\leq C\langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)^3 e^{B(\eta_t)}\xi\rangle \\ &\leq C\langle \xi, (\mathcal{N}+1)^3\xi\rangle = C\|(\mathcal{N}+1)^{3/2}\xi\|^2 \end{aligned}$$

Inserting the last two bounds in the r.h.s. of (3.137), we conclude that

$$N^{-1/2}\left|\langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)b(\widetilde{h}_{N,t})e^{B(\eta_t)}\xi\rangle\right| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

for all $\xi \in \mathcal{F}^{\leq N}$. Similarly, we can control the commutator of the second line on the r.h.s. of (3.135) with $\mathcal{N}$ and with $a^*(g_1)a(g_2)$ and its time-derivative.

We still have to show (3.136). To this end, we use Lemma 3.2.4 to expand

$$e^{-B(\eta_t)}b(g)e^{B(\eta_t)} = \sum_{n\geq 0}\frac{(-1)^n}{n!}\mathrm{ad}^{(n)}_{B(\eta_t)}(b(g)) \tag{3.138}$$

According to Lemma 3.2.3, the nested commutator $\text{ad}^{(n)}_{B(\eta_t)}(b(g))$ can be written as a sum of $2^n n!$ terms, having the form

$$\Lambda_1 \dots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(s)}_{t,\natural_{k+1}}(g_\Delta)) \tag{3.139}$$

where each Λ_m is either $(N-\mathcal{N})/N$, $(N-\mathcal{N}+1)/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p} \Pi^{(2)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_p)}_{t,\natural'_p}) \tag{3.140}$$

Exactly one of these $2^n n!$ terms has the form

$$\begin{cases} \frac{(N-\mathcal{N})^r}{N^r} \frac{(N+1-\mathcal{N})^r}{N^r} b(\eta^{(2r)}_t(g)) & \text{if } n = 2r \text{ is even} \\ -\frac{(N-\mathcal{N})^{r+1}}{N^{r+1}} \frac{(N+1-\mathcal{N})^r}{N^r} b^*(\eta^{(2r+1)}_t(\bar{g})) & \text{if } n = 2r+1 \text{ is odd} \end{cases} \tag{3.141}$$

All other terms are of the form (3.139), with either $k > 0$ or with at least one factor Λ_i being of the form (3.140). Let us suppose that $n = 2r$ is even. Then we write (3.141) as

$$\begin{aligned} &\frac{(N-\mathcal{N})^r}{N^r} \frac{(N+1-\mathcal{N})^r}{N^r} b(\eta^{(2r)}_t(g)) \\ &\qquad = b(\eta^{(2r)}_t(g)) + \left[\frac{(N-\mathcal{N})^r}{N^r} \frac{(N+1-\mathcal{N})^r}{N^r} - 1 \right] b(\eta^{(2r)}_t(g)) \end{aligned} \tag{3.142}$$

Inserting the term $b(\eta^{(2r)}_t(g))$ on the r.h.s. of (3.138) and summing over all $r \in \mathbb{N}$, we reconstruct

$$\sum_{r \geq 0} \frac{1}{(2r)!} b(\eta^{(2r)}_t(g)) = b(\cosh_{\eta_t}(g))$$

On the other hand, the contribution of the second term on the r.h.s. of (3.142) has matrix elements bounded by

$$\begin{aligned} &\left| \langle \xi_1, \left[\frac{(N-\mathcal{N})^r}{N^r} \frac{(N+1-\mathcal{N})^r}{N^r} - 1 \right] b(\eta^{(2r)}_t(g)) \xi_2 \rangle \right| \\ &\qquad \leq \left\| \left[\frac{(N-\mathcal{N})^r}{N^r} \frac{(N+1-\mathcal{N})^r}{N^r} - 1 \right] \xi_1 \right\| \|b(\eta^{(2r)}_t(g)) \xi_2\| \\ &\qquad \leq 2r N^{-1/2} \|\eta_t\|^{2r} \|g\| \|(\mathcal{N}+1)^{1/2} \xi_1\| \|(\mathcal{N}+1)^{1/2} \xi_2\| \end{aligned} \tag{3.143}$$

since $1 - (1-x)^r \leq rx$ for all $0 \leq x \leq 1$. Similarly, the contribution (3.141) with $n = 2r+1$ odd can be shown to reconstruct the operator $b^*(\sinh_{\eta_t}(\bar{g}))$, up to an error that can be estimated as in (3.143).

As for the other terms of the form (3.139), excluding (3.141), we can bound their matrix elements using part i) of Lemma 3.4.2. We obtain

$$\begin{aligned} &\left| \langle \xi_1, \Lambda_1 \dots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(s)}_{t,\natural_{k+1}}) \xi_2 \rangle \right| \\ &\qquad \leq \|(\mathcal{N}+1)^{1/2} \xi_1\| \left\| (\mathcal{N}+1)^{-1/2} N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(s)}_{t,\natural_{k+1}}(g_\Delta)) \xi_2 \right\| \\ &\qquad \leq C^n \|\eta_t\|^n N^{-1/2} \|g\| \|(\mathcal{N}+1)^{1/2} \xi_1\| \|(\mathcal{N}+1)^{1/2} \xi_2\| \end{aligned} \tag{3.144}$$

We conclude that

$$\begin{aligned}
&\Big|\langle\xi_1, \Big\{e^{-B(\eta_t)}b(g)e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t})(\bar{g}))\Big\}\,\xi_2\rangle\Big| \\
&\qquad\qquad \leq N^{-1/2}\|g\|\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\|\sum_{n\geq 2} nC^n\|\eta_t\|^n \\
&\qquad\qquad \leq CN^{-1/2}\|g\|\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\|
\end{aligned} \tag{3.145}$$

if the parameter $\ell > 0$ in the definition (3.76) of the kernel η_t is small enough.

Since, by Lemma 3.4.1, part i), the commutator of every term of the form (3.139) with $\mathcal{N}$ is again a term of the same form, just multiplied with a constant $\kappa \in \{0, \pm 1, \pm 2\}$, we conclude that

$$\begin{aligned}
&\Big|\langle\xi_1, \Big[\mathcal{N}, \Big\{e^{-B(\eta_t)}b(g)e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t})(\bar{g}))\Big\}\Big]\,\xi_2\rangle\Big| \\
&\qquad\qquad \leq CN^{-1/2}\|g\|\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\|
\end{aligned} \tag{3.146}$$

Since, again by Lemma 3.4.1, part ii) and iii), the commutator of every term of the form (3.139) with $a^*(g_1)a(g_2)$ can be written as a sum of at most $2n$ terms having again the form (3.139), just with one of the η_t-kernels or with the function $\eta^{(s)}_{t,\natural_{k+1}}(g_\Delta)$ appearing in the $\Pi^{(1)}$-operator replaced according to (3.91) and (3.96), we also find that

$$\begin{aligned}
&\Big|\langle\xi_1, \Big[a^*(g_1)a(g_2), \Big\{e^{-B(\eta_t)}b(g)e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t})(\bar{g}))\Big\}\Big]\,\xi_2\rangle\Big| \\
&\qquad\qquad \leq CN^{-1/2}\|g\|\|g_1\|\|g_2\|\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\|
\end{aligned} \tag{3.147}$$

Finally, since by Lemma 3.4.1, part iv), the time-derivative of each term of the form (3.139) can be written as a sum of at most $(n+1)$ terms having again the form (3.139), but with one of the η_t-kernels or the function $\eta^{(s)}_{t,\natural_{k+1}}(g_\Delta)$ appearing in the $\Pi^{(1)}$-operator replaced by their time-derivative, we get (since $\|\dot{\eta}_t\| \leq Ce^{c|t|}$)

$$\begin{aligned}
&\Big|\partial_t\langle\xi_1, \Big[e^{-B(\eta_t)}b(g)e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t})(\bar{g}))\Big]\,\xi_2\rangle\Big| \\
&\qquad\qquad \leq CN^{-1/2}e^{c|t|}(\|g\| + \|\dot{g}\|)\|(\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\|
\end{aligned} \tag{3.148}$$

□

3.4.4 Analysis of $e^{-B(\eta_t)}\mathcal{L}^{(2)}_{N,t}e^{B(\eta_t)}$

Recall that

$$\begin{aligned}
\mathcal{L}^{(2)}_{N,t} = \mathcal{K} &+ \int dxdy\, N^3V(N(x-y))|\widetilde{\varphi}_{\xi_t}(y)|^2\left[b_x^*b_x - \frac{1}{N}a_x^*a_x\right] \\
&+ \int dxdy\, N^3V(N(x-y))\widetilde{\varphi}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)\left[b_x^*b_y - \frac{1}{N}a_x^*a_y\right] \\
&+ \frac{1}{2}\int dxdyN^3V(N(x-y))\left[\widetilde{\varphi}_{\xi_t}(x)\widetilde{\varphi}_{\xi_t}(y)b_x^*b_y^* + \text{h.c.}\right]
\end{aligned} \tag{3.149}$$

with the notation

$$\mathcal{K} = \int dx\, \nabla_x a_x^* \nabla_x a_x$$

for the kinetic energy operator.

In the next two subsections we consider first the conjugation of the kinetic energy operator and then of the rest of $\mathcal{L}_{N,t}^{(2)}$ with $e^{B(\eta_t)}$.

Analysis of $e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)}$

We write

$$\begin{aligned} e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)} &= \mathcal{K} + \int |\nabla_x k_t(x;y)|^2 dxdy \\ &+ \int dxdy\, (\Delta w_\ell)(N(x-y))\, \left[\widetilde{\varphi_{\xi}}_t(x)\widetilde{\varphi_{\xi}}_t(y) b_x^* b_y^* + \text{h.c.}\right] \\ &+ \mathcal{E}_{N,t}^{(K)} \end{aligned} \tag{3.150}$$

In the next proposition, we collect important properties of the error term $\mathcal{E}_{N,t}^{(K)}$ defined in (3.150).

Proposition 3.4.6. *Under the same assumptions as in Theorem 3.3.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned} \left|\langle \xi, \mathcal{E}_{N,t}^{(K)} \xi\rangle\right| &\leq Ce^{c|t|} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2}\xi\| \|(\mathcal{N}+1)^{1/2}\xi\| \\ \left|\langle \xi, \left[\mathcal{N}, \mathcal{E}_{N,t}^{(K)}\right] \xi\rangle\right| &\leq Ce^{c|t|} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2}\xi\| \|(\mathcal{N}+1)^{1/2}\xi\| \\ \left|\langle \xi, \left[a^*(g_1)a(g_2), \mathcal{E}_{N,t}^{(K)}\right] \xi\rangle\right| &\leq Ce^{c|t|} \|g_1\|_{H^1}\|g_2\|_{H^1} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2}\xi\| \|(\mathcal{N}+1)^{1/2}\xi\| \\ \left|\partial_t\langle \xi, \mathcal{E}_{N,t}^{(K)} \xi\rangle\right| &\leq Ce^{c|t|} \|(\mathcal{H}_N + \mathcal{N} + 1)^{1/2}\xi\| \|(\mathcal{N}+1)^{1/2}\xi\| \end{aligned} \tag{3.151}$$

where we used the notation $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$, with

$$\mathcal{V}_N = \frac{1}{2}\int dxdy N^2 V(N(x-y)) a_x^* a_y^* a_y a_x \tag{3.152}$$

Proof. We write

$$\begin{aligned} e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)} - \mathcal{K} &= \int_0^1 e^{-sB(\eta_t)}\left[\mathcal{K}, B(\eta_t)\right] e^{sB(\eta_t)} \\ &= \int_0^1 ds \int dx\, e^{-sB(\eta_t)}\left[\nabla_x a_x^* \nabla_x a_x, B(\eta_t)\right] e^{sB(\eta_t)} \end{aligned}$$

From (3.43), we find

$$\begin{aligned}
&e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)} - \mathcal{K} \\
&\qquad = \int_0^1 ds \int dx \, \left[e^{-sB(\eta_t)} b(\nabla_x \eta_x)\nabla_x b_x e^{sB(\eta_t)} + \text{h.c.}\right] \\
&\qquad = \sum_{k,n\geq 0} \frac{(-1)^{k+n}}{k!n!(k+n+1)} \int dx \, \left[\text{ad}^{(n)}_{B(\eta_t)}(b(\nabla_x\eta_x))\text{ad}^{(k)}_{B(\eta_t)}(\nabla_x b_x) + \text{h.c.}\right]
\end{aligned}$$

From the sum on the r.h.s. we extract the term with $k = n = 0$ and also the term with $n = 0, k = 1$. We obtain

$$\begin{aligned}
&e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)} - \mathcal{K} \\
&\quad = \int dx \, [b(\nabla_x \eta_x)\nabla_x b_x + \text{h.c.}] \\
&\qquad + \int dx \, b(\nabla_x\eta_x) b^*(\nabla_x \eta_x) - \frac{1}{N}\int dx \, b(\nabla_x \eta_x)\mathcal{N} b^*(\nabla_x\eta_x) \\
&\qquad - \frac{1}{2N}\int dxdzdy \, \left[\eta_t(z,y) b(\nabla_x\eta_x) b^*_y a^*_z \nabla_x a_x + \text{h.c.}\right] \\
&\qquad + \sum^*_{k,n} \frac{(-1)^{k+n}}{k!n!(k+n+1)} \int dx \, \left[\text{ad}^{(n)}_{B(\eta_t)}(b(\nabla_x\eta_x))\text{ad}^{(k)}_{B(\eta_t)}(\nabla_x b_x) + \text{h.c.}\right]
\end{aligned} \tag{3.153}$$

where $\sum^*$ denotes the sum over all indices $k, n \geq 0$, excluding the two pairs $(k,n) = (0,0)$ and $(k,n) = (1,0)$. We discuss now the terms on the r.h.s. of (3.153) separately.

The first term on the r.h.s. of (3.153) can be decomposed as in (3.77), giving

$$\int dx \, b(\nabla_x\eta_x)\nabla_x b_x = \int dx \, b(\nabla_x k_x)\nabla_x b_x + \int dx \, b(\nabla_x \mu_x)\nabla_x b_x \tag{3.154}$$

The second term on the r.h.s. of (3.154) contributes to the error $\mathcal{E}^{(K)}_{N,t}$. Its expectation is bounded by

$$\begin{aligned}
\left|\int dx \langle \xi, b(\nabla_x\mu_x)\nabla_x b_x \xi\rangle\right| &\leq \|(\mathcal{N}+1)^{1/2}\xi\| \int dx \, \|\nabla_x \mu_x\| \|\nabla_x b_x \xi\| \\
&\leq \|\nabla_x \mu\| \|(\mathcal{N}+1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\| \leq C\|(\mathcal{N}+1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|
\end{aligned}$$

The expectation of the commutator of this term with $\mathcal{N}$ and with $a^*(g_1)a(g_2)$ and also its time-derivative can be bounded similarly, using the formula

$$[a^*(g_1)a(g_2), b(\nabla_x\mu_x)\nabla_x b_x] = \langle g_1, \nabla_x \mu_x\rangle b(g_2)\nabla_x b_x + b(\nabla_x\mu_x)\nabla g_1(x) b(g_2)$$

and the fact that $\|\partial_t \nabla_x \mu_t\| < Ce^{c|t|}$, uniformly in N.

As for the first term on the r.h.s. of (3.154), we integrate by parts and we use the definition (3.75), to write

$$\begin{aligned}\int dx\, b(\nabla_x k_x)\nabla_x b_x = &\int dxdy\, N^3(\Delta w_\ell)(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)\, b_x b_y \\ &+ 2\int dxdy\, N^2(\nabla w_\ell)(N(x-y))(\nabla\widetilde{\varphi}_{\xi_t})(x)\widetilde{\varphi}_{\xi_t}(y)\, b_x b_y \\ &+ \int dxdy\, N w_\ell(N(x-y))(\Delta\widetilde{\varphi}_{\xi_t})(x)\widetilde{\varphi}_{\xi_t}(y) b_x b_y\end{aligned} \tag{3.155}$$

The first term on the r.h.s. of (3.155) is exactly the (hermitian conjugate of the) contribution that we isolated on the second line of (3.150); it does not enter the error term $\mathcal{E}^{(K)}_{N,t}$. The second and third terms on the r.h.s. of (3.155), on the other hand, are included in $\mathcal{E}^{(K)}_{N,t}$. The expectation of the third term is bounded by

$$\begin{aligned}&\left|\int dxdy\, N w_\ell(N(x-y))(\Delta\widetilde{\varphi}_{\xi_t})(x)\widetilde{\varphi}_{\xi_t}(y)\,\langle\xi, b_x b_y\xi\rangle\right| \\ &\quad\leq \int dx\, |\Delta\widetilde{\varphi}_{\xi_t}(x)|\|b^*(N w_\ell(N(x-.))\widetilde{\varphi}_{\xi_t})\xi\|\, \|b_x\xi\| \\ &\quad\leq \sup_x \|N w_\ell(N(x-.))\widetilde{\varphi}_{\xi_t}\|\|\Delta\widetilde{\varphi}_{\xi_t}\|\|(\mathcal{N}+1)^{1/2}\xi\|^2 \leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|^2\end{aligned} \tag{3.156}$$

To bound the expectation of the second term on the r.h.s. of (3.155), we integrate by parts. We find

$$\begin{aligned}&\int dxdy\, N^2(\nabla w_\ell)(N(x-y))(\nabla\widetilde{\varphi}_{\xi_t})(x)\widetilde{\varphi}_{\xi_t}(y)\,\langle\xi, b_x b_y\xi\rangle \\ &\qquad = -\int dxdy N w_\ell(N(x-y))(\Delta\widetilde{\varphi}_{\xi_t})(x)\widetilde{\varphi}_{\xi_t}(y)\langle\xi, b_x b_y\xi\rangle \\ &\qquad\quad -\int dxdy N w_\ell(N(x-y))(\nabla\widetilde{\varphi}_{\xi_t})(x)\widetilde{\varphi}_{\xi_t}(y)\langle\xi, b_y\nabla_x b_x\xi\rangle\end{aligned}$$

Proceeding as in (3.156), we conclude that

$$\begin{aligned}&\left|\int dxdy\, N^2(\nabla w_\ell)(N(x-y))(\nabla\widetilde{\varphi}_{\xi_t})(x)\widetilde{\varphi}_{\xi_t}(y)\,\langle\xi, b_x b_y\xi\rangle\right| \\ &\leq \sup_x \|N w_\ell(N(x-.))\widetilde{\varphi}_{\xi_t}\|\,\left[\|\Delta\widetilde{\varphi}_{\xi_t}\|\|(\mathcal{N}+1)^{1/2}\xi\|^2 + \|\nabla\widetilde{\varphi}_{\xi_t}\|\|(\mathcal{N}+1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\right] \\ &\leq Ce^{c|t|}\left[\|(\mathcal{N}+1)^{1/2}\xi\|^2 + \|(\mathcal{N}+1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\right]\end{aligned}$$

Notice that the last estimate and the estimate (3.156) for the third term on the r.h.s. of (3.155) continue to hold, if we replace the operator whose expectation we are bounding, with its commutator with $\mathcal{N}$ or with $a^*(g_1)a(g_2)$ or with its time-derivative.

Now, let us consider the second term on the r.h.s. of (3.153). We observe that

$$\int dx\, b(\nabla_x\eta_x)b^*(\nabla_x\eta_x) = \|\nabla_x\eta_x\|^2 - \frac{\mathcal{N}}{N}\|\nabla_x\eta_x\|^2 + \int dxdydz \nabla_x\eta_t(x;z)\nabla_x\bar{\eta}_t(y;x)\left(b_z^*b_y - \frac{1}{N}a_z^*a_y\right) \tag{3.157}$$

Denoting by D the operator with the integral kernel

$$D(z;y) = \int dx\, \nabla_x\eta_t(z;x)\nabla_x\bar{\eta}_t(x;y) \tag{3.158}$$

we have

$$\left|\int dxdydz \nabla_x\eta_t(x;z)\nabla_x\bar{\eta}_t(y;x)\langle\xi, b_z^*b_y\xi\rangle\right| \leq |\langle\xi, d\Gamma(D)\xi\rangle| \leq \|D\|_2\|\mathcal{N}^{1/2}\xi\|^2 \tag{3.159}$$

Since, by Lemma 3.3.3, $\|D\|_2 \leq C$, we obtain

$$\left|\int dxdydz\, \nabla_x\eta_t(x;z)\nabla_x\bar{\eta}_t(y;x)\langle\xi, b_z^*b_y\xi\rangle\right| \leq C\|\mathcal{N}^{1/2}\xi\|^2$$

and similarly for the $a_z^*a_y$ term. As for the first term on the r.h.s. of (3.157), we decompose $\eta_t = k_t + \mu_t$. Since $\|\nabla_x\mu_t\|$ is finite, uniformly in N and in t, we find

$$\left|\int dx\|\nabla_x\eta_x\|^2 - \int dxdy\, |\nabla_x k_t(x;y)|^2\right| \leq C$$

The second term on the r.h.s. of (3.157) can be controlled using $N^{-1}\|\nabla_x\eta_x\|^2 \leq C$. Furthermore, one can show that

$$\int dx\, \langle\xi, [\mathcal{N}, b(\nabla_x\eta_x)b^*(\nabla_x\eta_x)]\xi\rangle = 0$$
$$\left|\int dx\, \langle\xi, [a^*(g_1)a(g_2), b(\nabla_x\eta_x)b^*(\nabla_x\eta_x)]\xi\rangle\right| \leq C\|g_1\|\|g_2\|\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

and

$$\left|\partial_t\left[\int dx\, \langle\xi, \partial_t[b(\nabla_x\eta_x)b^*(\nabla_x\eta_x)]\xi\rangle - \int dxdy|\nabla_x k_t(x;y)|^2\right]\right| \leq Ce^{K|t|}\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

Here we used the formula

$$\left[a^*(g_1)a(g_2), \int dx\, b(\nabla_x\eta_x)b^*(\nabla_x\eta_x)\right] = \int dx\, \langle\nabla_x\eta_x, g_1\rangle b(g_2)b^*(\nabla_x\eta_x) + \int dx, \langle g_2, \nabla_x\eta_x\rangle b(\nabla_x\eta_x)b^*(g_1)$$

for the commutator with $a^*(g_1)a(g_2)$ and the bounds in Proposition 3.3.2 for $\partial_t\widetilde{\varphi_{\xi_t}}$.

The third term on the r.h.s. of (3.153) can be controlled similarly.

To control the fourth term on the r.h.s. of (3.153) we proceed as follows. First of all, we commute the annihilation operator $b(\nabla_x \eta_x)$ to the right of the two creation operators $b_y^* a_z^*$. Using (1.20), we find

$$\begin{aligned}
&\frac{1}{2N}\int dxdydz\, \eta_t(z;y) b(\nabla_x \eta_x) b_y^* a_z^* \nabla_x a_x \\
&\qquad = \frac{1}{2N}\int dxdydz\, \eta_t(z;y) b_y^* a_z^* a(\nabla_x \eta_x)\nabla_x b_x \\
&\qquad\quad + \frac{1}{N}\int dxdydz\, \eta_t(z;y)\nabla_x \eta_t(x;y)\left(1-\frac{\mathcal{N}}{N}-\frac{1}{2N}\right) a_z^* \nabla_x a_x \\
&\qquad\quad - \frac{1}{2N^2}\int dxdydz\, \eta_t(z;y) a_y^* a(\nabla_x \eta_x) a_z^* \nabla_x a_x
\end{aligned} \tag{3.160}$$

To bound the expectation of the last term, we use the additional N^{-1} factor to compensate for $\|\nabla_x \eta_t\| \simeq N^{1/2}$. We find

$$\begin{aligned}
&\left|\frac{1}{2N^2}\int dxdydz\, \eta_t(z;y)\langle \xi, a_y^* a(\nabla_x \eta_x) a_z^* \nabla_x a_x \xi\rangle\right| \\
&\qquad \le \frac{1}{2N^2}\left[\int dxdydz |\eta_t(y;z)|^2 \|\nabla_x a_x \xi\|^2\right]^{1/2}\left[\int dxdydz \|a_z a^*(\nabla_x \eta_x) a_y \xi\|^2\right]^{1/2} \\
&\qquad \le \frac{\|\eta_t\|\|\nabla_x \eta_t\|}{2N^2}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}+1)^{3/2}\xi\| \\
&\qquad \le CN^{-1/2}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi\|
\end{aligned}$$

Similarly, the expectation of the second term on the r.h.s. of (3.160) is bounded by

$$\begin{aligned}
&\left|\frac{1}{N}\int dxdydz\, \eta_t(z;y)\nabla_x \eta_t(x;y)\left\langle \xi, \left(1-\frac{\mathcal{N}}{N}-\frac{1}{2N}\right) a_z^* \nabla_x a_x \xi\right\rangle\right| \\
&\qquad \le \frac{1}{N}\left[\int dxdydz |\eta_t(z;y)|^2 \|\nabla_x a_x \xi\|^2\right]^{1/2}\left[\int dxdydz |\nabla_x \eta_t(x;y)|\|^2 \|a_z \xi\|^2\right]^{1/2} \\
&\qquad \le \frac{\|\eta_t\|\|\nabla_x \eta_t\|}{N}\|(\mathcal{N}+1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\| \\
&\qquad \le CN^{-1/2}\|(\mathcal{N}+1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|
\end{aligned}$$

We are left with the first term on the r.h.s. of (3.160). Here, we decompose

$$\begin{aligned}
&\frac{1}{2N}\int dxdydz\, \eta_t(z;y) b_y^* a_z^* a(\nabla_x \eta_x)\nabla_x b_x \\
&\qquad = \frac{1}{2N}\int dxdydz\, \eta_t(z;y) b_y^* a_z^* a(\nabla_x k_x)\nabla_x b_x \\
&\qquad\quad + \frac{1}{2N}\int dxdydz\, \eta_t(z;y) b_y^* a_z^* a(\nabla_x \mu_x)\nabla_x b_x =: \mathrm{M}_1 + \mathrm{M}_2
\end{aligned} \tag{3.161}$$

Since $\nabla_x \mu_t \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, with norm bounded uniformly in N and t, we easily find

$$|\langle \xi, M_2 \xi \rangle| \leq C N^{-1/2} \|(\mathcal{N}+1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\|$$

To control the term M_1, on the other hand, we integrate by parts. We obtain

$$\begin{aligned}
\mathrm{M}_1 &= \frac{1}{2N} \int dxdydzdw\, \eta_t(z;y)(-\Delta_x k_t)(x;w) b_y^* a_z^* a_w b_x \\
&= \frac{N^2}{2} \int dxdydzdw\, \eta_t(z;y)(\Delta w_\ell)(N(x-w)) \widetilde{\varphi}_{\xi_t}(x) \widetilde{\varphi}_{\xi_t}(w) b_y^* a_z^* a_w b_x \\
&\quad + \frac{N}{2} \int dxdydzdw \eta_t(z;y)(\nabla w_\ell)(N(x-w)) \nabla \widetilde{\varphi}_{\xi_t}(x) \widetilde{\varphi}_{\xi_t}(w) b_y^* a_z^* a_w b_x \\
&\quad + \frac{1}{2} \int dxdydzdw\, \eta_t(z;y) w_\ell(N(x-w)) \Delta \widetilde{\varphi}_{\xi_t}(x) \widetilde{\varphi}_{\xi_t}(w) b_y^* a_z^* a_w b_x \\
&= \mathrm{M}_{11} + \mathrm{M}_{12} + \mathrm{M}_{13}
\end{aligned} \tag{3.162}$$

Since $|(\nabla w_\ell)(Nx)| \leq C/(N^2|x|^2)$, we have

$$\begin{aligned}
|\langle \xi, \mathrm{M}_{12} \xi \rangle| &\leq C N^{-1} \int dxdydzdw\, |\eta_t(z;y)| \frac{|\nabla \widetilde{\varphi}_{\xi_t}(x)| |\widetilde{\varphi}_{\xi_t}(w)|}{|x-w|^2} \|a_z b_y \xi\| \|a_w b_x \xi\| \\
&\leq C N^{-1} \left[\int dxdydzdw\, \frac{|\nabla \widetilde{\varphi}_{\xi_t}(x)|^2 |\widetilde{\varphi}_{\xi_t}(w)|^2}{|x-w|^2} \|a_z b_y \xi\|^2 \right]^{1/2} \\
&\qquad\qquad \times \left[\int dxdydzdw\, \frac{|\eta_t(y;z)|^2}{|x-w|^2} \|a_w b_x \xi\|^2 \right]^{1/2} \\
&\leq C N^{-1} \|\eta_t\| \|(\mathcal{N}+1)\xi\| \|(\mathcal{N}+1)^{1/2} (\mathcal{K}+\mathcal{N})^{1/2} \xi\| \\
&\leq C \|(\mathcal{N}+1)^{1/2} \xi\| \|(\mathcal{K}+\mathcal{N})^{1/2} \xi\|
\end{aligned}$$

where we used Hardy's inequality $|x|^{-2} \leq C(1-\Delta)$. The expectation of M_{13} can be bounded analogously. Let us focus now on the term M_{11}. Here we use the fact that $f_\ell = 1 - w_\ell$ solves the Neumann problem (3.61) to write

$$\begin{aligned}
\mathrm{M}_{11} &= -\frac{N^2}{2} \int dxdydzdw\, \eta_t(z;y) V(N(x-w)) f_\ell(N(x-w)) \widetilde{\varphi}_{\xi_t}(x) \widetilde{\varphi}_{\xi_t}(w) b_y^* a_z^* a_w b_x \\
&\quad + N^2 \lambda_\ell \int dxdydzdw\, \eta_t(z;y) f_\ell(N(x-w)) \chi(|x-w| \leq \ell) \widetilde{\varphi}_{\xi_t}(x) \widetilde{\varphi}_{\xi_t}(w) b_y^* a_z^* a_w b_x \\
&=: \mathrm{M}_{111} + \mathrm{M}_{112}
\end{aligned} \tag{3.163}$$

Since, by Lemma 4.3.1, $\lambda_\ell \leq C N^{-3}$ and $0 \leq f_\ell \leq 1$, it is easy to check that

$$|\langle \xi, \mathrm{M}_{112} \xi \rangle| \leq C \|(\mathcal{N}+1)^{1/2} \xi\|^2$$

As for the first term on the r.h.s. of (3.163), it can be estimated by

$$\begin{aligned}|\langle\xi, \mathrm{M}_{111}\xi\rangle| &\leq \int dxdydzdw|\eta_t(z;y)|N^2V(N(x-w))|\widetilde{\varphi}_{\xi_t}(w)||\widetilde{\varphi}_{\xi_t}(x)|\|a_zb_y\xi\|\|a_wb_x\xi\| \\ &\leq \left[\int dxdydzdw|\eta_t(z;y)|^2N^2V(N(x-w))\|a_wb_x\xi\|^2\right]^{1/2} \\ &\qquad\qquad \times\left[\int dxdydzdw\, N^2V(N(x-w))|\widetilde{\varphi}_{\xi_t}(w)|^2|\widetilde{\varphi}_{\xi_t}(x)|^2\|a_zb_y\xi\|^2\right]^{1/2} \\ &\leq CN^{-1/2}\|\eta_t\|\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}+1)\xi\| \leq C\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi\|\end{aligned}$$

where we used the fact that $0 \leq f_\ell \leq 1$ and the notation (3.152).

Summarizing, we have shown that the expectation of the fourth term on the r.h.s. of (3.153) can be bounded by

$$\left|\frac{1}{2N}\int dxdydz\,\eta_t(y;z)\langle\xi, b(\nabla_x\eta_x)b_y^*a_z^*\nabla_xa_x\xi\rangle\right| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{K}+\mathcal{N}+\mathcal{V}_N+1)^{1/2}\xi\| \tag{3.164}$$

Also in this case, it is also easy to check that the same estimate holds true for the expectation of the commutator of the fourth term on the r.h.s. of (3.153) with $\mathcal{N}$ and with $a^*(g_1)a(g_2)$ and for the expectation of its time-derivative.

Finally, we have to deal with the last term on the r.h.s. of (3.153). According to Lemma 3.2.3, the operator

$$\int dx\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b(\nabla_x\eta_x))\mathrm{ad}^{(k)}_{B(\eta_t)}(\nabla_xb_x)$$

is given by the sum of $2^{n+k}n!k!$ terms, all having the form

$$\begin{aligned}\mathrm{E} := \int dx\,\Lambda_1\ldots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\nabla_x\eta^{(\ell_1+1)}_{x,\Diamond}) \\ \times\Lambda'_1\ldots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\ldots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\nabla_x\eta^{(\ell_2)}_{x,\Diamond'})\end{aligned} \tag{3.165}$$

with $k_1, k_2, \ell_1, \ell_2 \geq 0$, $j_1, \ldots, j_{k_1}, m_1, \ldots, m_{k_2} \geq 1$, and where each operator Λ_i or Λ'_i is either a factor $(N-\mathcal{N})/N$, $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1},\ldots,\eta^{(q_p)}_{t,\underline{\natural}_p}) \tag{3.166}$$

with $p, q_1, \ldots, q_p \geq 1$. Here we used the fact that $\eta^{(\ell_1)}_\natural(\nabla_x\eta_{x,\Diamond}) = \nabla_x\eta^{(\ell_1+1)}_{x,\Diamond'}$ for an appropriate choice of $\Diamond' \in \{\cdot,*\}^{\ell_1+1}$.

We study the expectation of a term of the form (3.165), distinguishing several cases, depending on the values of $\ell_1, \ell_2 \in \mathbb{N}$.

Case 1: $\ell_1 \geq 1$, $\ell_2 \geq 2$. In this case, $\nabla_x\eta^{(\ell_1+1)}_{t,\Diamond}, \nabla_x\eta^{(\ell_2)}_{t,\Diamond} \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$, with norm bounded uniformly in N and t. Hence, with Lemma 3.2.1, we can bound

$$|\langle\xi, \mathrm{E}\xi\rangle| \leq C^{k+n}\|\eta_t\|^{k+n-\ell_1-\ell_2}\|\nabla_x\eta_t^{(\ell_1+1)}\|\|\nabla_x\eta_t^{(\ell_2)}\|\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

Now we observe that, for example,

$$\|\nabla_x \eta_t^{(\ell_2)}\| \le \|\nabla_x \eta_t^{(2)}\| \|\eta_t^{(\ell_2-2)}\| \le \|\nabla_x \eta_t^{(2)}\| \|\eta_t\|^{\ell_2-2} \le C\|\eta_t\|^{\ell_2-2}$$

Similarly, $\|\nabla_x \eta_t^{(\ell_1+1)}\| \le C\|\eta_t\|^{\ell_1-1}$. Hence, in this case,

$$|\langle \xi, \mathrm{E}\xi\rangle| \le C^{k+n}\|\eta_t\|^{k+n-3}\|(\mathcal{N}+1)^{1/2}\xi\|^2 .$$

Case 2: $\ell_1 \ge 1$, $\ell_2 = 1$. In this case we integrate by parts, writing

$$\begin{aligned}\langle \xi, \mathrm{E}\xi\rangle = \int dx\, \langle \xi, \Lambda_1 \dots \Lambda_{i_1} N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; -\Delta_x \eta^{(\ell_1+1)}_{x,\Diamond}) \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta_{x,\Diamond'})\xi\rangle\end{aligned}$$

Since, by Lemma 3.3.3, $\|\Delta_x \eta_t^{(2)}\| \le Ce^{c|t|}$, we conclude by Lemma 3.2.1 that, in this case,

$$|\langle \xi, \mathrm{E}\xi\rangle| \le C^{k+n}\|\eta_t\|^{k+n-1}\|\Delta_x\eta_t^{(2)}\|\|(\mathcal{N}+1)^{1/2}\xi\|^2 \le C^{k+n}e^{c|t|}\|\eta_t\|^{k+n-1}\|(\mathcal{N}+1)^{1/2}\xi\|^2 .$$

Case 3: $\ell_1 \ge 1$, $\ell_2 = 0$. In this case, the second $\Pi^{(1)}$-operator in (3.165) has the form

$$\begin{aligned}&N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \nabla_x \delta_x) \\ &\qquad = N^{-k_2}\int b^{\flat_0}_{x_1} \prod_{j=1}^{k_2-1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_{k_2}}_{y_{k_2}} \nabla_x a_x \prod_{j=1}^{k_2} \eta^{(m_j)}_{t,\natural'_j}(x_j;y_j)dx_j dy_j\end{aligned}$$

Here we used part v) of Lemma 3.2.3 to conclude that the last field on the right, the one carrying the derivative, must be an annihilation operator (or possibly a b-operator). Repeatedly applying Lemma 1.2.1 on pairs of creation and annihilation operators, but leaving the last annihilation operator $\nabla_x a_x$ untouched, we find

$$\begin{aligned}|\langle \xi, E\xi\rangle| &\le C^{k+n}\|\eta_t\|^{k+n-\ell_1}\|(\mathcal{N}+1)^{1/2}\xi\| \int dx\, \|\nabla_x \eta_x^{(\ell_1+1)}\|\|\nabla_x a_x \xi\| \\ &\le C^{k+n}\|\eta_t\|^{k+n-\ell_1}\|\nabla_x\eta_t^{(\ell_1+1)}\|\|(\mathcal{N}+1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\| \\ &\le C^{k+n}\|\eta_t\|^{k+n-1}\|(\mathcal{N}+1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\end{aligned}$$

Case 4: $\ell_1 = 0$, $\ell_2 \ge 2$. Here we proceed as in Case 2, integrating by parts and moving the derivative over x from $\nabla_x \eta_{x,\Diamond}$ (whose L^2 norm blows up) to $\nabla_x \eta^{(\ell_2)}_{x,\Diamond'}$ (using the fact that $\|\Delta_x \eta_t^{(2)}\| < \infty$).

Case 5: $\ell_1 = 0$, $\ell_2 = 1$. In this case, by part v) of Lemma 3.2.3, the two $\Pi^{(1)}$-operators in (3.165) have the form

$$\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \nabla_x \eta^{(\ell_1+1)}_{x,\Diamond}) = \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} a(\nabla_x \eta_x) \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i;y_i)dx_i dy_i \tag{3.167}$$

and

$$\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\nabla_x\eta^{(\ell_2)}_{x,\Diamond'}) = \int b^{\flat'_0}_{x_1}\prod_{j=1}^{k_2} a^{\sharp'_j}_{y_j} a^{\flat'_j}_{x_{j+1}} a^{\sharp'_n}_{y_n} a^*(\nabla_x\eta_x)\prod_{i=1}^{k_2}\eta^{(m_i)}_{t,\natural_i}(x_i;y_i)dx_i dy_i \tag{3.168}$$

Since $\|\nabla_x\eta_t\| \simeq N^{1/2}$ blows up as $N\to\infty$, to estimate (3.165) in this case we first have to commute the annihilation operator $a(\nabla_x\eta_{x,\Diamond})$ in (3.167) with the creation operator $a^*(\nabla_x\eta_{x,\Diamond'})$ in (3.168). We proceed similarly as we did to bound the second term on the r.h.s. of (3.153) in the case $n=0$, $k=1$, starting in (3.157). Here, however, we first have to commute the annihilation operator $a(\nabla_x\eta_{x,\Diamond})$ through the Λ'_i operators and through the creation operators in (3.168).

If $\Lambda'_i = (N-\mathcal{N})/N$ or $\lambda'_i = (N+1-\mathcal{N})/N$, we just pull the annihilation operator $a(\nabla_x\eta_{x,\Diamond})$ through, using the fact that $a(\nabla_x\eta_{x,\Diamond})\mathcal{N} = (\mathcal{N}+1)a(\nabla_x\eta_{x,\Diamond})$. On the other hand, to commute $a(\nabla_x\eta_{x,\Diamond})$ through the Λ'_i operators having the form (3.166) and through the creation operators in (3.168) (excluding the very last one on the right), we use the canonical commutation relations (1.6). The important observation here is the fact that every creation operator appearing in (3.166) and in (3.168) is associated with an η_t-kernel; the commutator produces a new creation or annihilation operator, this time with a wave function whose L^2-norm remains bounded, uniformly in N. For example, we have

$$\left[a(\nabla_x\eta_x), \int a^*_{x_i}a_{y_i}\eta^{(m_i)}(x_i;y_i)dx_idy_i\right] = a(\nabla_x\eta_x^{(m_i+1)}) \tag{3.169}$$

Since $m_i+1\geq 2$, $\|\nabla_x\eta^{(m_i+1)}\|\leq C$, uniformly in N. Similar formulas hold for commutators of $a(\nabla_x\eta_x)$ with a pair of not normally ordered creation and annihilation operators or with the product of two creation operators. In fact, not only the L^2-norm but even the H^1-norm of the wave function of the annihilation operator on the r.h.s. of (3.169) is bounded, uniformly in N. This means that terms resulting from commutators like (3.169) can be bounded integrating by parts and moving the derivative in (3.168) to the argument of the annihilation operator in (3.169). We conclude that $E = \mathrm{F}_1+\mathrm{F}_2$, where

$$\begin{aligned}\mathrm{F}_1 = \int dx\,\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\int b^{\flat_0}_{x_1}\prod_{i=1}^{k_1} a^{\sharp_j}_{y_j}a^{\flat_j}_{x_{j+1}}a^{\sharp_n}_{y_n}\prod_{i=1}^{k_1}\eta^{(j_i)}_{t,\natural_i}(x_i;y_i)dx_idy_i\\ \times\Lambda'_1\dots\Lambda'_{i_2}N^{-k_2}\int b^{\flat'_0}_{x'_1}\prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j}a^{\flat'_j}_{x'_{j+1}}a^{\sharp'_n}_{y'_n}\prod_{i=1}^{k_1}\eta^{(j_i)}_{t,\natural_i}(x'_i;y'_i)dx'_idy'_i\\ \times a(\nabla_x\eta_{x,\Diamond})a^*(\nabla_x\eta_{x,\Diamond'})\end{aligned}$$

while F_2, which contains the contribution of all commutators, is bounded by

$$|\langle\xi,\mathrm{F}_2\xi\rangle| \leq nC^{k+n}\|\eta_t\|^{k+n-1}\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

To estimate F_1, we write it as $\mathrm{F}_1 = \mathrm{F}_{11} + \mathrm{F}_{12}$, with

$$\begin{aligned} \mathrm{F}_{11} = \|\nabla_x \eta_t\|^2 \, \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i; y_i) dx_i dy_i \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j} a^{\flat'_j}_{x'_{j+1}} a^{\sharp'_n}_{y'_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x'_i; y'_i) dx'_i dy'_i \end{aligned} \tag{3.170}$$

and

$$\begin{aligned} \mathrm{F}_{12} = \int dx \, \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i; y_i) dx_i dy_i \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j} a^{\flat'_j}_{x'_{j+1}} a^{\sharp'_n}_{y'_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x'_i; y'_i) dx'_i dy'_i \\ \times a^*(\nabla_x \eta_{x,\diamondsuit'}) a(\nabla_x \eta_{x,\diamondsuit}) \end{aligned} \tag{3.171}$$

The contribution F_{11} can be estimated by

$$|F_{11}| \leq C^{k+n} \|\eta_t\|^{k+n-1} \|\nabla_x \eta_t\|^2 N^{-\alpha} \|(\mathcal{N}+1)^{\alpha/2} \xi\|^2 \tag{3.172}$$

where $\alpha = k_1 + p_1 + \dots + p_r + k_2 + p'_1 + \dots + p'_{r'}$, if r of the operators $\Lambda_1, \dots, \Lambda_{i_1}$ and r' of the operators $\Lambda'_1, \dots, \Lambda'_{i_2}$ are $\Pi^{(2)}$-operators of the form (3.166), with orders $p_1, \dots, p_r > 0$ and, respectively, $p'_1, \dots, p'_{r'} > 0$. Now observe that, since $\ell_2 = 1$, we must have $k \geq 1$. Since we are excluding here the case $n = 0, k = 1$, we must either have $n \geq 1$ and $k = 1$, or $k \geq 2$. In both cases $k + n \geq 2$. According to Lemma 3.2.3, the total number of η_t-kernels in every term of the form (3.165) is equal to $k + n + 1 \geq 3$. This implies that there is at least one η_t-kernel, additional to the two η_t-kernels which produced the commutator $\|\nabla_x \eta_t\|^2$ in (3.170). We conclude that, in (3.172), we have $\alpha \geq 1$, and therefore, on $\mathcal{F}^{\leq N}$,

$$|\mathrm{F}_{11}| \leq C^{k+n} \|\eta_t\|^{k+n-1} \|\nabla_x \eta_t\|^2 N^{-1} \|(\mathcal{N}+1)^{1/2} \xi\|^2 \leq C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2} \xi\|^2$$

since $\|\nabla_x \eta_t\|^2 \leq CN$ by Lemma 3.3.3. To control F_{12} we notice that, with the operator D defined in (3.158),

$$0 \leq \int dx \, a^*(\nabla_x \eta_{x,\diamondsuit'}) a(\nabla_x \eta_{x,\diamondsuit}) = d\Gamma(D) \leq \|D\|_2 \mathcal{N} \leq C\mathcal{N}$$

This easily implies that

$$|\langle \xi, \mathrm{F}_{12} \xi \rangle| \leq C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2} \xi\|^2$$

We conclude that, in this case,

$$|\langle \xi, \mathrm{E} \xi \rangle| \leq n C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2} \xi\|^2$$

Case 6: $\ell_1 = 0$, $\ell_2 = 0$. In this case, the term (3.165) has the form

$$\begin{aligned}\mathrm{E} = \int dx\, \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} a^{\sharp_n}_{y_n} a(\nabla_x \eta_{x,\diamondsuit}) \prod_{i=1}^{k_1} \eta^{(j_i)}(x_i; y_i) dx_i dy_i \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_i}_{y'_i} a^{\flat'_i}_{x'_{i+1}} a^{\sharp'_n}_{y'_n} \nabla_x a_x \prod_{i=1}^{k_2} \eta^{(m_i)}(x'_i; y'_i) dx'_i dy'_i\end{aligned} \tag{3.173}$$

Notice that a term of this form (with $n = 0$ and $k = 1$) already appears in the fourth line of (3.153) and was studied starting in (3.160) (to be more precise, in this case the first $\Pi^{(1)}$-operator in (3.165) is of order zero (for $n = 0$, there is no other choice), and therefore the operator $a(\nabla_x \eta_{x,\diamondsuit})$ appearing in (3.173) is replaced by $b(\nabla_x \eta_{x,\diamondsuit})$). We will bound (3.173) following the same strategy used in (3.160). First we have to commute the operator $a(\nabla_x \eta_{x,\diamondsuit})$ in (3.173) to the right, close to the $\nabla_x a_x$ operator. As already explained in Case 5, the annihilation and creation operators produced while commuting $a(\nabla_x \eta_{x,\diamondsuit})$ to the right will have wave function with H^1-norm bounded, uniformly in N. Integrating by parts over x, we obtain $\mathrm{E} = \mathrm{G}_1 + \mathrm{G}_2$, with

$$\begin{aligned}\mathrm{G}_1 = \int dx\, \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i; y_i) dx_i dy_i \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j} a^{\flat'_j}_{x'_{j+1}} a^{\sharp'_n}_{y'_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x'_i; y'_i) dx'_i dy'_i\, a(\nabla_x \eta_{x,\diamondsuit}) \nabla_x a_x\end{aligned}$$

and

$$|\langle \xi, \mathrm{G}_2 \xi \rangle| \leq n C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2} \xi\|^2$$

To bound G_1, we proceed exactly as we did starting in (3.161). Decomposing $\eta_t = \mu_t + k_t$, and using the fact that $\nabla_x \mu_t$ has bounded L^2-norm, uniformly in N, we conclude that $\mathrm{G}_1 = \mathrm{G}_{11} + \mathrm{G}_{12}$, with

$$\begin{aligned}\mathrm{G}_{11} = \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i; y_i) dx_i dy_i \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j} a^{\flat'_j}_{x'_{j+1}} a^{\sharp'_n}_{y'_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x'_i; y'_i) dx'_i dy'_i \\ \times \int dx\, (-\Delta_x k_t)(x; y)\, a_x a_y\end{aligned}$$

and

$$|\langle \xi, \mathrm{G}_{12} \xi \rangle| \leq C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\|$$

By Cauchy-Schwarz, the term G_{11} is bounded by

$$|\langle \xi, \mathrm{G}_{11} \xi \rangle| \leq C^{k+n} \|\eta_t\|^{k+n-1} N^{-\alpha} \|(\mathcal{N}+1)^{\alpha} \xi\| \int dx dy |\Delta_x k_t(x; y)| \|a_x a_y \xi\| \tag{3.174}$$

where $\alpha = k_1 + p_1 + \cdots + p_r + k_2 + p'_1 + \cdots + p'_{r'}$, if r of the operators $\Lambda_1, \dots, \Lambda_{i_1}$ and r' of the operators $\Lambda'_1, \dots, \Lambda'_{i_2}$ are $\Pi^{(2)}$-operators of the form (3.166), with orders $p_1, \dots, p_r > 0$ and, respectively, $p'_1, \dots, p'_r > 0$. The important observation now is that, since we excluded the case $k = n = 0$, we have $k + n \geq 1$, and therefore every term of the form (3.165) must have at least two η_t-kernels in it. This implies that, in (3.174), $\alpha \geq 1$, and therefore that

$$|\mathrm{G}_{11}| \leq C^{k+n} \|\eta_t\|^{k+n-1} N^{-1/2} \|(\mathcal{N}+1)^{1/2}\xi\| \int dxdy |\Delta_x k_t(x;y)| \|a_x a_y \xi\|$$

Proceeding as we did from (3.162) to (3.164), we conclude that

$$|\mathrm{G}_{11}| \leq C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{H}_N + \mathcal{N}+1)^{1/2}\xi\|$$

Summarizing, we proved that the last term on the r.h.s. of (3.153) is a sum over all $(k,n) \neq (0,0), (1,0)$ of $2^{n+k} n! k!$ terms of the form (3.165), each of them having expectation bounded by

$$|\langle \xi, \mathrm{E}\xi \rangle| \leq C^{k+n} e^{c|t|} \|\eta_t\|^{\max(0,k+n-3)} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{H}_N + \mathcal{N}+1)^{1/2}\xi\|$$

Similarly, one can show that

$$\begin{aligned}
|\langle \xi, [\mathcal{N}, \mathrm{E}]\xi \rangle| &\leq C^{k+n} e^{c|t|} \|\eta_t\|^{\max(0,k+n-3)} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{H}_N + \mathcal{N}+1)^{1/2}\xi\| \\
|\langle \xi, [a^*(g_1)a(g_2), \mathrm{E}]\xi \rangle| &\leq C^{k+n} e^{c|t|} \|\eta_t\|^{\max(0,k+n-3)} \|g_1\|_{H^1} \|g_2\|_{H^1} \\
&\qquad \times \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{H}_N + \mathcal{N}+1)^{1/2}\xi\| \\
|\langle \xi, \partial_t[\mathrm{E}]\xi \rangle| &\leq C^{k+n} e^{c|t|} \|\eta_t\|^{\max(0,k+n-3)} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{H}_N + \mathcal{N}+1)^{1/2}\xi\|
\end{aligned}$$

Inserting in (3.153) we conclude that, if $\sup_{t\in\mathbb{R}} \|\eta_t\|$ is small enough, the operator $\mathcal{E}^{(K)}_{N,t}$ defined in (3.150) satisfies the bounds in (3.151). □

Analysis of $e^{-B(\eta_t)}(\mathcal{L}^{(2)}_{N,t} - \mathcal{K})e^{B(\eta_t)}$

Recall that

$$\begin{aligned}
\mathcal{L}^{(2)}_{N,t} - \mathcal{K} = & \int dx (N^3 V(N.) * |\widetilde{\varphi}_{\xi_t}|^2)(x) \left[b^*_x b_x - N^{-1} a^*_x a_x \right] \\
& + \int dxdy N^3 V(N(x-y)) \widetilde{\varphi}_{\xi_t}(x) \bar{\widetilde{\varphi}}_{\xi_t}(y) \left[b^*_x b_y - N^{-1} a^*_x a_y \right] \\
& + \frac{1}{2} \int dxdy\, N^3 V(N(x-y)) \left[\widetilde{\varphi}_{\xi_t}(x) \widetilde{\varphi}_{\xi_t}(y) b^*_x b^*_y + \text{h.c.} \right]
\end{aligned} \tag{3.175}$$

We define the error term $\mathcal{E}^{(2)}_{N,t}$ through the equation

$$\begin{aligned}
e^{-B(\eta_t)}(\mathcal{L}^{(2)}_{N,t} - \mathcal{K})e^{B(\eta_t)} = & \,\mathrm{Re} \int dxdy\, N^3 V(N(x-y)) \bar{\widetilde{\varphi}}_{\xi_t}(x) \bar{\widetilde{\varphi}}_{\xi_t}(y) k_t(y;x) \\
& + \frac{1}{2} \int dxdy\, N^3 V(N(x-y)) \left[\widetilde{\varphi}_{\xi_t}(x) \widetilde{\varphi}_{\xi_t}(y) b^*_x b^*_y + \text{h.c.} \right] \\
& + \mathcal{E}^{(2)}_{N,t}
\end{aligned} \tag{3.176}$$

The properties of the error term $\mathcal{E}^{(2)}_{N,t}$ are described in the next proposition.

Proposition 3.4.7. *Under the same assumptions as in Theorem 3.3.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned}
\left|\left\langle \xi, \mathcal{E}^{(2)}_{N,t}\xi\right\rangle\right| &\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\| \\
\left|\left\langle \xi, \left[\mathcal{N}, \mathcal{E}^{(2)}_{N,t}\right]\xi\right\rangle\right| &\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\| \\
\left|\left\langle \xi, \left[a^*(g_1)a(g_2), \mathcal{E}^{(2)}_{N,t}\right]\xi\ \right\rangle\right| &\leq Ce^{c|t|}\|g_1\|_{H^2}\|g_2\|_{H^2}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N^{1/2}+\mathcal{N}+1)^{1/2}\xi\| \\
\left|\partial_t\left\langle \xi, \mathcal{E}^{(2)}_{N,t}\xi\right\rangle\right| &\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N^{1/2}+\mathcal{N}+1)^{1/2}\xi\|
\end{aligned} \tag{3.177}$$

for all $\xi \in \mathcal{F}^{\leq N}$.

Proof. The conjugation of the first two terms on the r.h.s of (3.175) can be controlled with Lemma 3.4.4, taking r to be multiplication operator with the convolution $N^3V(N.)*|\widetilde{\varphi}_{\xi_t}|^2$ in the first case (so that $\|r\|_{\text{op}} = \|N^3V(N.)*|\widetilde{\varphi}_{\xi_t}|^2\|_\infty \leq C\|\widetilde{\varphi}_{\xi_t}\|^2_\infty \leq Ce^{c|t|}$) and the operator with integral kernel $r(x;y) = N^3V(N(x-y))\widetilde{\varphi}_{\xi_t}(x)\widetilde{\varphi}_{\xi_t}(y)$ in the second case (then $\|r\|_{\text{op}} \leq \sup_x \int |r(x;y)|dy \leq Ce^{c|t|}$, uniformly in N). Hence, to show Prop. 3.4.7 it is enough to prove the bounds (3.177), with $\mathcal{E}^{(2)}_{N,t}$ replaced by

$$\begin{aligned}
\widetilde{\mathcal{E}}^{(2)}_{N,t} = \frac{1}{2}\int dxdy\, N^3V(N(x-y))\left[\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)e^{-B(\eta_t)}b_xb_ye^{B(\eta_t)} + \text{h.c.}\right] \\
- \text{Re}\int dxdy\, N^3V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)k_t(x;y) \\
- \frac{1}{2}\int dxdy N^3V(N(x-y))\left[\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)b_xb_y + \text{h.c.}\right]
\end{aligned} \tag{3.178}$$

By Lemma 3.2.4, we can write

$$\begin{aligned}
&\int dxdy\, N^3V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)e^{-B(\eta_t)}b_xb_ye^{B(\eta_t)} \\
&\quad= \sum_{n,k\geq 0}\frac{(-1)^{k+n}}{k!n!}\int dxdy N^3V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)\text{ad}^{(n)}_{B(\eta_t)}(b_x)\text{ad}^{(k)}_{B(\eta_t)}(b_y) \\
&\quad= \int dxdy N^3V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)b_xb_y \\
&\qquad- \int dxdy N^3V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)b_x[B(\eta_t), b_y] \\
&\qquad+ \sum^*_{n,k}\frac{(-1)^{k+n}}{k!n!}\int dxdy N^3V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)\text{ad}^{(n)}_{B(\eta_t)}(b_x)\text{ad}^{(k)}_{B(\eta_t)}(b_y)
\end{aligned} \tag{3.179}$$

where we isolated the terms with $(n,k) = (0,0)$ and $(n,k) = (0,1)$ and the sum $\sum^*$ runs over all other pairs $(n,k) \in \mathbb{N} \times \mathbb{N}$. The first term on the r.h.s. of (3.179) (the one associated with $(k,n) = (0,0)$) is subtracted in (3.178) and does not enter the error term $\widetilde{\mathcal{E}}^{(2)}_{N,t}$. The second term on the r.h.s. of (3.179), on the other hand, is given by

$$\begin{aligned}
\mathrm{P} := & -\int dxdy\, N^3 V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y) b_x [B(\eta_t), b_y] \\
= & \frac{N-1-\mathcal{N}}{N} \int dxdy\, N^3 V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)\, b_x b^*(\eta_y) \\
& - \frac{1}{N} \int dxdydwdz\, N^3 V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)\, \eta_t(z;w)\, b_x b^*_z a^*_w a_y
\end{aligned}$$

Commuting in both terms the annihilation field b_x to the right, we find

$$\begin{aligned}
\mathrm{P} = & \frac{N-1-\mathcal{N}}{N}\frac{N-\mathcal{N}}{N} \int dxdy\, N^3 V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)\, \eta_t(x;y) \\
& + \frac{N-1-\mathcal{N}}{N} \int dxdydz\, N^3 V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y) \left[b^*(\eta_y) b_x - \frac{1}{N} a^*(\eta_y) a_x \right] \\
& - 2\frac{N-\mathcal{N}}{N^2} \int dxdydz\, N^3 V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)\, a^*(\eta_y) a_x \\
& - \frac{N-\mathcal{N}}{N^2} \int dxdydzdw\, N^3 V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)\, \eta_t(z;w) a^*_w a^*_z a_x a_y \\
=: & \, \mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3 + \mathrm{P}_4
\end{aligned} \tag{3.180}$$

Writing $\eta_t = k_t + \mu_t$, and using the pointwise bounds $|\mu_t(x;y)| \leq C|\widetilde{\varphi}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)|$ and $|k_t(x;y)| \leq CN|\widetilde{\varphi}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)|$ from Lemma 3.3.3, we obtain that

$$\left| \langle \xi, \mathrm{P}_1 \xi \rangle - \int dxdy N^3 V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y) k_t(x;y) \right| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

The expectation of the operator P_2, and analogously the expectation of the operator P_3, can be bounded by

$$\begin{aligned}
|\langle \xi, \mathrm{P}_2 \xi \rangle| \leq & \|(\mathcal{N}+1)^{1/2}\xi\| \int dxdy N^3 V(N(x-y)) |\widetilde{\varphi}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)| \|\eta_y\| \|b_x \xi\| \\
\leq & \|\widetilde{\varphi}_{\xi_t}\|^2_\infty \, \|(\mathcal{N}+1)^{1/2}\xi\| \left[\int dxdy N^3 V(N(x-y)) \|\eta_y\|^2 \right]^{1/2} \\
& \times \left[\int dxdy N^3 V(N(x-y)) \|b_x \xi\|^2 \right]^{1/2} \\
\leq & \, Ce^{c|t|} \|\eta_t\| \|(\mathcal{N}+1)^{1/2}\xi\|^2
\end{aligned}$$

As for the last term on the r.h.s. of (3.180), its expectation is estimated by

$$\begin{aligned}|\langle\xi, \mathrm{P}_3\xi\rangle| &\leq \|\eta_t\|\|(\mathcal{N}+1)\xi\| \int dxdy N^2 V(N(x-y))|\widetilde{\varphi}_{\xi_t}(x)||\widetilde{\varphi}_{\xi_t}(y)|\|a_x a_y\xi\| \\ &\leq \|\eta_t\|\|(\mathcal{N}+1)\xi\| \left[\int dxdy N^2 V(N(x-y))\|a_x a_y \xi\|^2\right]^{1/2} \\ &\qquad \times \left[\int dxdy N^2 V(N(x-y))|\widetilde{\varphi}_{\xi_t}(x)|^2|\widetilde{\varphi}_{\xi_t}(y)|^2\right]^{1/2} \\ &\leq C\|\eta_t\|\|(\mathcal{N}+1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\end{aligned}$$

We conclude that

$$\begin{aligned}\Big|\langle\xi, \mathrm{P}\xi\rangle - &\int dxdy N^3 V(N(x-y))\bar{\widetilde{\varphi}}_{\xi_t}(x)\bar{\widetilde{\varphi}}_{\xi_t}(y)k_t(x;y)\Big| \\ &\leq Ce^{c|t|}\|\eta_t\|\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|\end{aligned} \tag{3.181}$$

Let us now consider the terms in the sum on the last line of (3.179), where we excluded the pairs $(k,n)=(0,0)$ and $(k,n)=(0,1)$. By Lemma 3.2.3, the operator

$$\int dxdy N^3 V(N(x-y))\widetilde{\varphi}_{\xi_t}(x)\widetilde{\varphi}_{\xi_t}(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y) \tag{3.182}$$

can be expressed as the sum of $2^{n+k}n!k!$ terms having the form

$$\begin{aligned}\mathrm{E} = \int dxdy N^3 V(N(x-y))\widetilde{\varphi}_{\xi_t}(x)\widetilde{\varphi}_{\xi_t}(y)\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{x,\diamondsuit}) \\ \times \Lambda'_1\dots\Lambda'_{i_2}N^{-k_2}\Pi^{(2)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{y,\diamondsuit'})\end{aligned} \tag{3.183}$$

where $k_1,k_2,i_1,i_2\geq 0$, $j_1,\dots,j_{k_1},m_1,\dots,m_{k_2}>0$ and where each Λ_i and Λ'_i is either a factor $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\,\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1},\dots,\eta^{(q_p)}_{t,\underline{\natural}_p}) \tag{3.184}$$

With Lemma 3.4.1, we obtain

$$\begin{aligned}|\langle\xi, \mathrm{E}\xi\rangle| &\leq \|(\mathcal{N}+1)^{1/2}\xi\| \int dxdy N^3 V(N(x-y))\widetilde{\varphi}_{\xi_t}(x)\widetilde{\varphi}_{\xi_t}(y) \\ &\qquad \times \Big\|(\mathcal{N}+1)^{-1/2}\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{x,\diamondsuit}) \\ &\qquad\qquad \times \Lambda'_1\dots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{y,\diamondsuit'})\xi\Big\| \\ &\leq C^{k+n}\|\eta_t\|^{n+k-2}\|(\mathcal{N}+1)^{1/2}\xi\| \int dxdy N^3 V(N(x-y))\widetilde{\varphi}_{\xi_t}(x)\widetilde{\varphi}_{\xi_t}(y) \\ &\qquad \times \Big\{n\|\eta_x\|\|\eta_y\|\|(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|\|\eta_y\|\|a_x\xi\| \\ &\qquad\qquad + Ce^{c|t|}\|\eta_t\|\|(\mathcal{N}+1)^{1/2}\xi\| + N^{-1/2}\|\eta_t\|^2\|a_x a_y\xi\|\Big\}\end{aligned}$$

where (in the last term in the parenthesis) we used the pointwise bound $N^{-1}|\eta_t(x;y)| \leq Ce^{c|t|}$ from Lemma 3.3.3. The contribution of the first three terms in the parenthesis can be bounded by Cauchy-Schwarz, since $\|\widetilde{\varphi}_{\xi_t}\|_\infty \leq Ce^{c|t|}$. We find

$$|\langle \xi, \mathrm{E}\xi \rangle| \leq C^{k+n} n e^{c|t|} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\|$$

Since the expectation of (3.182) is the sum of $2^{n+k} k! n!$ such contributions, inserting in (3.179) and taking into account also (3.181), we conclude that

$$\left| \langle \xi, \widetilde{\mathcal{E}}^{(2)}_{N,t} \xi \rangle \right| \leq Ce^{c|t|} \, \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\|$$

if $\sup_t \|\eta_t\|$ is small enough. As usual, we can prove similarly that the same bounds hold true for the expectation of the commutators of $\widetilde{\mathcal{E}}^{(2)}_{N,t}$ with the number of particles operator $\mathcal{N}$ and with $a^*(g_1)a(g_2)$, for arbitrary $g_1, g_2 \in H^2(\mathbb{R}^3)$ (this assumption allows us to extract $\|g_j\|_\infty \leq C\|g_j\|_{H^2}$) and also for the time derivative of $\widetilde{\mathcal{E}}^{(2)}_{N,t}$. □

3.4.5 Analysis of $e^{-B(\eta_t)} \mathcal{L}^{(3)}_{N,t} e^{B(\eta_t)}$

Recall from (3.90) that

$$\mathcal{L}^{(3)}_{N,t} = \int dxdy \, N^{5/2} V(N(x-y)) \widetilde{\varphi}_{\xi_t}(y) \left[b^*_x a^*_y a_x + \text{h.c.} \right]$$

We conjugate $\mathcal{L}^{(3)}_{N,t}$ with the unitary operator $e^{B(\eta_t)}$. We define the error term $\mathcal{E}^{(3)}_{N,t}$ through the equation

$$e^{-B(\eta_t)} \mathcal{L}^{(3)}_{N,t} e^{B(\eta_t)} = -\sqrt{N} \, \left[b(\cosh_{\eta_t}(h_{N,t})) + b^*(\sinh_{\eta_t}(\bar{h}_{N,t})) + \text{h.c.} \right] + \mathcal{E}^{(3)}_{N,t} \quad (3.185)$$

where we recall, from (3.132) that, $h_{N,t} = (N^3 V(N.) w_\ell(N.) * |\widetilde{\varphi}_{\xi_t}|^2) \widetilde{\varphi}_{\xi_t}$. In the next proposition we collect the important properties of the error term $\mathcal{E}^{(3)}_{N,t}$

Proposition 3.4.8. *Under the same assumptions as in Theorem 3.3.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned} \left| \langle \xi, \mathcal{E}^{(3)}_{N,t} \xi \rangle \right| &\leq Ce^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\| \\ \left| \langle \xi, \left[\mathcal{N}, \mathcal{E}^{(3)}_{N,t} \right] \xi \rangle \right| &\leq Ce^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\| \\ \left| \langle \xi, \left[a^*(g_1)a(g_2), \mathcal{E}^{(3)}_{N,t} \right] \xi \rangle \right| &\leq Ce^{c|t|} \|g_1\|_{H^2} \|g_2\|_{H^2} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\| \\ \left| \partial_t \langle \xi, \mathcal{E}^{(3)}_{N,t} \xi \rangle \right| &\leq Ce^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\| \end{aligned} \quad (3.186)$$

for all $\xi \in \mathcal{F}^{\leq N}$.

Proof. We start by writing

$$\begin{aligned}
e^{-B(\eta_t)}a_y^*a_xe^{B(\eta_t)} &= a_y^*a_x + \int_0^1 ds\, e^{-sB(\eta_t)}[a_y^*a_x, B(\eta_t)]e^{sB(\eta_t)} \\
&= a_y^*a_x + \int_0^1 e^{-sB(\eta_t)}\left[b_y^*b^*(\eta_x) + b(\eta_y)b_x\right]e^{sB(\eta_t)}
\end{aligned}$$

From Lemma 3.2.4, we conclude that

$$\begin{aligned}
e^{-B(\eta_t)}a_y^*a_xe^{B(\eta_t)} = a_y^*a_x + \sum_{k,r\geq 0} &\frac{(-1)^{k+r}}{k!r!(k+r+1)} \\
&\times \left[\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y^*)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) + \mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\right]
\end{aligned}$$

Inserting in the expression for $\mathcal{L}^{(3)}_{N,t}$, we conclude that

$$\begin{aligned}
&e^{-B(\eta_t)}\mathcal{L}^{(3)}_{N,t}e^{B(\eta_t)} \\
&\quad= \sum_{n\geq 0}\frac{(-1)^n}{n!}\int dxdyN^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)a_y^*a_x \\
&\qquad+ \sum_{n,k,r\geq 0}\frac{(-1)^{n+k+r}}{n!k!r!(k+r+1)}\int dxdyN^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*) \\
&\qquad\qquad\times\left[\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y^*)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) + \mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\right] \\
&\qquad+ \text{h.c.}
\end{aligned}$$

We divide the triple sum in several parts. We find

$$\begin{aligned}
&e^{-B(\eta_t)}\mathcal{L}^{(3)}_{N,t}e^{B(\eta_t)} \\
&\quad= \sum_{n\geq 0}\frac{(-1)^n}{n!}\int dxdyN^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)a_y^*a_x \\
&\qquad+ \sum_{n,r\geq 0}\frac{(-1)^{n+r}}{n!(r+1)!}\int dxdyN^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)b_y^*\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\
&\qquad+ \sum_{n,r\geq 0,k\geq 1}\frac{(-1)^{n+k+r}}{n!k!r!(k+r+1)}\int dxdyN^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y) \\
&\qquad\qquad\qquad\times\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)\mathrm{ad}^{(k)}(b_y^*)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\
&\qquad+ \sum_{n,r\geq 0,k\geq 1}\frac{(-1)^{n+k+r}}{n!k!r!(k+r+1)}\int dxdyN^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y) \\
&\qquad\qquad\qquad\times\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x) \\
&\qquad+ \text{h.c.}
\end{aligned}$$

In the terms with $k=0$, we distinguish furthermore the case $n=1$ from $n\neq 1$. We find

$$
\begin{aligned}
&e^{-B(\eta_t)}\mathcal{L}^{(3)}_{N,t}e^{B(\eta_t)}\\
&\quad = -\int dxdy N^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)[B(\eta_t),b^*_x]a^*_y a_x\\
&\qquad -\sum_{r\geq 0}\frac{(-1)^r}{(r+1)!}\int dxdy N^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)[B(\eta_t),b^*_x]b^*_y\,\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\\
&\qquad +\sum_{n\neq 1}\frac{(-1)^n}{n!}\int dxdy N^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)a^*_y a_x\\
&\qquad +\sum_{n\neq 1,r\geq 0}\frac{(-1)^{n+r}}{n!(r+1)!}\int dxdy N^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)b^*_y\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\\
&\qquad +\sum_{n,r\geq 0,k\geq 1}\frac{(-1)^{n+k+r}}{n!k!r!(k+r+1)}\int dxdy N^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)\\
&\qquad\qquad\qquad\times\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\\
&\qquad +\sum_{n,r\geq 0,k\geq 1}\frac{(-1)^{n+k+r}}{n!k!r!(k+r+1)}\int dxdy N^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)\\
&\qquad\qquad\qquad\times\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\\
&\qquad +\text{h.c.}
\end{aligned}
\tag{3.187}
$$

We start by estimating the contribution of the last term on the r.h.s. of (3.187). We are interested in the expectation

$$
\begin{aligned}
&\left|\int dxdy\, N^{5/2}V(N(x-y))\widetilde{\varphi_{\xi_t}}(y)\,\langle\xi,\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\xi\rangle\right|\\
&\qquad\leq\int dxdy\, N^{5/2}V(N(x-y))|\widetilde{\varphi_{\xi_t}}(y)|\|\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|\|\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\xi\|
\end{aligned}
$$

for $n,r\geq 0$ and $k\geq 1$. According to Lemma 3.2.4, the norm $\|\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ is bounded by the sum of $2^n n!$ terms of the form

$$\mathrm{P}_1=\|\Lambda_1\dots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(s)}_{x,\diamondsuit})\xi\|$$

for $i,k,s\geq 0$, $j_1,\dots,j_k\geq 1$, where each Λ_i is either a factor $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\sharp',\flat'}(\eta^{(q_1)}_{t,\natural'_1},\dots,\eta^{(q_p)}_{t,\natural'_p})\tag{3.188}$$

From Lemma 3.4.1, we find

$$\mathrm{P}_1\leq\begin{cases}C^n\|\eta\|^{n-1}\|\eta_x\|\|(\mathcal{N}+1)^{1/2}\xi\| & \text{if } s\geq 1\\ C^n\|\eta\|^n\|a_x\xi\| & \text{if } s=0\end{cases}\tag{3.189}$$

Similarly, the norm $\|\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\xi\|$ is bounded by the sum of $2^{k+r}k!r!$ terms having the form

$$\begin{aligned}\mathrm{P}_2 = \Big\| \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \eta^{(\ell_1+1)}_{y,\diamondsuit}) \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})\xi \Big\|\end{aligned}$$

which can be estimated (again with Lemma 3.4.1) by

$$\mathrm{P}_2 \le \begin{cases} C^{k+r}\|\eta_t\|^{k+r-2}\|\eta_x\|\|\eta_y\|\|(\mathcal{N}+1)\xi\| & \text{if } \ell_2 \ge 1 \\ C^{k+r}\|\eta_t\|^{k+r-1}\|\eta_y\|\|a_x(\mathcal{N}+1)^{1/2}\xi\| & \text{if } \ell_2 = 0 \end{cases}$$

Combining this estimate with (3.189), distinguishing different cases depending on the values of s and ℓ_2, and using the estimate $\sup_y \|\eta_y\| \le Ce^{c|t|} < \infty$ from Lemma 3.3.3, we easily find by Cauchy-Schwarz that

$$\begin{aligned}&\left| \int dxdy\, N^{5/2}V(N(x-y))\widetilde{\varphi}_{\xi_t}(y)\, \langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\xi\rangle \right| \\ &\qquad \le n!k!r!\, C^{n+k+r}N^{-1/2}\|\eta_t\|^{k+r-1}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{N}+1)\xi\| \\ &\qquad \le n!k!r!\, C^{n+k+r}\|\eta_t\|^{k+r-1}\|(\mathcal{N}+1)^{1/2}\xi\|^2\end{aligned} \tag{3.190}$$

for all $\xi \in \mathcal{F}^{\le N}$.

Let us now consider the fifth sum on the r.h.s. of (3.187). The expectation of every term in this sum is bounded by

$$\begin{aligned}&\left| \int dxdy\, N^{5/2}V(N(x-y))\widetilde{\varphi}_{\xi_t}(y)\langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\rangle \right| \\ &\quad \le \int dxdy\, N^{5/2}V(N(x-y))|\widetilde{\varphi}_{\xi_t}(y)|\, \|\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|\, \|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|\end{aligned} \tag{3.191}$$

where we assume $k \ge 1$, $n, r \ge 0$. According to Lemma 3.2.3, $\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|$ is bounded by the sum of $2^r r!$ terms of the form

$$\mathrm{Q}_1 = \|\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \eta^{(\ell_1+1)}_{x})\,\xi\|$$

for a $i_1, k_1, \ell_1 \ge 0$ and $j_1, \dots, j_{k_1} \ge 1$. Each Λ_i is either a factor $(N-\mathcal{N})/N$, a factor $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form (3.188). From Lemma 3.4.1, we have

$$\mathrm{Q}_1 \le C^r\|\eta_t\|^r\|\eta_x\|\|(\mathcal{N}+1)^{1/2}\xi\|$$

On the other hand, using again Lemma 3.2.3 the norm $\|\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ is bounded by the sum of $2^{n+k}k!n!$ terms having the form

$$\begin{aligned}\mathrm{Q}_2 = \|\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_2}}; \eta^{(\ell_1)}_{y,\diamondsuit}) \\ \times \Lambda'_1 \dots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})\xi\|\end{aligned}$$

where $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \geq 0$ and $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \geq 1$ and where each Λ_i and Λ'_i operator is either a factor $(N - \mathcal{N})/N$, $(N - \mathcal{N} + 1)/N$ or a $\Pi^{(2)}$-operator of the form (3.188). Using part iv) of Lemma 3.4.1, we obtain (using the assumption $k \geq 1$ to apply (3.108) and using (3.109) with $\alpha = 1$)

$$\begin{aligned} \mathrm{Q}_2 \leq C^{n+k} \|\eta_t\|^{n+k-2} \Big\{ & \big[(n+1)\|\eta_x\|\|\eta_y\| + \|\eta_t\| N^{-1} |\eta_t(x;y)\big] \, \|(\mathcal{N}+1)\xi\| \\ & + \|\eta_y\|\|\eta_t\|\|a_x(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|^2 \|a_x a_y \xi\| \Big\} \end{aligned}$$

With the bound $\sup_x \|\eta_x\|, \sup_{x,y} N^{-1}|\eta_t(x;y)| \leq Ce^{c|t|}$ from Lemma 3.3.3, we conclude that

$$\begin{aligned} & \left| \int dxdy \, N^{5/2} V(N(x-y)) \widetilde{\varphi_\xi}_t(y) \langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x) \mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y) \mathrm{ad}^{(r)}_{B(\eta_t)}(\eta_x)\xi \rangle \right| \\ & \qquad \leq n!k!r! \, C^{n+k+r} e^{c|t|} \|\eta_t\|^{n+k+r} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\| \end{aligned} \tag{3.192}$$

for all $\xi \in \mathcal{F}^{\leq N}$.

Let us now study the fourth term on the r.h.s. of (3.187). As we did for the other terms, we bound the expectation

$$\begin{aligned} & \left| \int dxdy \, N^{5/2} V(N(x-y)) \widetilde{\varphi_\xi}_t(y) \langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x) b^*_y \mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi \rangle \right| \\ & \qquad \leq \int dxdy \, N^{5/2} V(N(x-y)) |\widetilde{\varphi_\xi}_t(y)| \|b_y \mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\| \|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\| \end{aligned} \tag{3.193}$$

where we assume that $n \neq 1$, $r \geq 0$. According to Lemma 3.2.3, $\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|$ can be bounded by the sum of $2^r r!$ terms of the form

$$\mathrm{R}_1 = \|\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \eta^{(\ell_1+1)}_{x,\diamond})\xi\|$$

for $i_1, k_1, \ell_1 \geq 0$ and $j_1, \dots, j_{k_1} \geq 1$. According to Lemma 3.4.1, such a term can always be estimated by

$$\mathrm{R}_1 \leq C^r \|\eta_t\|^r \|\eta_x\| \|(\mathcal{N}+1)^{1/2}\xi\| \tag{3.194}$$

On the other hand, the norm $\|b_y \mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ can be bounded by the sum of $2^n n!$ contributions having the form

$$\mathrm{R}_2 = \|b_y \Lambda_1 \dots \Lambda_{i_1} \Pi^{(k_1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \eta^{(\ell_1)}_{x,\diamond})\xi\| \tag{3.195}$$

for $i_1, k_1, \ell_1 \geq 0$ and $j_1, \dots, j_{k_1} \geq 1$. With Lemma 3.4.1, we find that

$$\begin{aligned} \mathrm{R}_2 \leq C^n \|\eta_t\|^{n-2} \Big\{ & \big[(1+n/N)\|\eta_x\|\|\eta_y\| + \|\eta_t\| N^{-1} |\eta_t(x;y)|\big] \, \|(\mathcal{N}+1)\xi\| \\ & + \|\eta_t\|\|\eta_x\|\|a_y(\mathcal{N}+1)^{1/2}\xi\| + (n/N)\|\eta_t\|\|\eta_y\|\|a_x(\mathcal{N}+1)^{1/2}\xi\| \\ & + \|\eta_t\|^2 \|a_x a_y \xi\| \Big\} \end{aligned}$$

With $\|\widetilde{\varphi}_{\xi_t}\|_\infty \leq Ce^{c|t|}$ and $\sup_{x,y} N^{-1}|\eta_t(x;y)| \leq Ce^{c|t|}$ we conclude, similarly to (3.192), that

$$\begin{aligned}\left|\int dxdy\, N^{5/2}V(N(x-y))\widetilde{\varphi}_{\xi_t}(y)\langle\xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)b^*_y\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\rangle\right| \\ \leq (n+1)!r!C^{n+r}e^{c|t|}\|\eta_t\|^{r+n}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|\end{aligned} \tag{3.196}$$

The expectation of terms in the third sum on the r.h.s. of (3.187) are bounded by

$$\begin{aligned}&\left|\int dxdy\, N^{5/2}V(N(x-y))\widetilde{\varphi}_{\xi_t}(y)\langle\xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)a^*_ya_x\xi\rangle\right| \\ &\qquad\leq \int dxdy\, N^{5/2}V(N(x-y))|\widetilde{\varphi}_{\xi_t}(y)|\|a_y\mathrm{ad}^{(n)}_{B(\eta_t)}\xi\|\|a_x\xi\|\end{aligned}$$

which is similar to the r.h.s. of (3.193), the only difference being that instead of $\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|$ we have $\|a_x\xi\|$ (and the fact that in the other norm, we have the field a_y instead of b_y; it is clear, however, that both fields can be treated similarly). Analogously to (3.196), we conclude that

$$\begin{aligned}\left|\int dxdy\, N^{5/2}V(N(x-y))\widetilde{\varphi}_{\xi_t}(y)\langle\xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)a^*_ya_x\xi\rangle\right| \\ \leq (n+1)!\, C^ne^{c|t|}\|\eta_t\|^{n-1}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|\end{aligned} \tag{3.197}$$

Let us now switch to the second term on the r.h.s. of (3.187) (the sum over $r \geq 0$). First of all, we compute the commutator

$$[B(\eta_t), b^*_x] = -b(\eta_x)\left(1-\frac{\mathcal{N}}{N}\right) + \frac{1}{N}\int dzdw\bar{\eta}(z;w)a^*_xa_wb_z$$

Hence the r-th term in the sum is proportional to

$$\begin{aligned}&-\int dxdyN^{5/2}V(N(x-y))\widetilde{\varphi}_{\xi_t}(y)\frac{(N-1-\mathcal{N})}{N}b(\eta_x)b^*_y\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ &\qquad+\int dxdyN^{5/2}V(N(x-y))\widetilde{\varphi}_{\xi_t}(y)N^{-1}\Pi^{(1)}_{(*,\cdot),*}(\eta_t,\delta_x)^*b^*_y\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ &\quad=: \mathrm{S}_1+\mathrm{S}_2\end{aligned} \tag{3.198}$$

The expectation of S_2 can be bounded as follows.

$$|\langle\xi,\mathrm{S}_2\xi\rangle| \leq \int dxdy\, N^{5/2}V(N(x-y))|\widetilde{\varphi}_{\xi_t}(y)|\|b_yN^{-1}\Pi^{(1)}_{(*,\cdot),*}(\eta_t,\delta_x)\xi\|\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|$$

As in (3.194), we find

$$\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\| \leq C^rr!\|\eta_t\|^r\|\eta_x\|\|(\mathcal{N}+1)^{1/2}\xi\|$$

Since, on the other hand,

$$\|b_y N^{-1} \Pi^{(1)}_{(*,\cdot),*}(\eta_t, \delta_x)\xi\| \leq CN^{-1}\|\eta_y\| \|a_x(\mathcal{N}+1)^{1/2}\xi\| + C\|\eta_t\| \|a_x a_y \xi\|$$

we conclude that

$$|\langle \xi, \mathrm{S}_2 \xi\rangle| \leq C^r e^{c|t|} \|\eta_t\|^{r+1} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\|$$

for all $\xi \in \mathcal{F}^{\leq N}$. We are left with the operator S_1 defined in (3.198). Commuting $b(\eta_x)$ with b^*_y we write it as

$$\begin{aligned}
\mathrm{S}_1 = & -\int dxdy N^{5/2} V(N(x-y)) \eta_t(x;y) \widetilde{\varphi_{\xi}}_t(y) \frac{(N-\mathcal{N})(N-\mathcal{N}-1)}{N^2} \mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\
& - \int dxdy N^{5/2} V(N(x-y)) \widetilde{\varphi_{\xi}}_t(y) \frac{(N-\mathcal{N}-1)}{N} \left[b^*_y b(\eta_x) - \frac{1}{N} a^*_y a(\eta_x) \right] \mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\
=: & \ \mathrm{S}_{11} + \mathrm{S}_{12}
\end{aligned}$$

The expectation of S_{12} is estimated by

$$|\langle \xi, \mathrm{S}_{12}\xi\rangle| \leq C^r e^{c|t|} \|\eta_t\|^{r+1} \|(\mathcal{N}+1)^{1/2}\xi\|^2$$

As for S_{11}, we decompose

$$\begin{aligned}
\mathrm{S}_{11} = & -\int dxdy N^{5/2} V(N(x-y)) k_t(x;y) \widetilde{\varphi_{\xi}}_t(y) \frac{(N-\mathcal{N})(N-\mathcal{N}-1)}{N^2} \mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\
& - \int dxdy N^{5/2} V(N(x-y)) \mu_t(x;y) \widetilde{\varphi_{\xi}}_t(y) \frac{(N-\mathcal{N})(N-\mathcal{N}-1)}{N^2} \mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\
=: & \ \mathrm{S}_{111} + \mathrm{S}_{112}
\end{aligned}$$

Since $|\mu_t(x;y)| \leq Ce^{c|t|}$ from Lemma 3.3.3, it is easy to estimate the expectation of the term S_{112} by

$$|\langle \xi, \mathrm{S}_{112}\xi\rangle| \leq C^r e^{c|t|} \|\eta_t\|^{r+1} \|(\mathcal{N}+1)^{1/2}\xi\|^2$$

As for the term S_{111}, we use the fact that, by Lemma 3.2.3, the nested commutator $\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))$ is given by

$$\left(1 - \frac{\mathcal{N}-1}{N}\right)^m \left(1 - \frac{\mathcal{N}-2}{N}\right)^m b^*((\eta_t \bar{\eta}_t)^m \eta_x)$$

if $r = 2m$ is even and by

$$-\left(1 - \frac{\mathcal{N}+1}{N}\right)^{m+1} \left(1 - \frac{\mathcal{N}}{N}\right)^m b((\eta_t \bar{\eta}_t)^{m+1}_x)$$

if $r = 2m+1$ is odd, up to terms ($2^r r! - 1$ of them) having the form

$$\Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \eta^{(\ell_1+1)}_{x,\diamondsuit})$$

where either $k_1 \geq 1$ or at least one of the Λ-operators is a $\Pi^{(2)}$-operator of the form (3.188). We conclude that, if $r = 2m$ is even,

$$\mathrm{S}_{111} = \sqrt{N} \int dxdy N^3 V(N(x-y)) w_\ell(N(x-y)) |\widetilde{\varphi}_{\xi_t}(y)|^2 \widetilde{\varphi}_{\xi_t}(x) b^*((\eta_t \bar{\eta}_t)^m \eta_x) + \mathrm{S}_{1112} \tag{3.199}$$

while, if $r = 2m+1$ is odd,

$$\mathrm{S}_{111} = -\sqrt{N} \int dxdy N^3 V(N(x-y)) w_\ell(N(x-y)) |\widetilde{\varphi}_{\xi_t}(y)|^2 \widetilde{\varphi}_{\xi_t}(x) b^*((\eta_t \bar{\eta}_t)^{m+1}_x) + \mathrm{S}_{1112} \tag{3.200}$$

where, in both cases, the expectation of the error term S_{1112} is bounded by

$$\begin{aligned} |\langle \xi, \mathrm{S}_{1112} \xi \rangle| &\leq C^r \|\eta_t\|^r \int dxdy\, N^{3/2} V(N(x-y))\, |k_t(x;y)| \|\eta_x\| \|(\mathcal{N}+1)^{1/2} \xi\| \|(\mathcal{N}+1)\xi\| \\ &\leq C^r \|\eta_t\|^{r+1} \|(\mathcal{N}+1)^{1/2}\xi\|^2 \end{aligned}$$

for all $\xi \in \mathcal{F}^{\leq N}$. Here, once again, we used the fact that $N^{-1}|\eta_t(x;y)| \leq C$. Summing over all $r \geq 0$, we conclude that

$$\begin{aligned} -\sum_{r \geq 0} \frac{(-1)^r}{(r+1)!} \int dxdy\, N^{5/2} V(N(x-y)) \widetilde{\varphi}_{\xi_t}(y) [B(\eta_t), b_x^*] b_y^* \mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ = -\sqrt{N} \left[b((\cosh_{\eta_t} - 1)(h_{N,t})) + b^*(\sinh_{\eta_t}(h_{N,t})) \right] + \mathrm{S} \end{aligned}$$

where

$$|\langle \xi, S\xi \rangle| \leq e^{c|t|} \sum_{r \geq 0} (C\|\eta_t\|)^r \|(\mathcal{N}+1)^{1/2}\xi\|^2 \leq C e^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\|^2 \tag{3.201}$$

for all $\xi \in \mathcal{F}^{\leq N}$.

Finally, we consider the first term on the r.h.s. of (3.187). This term can be handled similarly as we did with the second term (the sum over $r \geq 0$). We obtain that

$$-\int dxdy N^{5/2} V(N(x-y)) \widetilde{\varphi}_{\xi_t}(y) [B(\eta_t), b_x^*] a_y^* a_x = -\sqrt{N} b(h_{N,t}) + \widetilde{S}$$

where the expectation of $\widetilde{S}$ can be bounded as we did with the expectation of S in (3.201).

Recalling the definition of $\mathcal{E}^{(3)}_{N,t}$ in (3.185), it follows from (3.190), (3.192), (3.196), (3.197) and (3.201) that

$$|\langle \xi, \mathcal{E}^{(3)}_{N,t} \xi \rangle| \leq C e^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2}\xi\|$$

The bounds in (3.186) for the expectation of the commutators $[\mathcal{N}, \mathcal{E}^{(3)}_{N,t}]$, $[a^*(g_1)a(g_2), \mathcal{E}^{(3)}_{N,t}]$ and of the time-derivative $\partial_t \mathcal{E}^{(3)}_{N,t}$ can be proven analogously. We omit the details. $\square$

3.4.6 Analysis of $e^{-B(\eta_t)}\mathcal{L}^{(4)}_{N,t}e^{B(\eta_t)}$

Recall from (3.90) that

$$\mathcal{L}^{(4)}_{N,t} = \mathcal{V}_N = \frac{1}{2}\int dxdy\, N^2V(N(x-y))a^*_x a^*_y a_y a_x$$

We conjugate $\mathcal{L}^{(4)}_{N,t}$ with the unitary operator $e^{B(\eta_t)}$. We define the error term $\mathcal{E}^{(4)}_{N,t}$ through the equation

$$\begin{aligned} e^{-B(\eta_t)}\mathcal{L}^{(4)}_{N,t}e^{B(\eta_t)} = \mathcal{V}_N &+ \frac{1}{2}\int dxdy\, N^2V(N(x-y))|k_t(x;y)|^2 \\ &+ \frac{1}{2}\int dxdy\, N^2V(N(x-y))\left[k_t(x;y)b^*_x b^*_y + \text{h.c.}\right] \\ &+ \mathcal{E}^{(4)}_{N,t} \end{aligned} \tag{3.202}$$

In the next proposition we collect some important properties of the operator $\mathcal{E}^{(4)}_{N,t}$.

Proposition 3.4.9. *Under the same assumptions as in Theorem 3.3.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned} \left|\langle \xi, \mathcal{E}^{(4)}_{N,t}\xi\rangle\right| &\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\| \\ \left|\langle \xi, \left[\mathcal{N}, \mathcal{E}^{(4)}_{N,t}\right]\xi\rangle\right| &\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\| \\ \left|\langle \xi, \left[a^*(g_1)a(g_2), \mathcal{E}^{(4)}_{N,t}\right]\xi\rangle\right| &\leq Ce^{c|t|}\|g_1\|_{H^2}\|g_2\|_{H^2}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\| \\ \left|\partial_t\langle \xi, \mathcal{E}^{(4)}_{N,t}\xi\rangle\right| &\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\| \end{aligned} \tag{3.203}$$

for all $\xi \in \mathcal{F}^{\leq N}$.

Proof. We start by writing

$$e^{-B(\eta_t)}a^*_x a^*_y a_y a_x e^{B(\eta_t)} = a^*_x a^*_y a_y a_x + \int_0^1 ds\, e^{-sB(\eta_t)}\left[a^*_x a^*_y a_y a_x, B(\eta_t)\right]e^{sB(\eta_t)}$$

A straighforward computation gives

$$\begin{aligned} &e^{-B(\eta_t)}a^*_x a^*_y a_y a_x e^{B(\eta_t)} \\ &\quad = a^*_x a^*_y a_y a_x + \int_0^1 ds\, e^{-sB(\eta_t)}\left[b^*_x b^*_y\,\left(a_x a^*(\eta_y) + a^*(\eta_x)a_y\right) + \text{h.c.}\right]e^{sB(\eta_t)} \end{aligned} \tag{3.204}$$

Now we observe that

$$\begin{aligned}
&e^{-sB(\eta_t)}\left[a_x a^*(\eta_y)+a^*(\eta_x)a_y\right]e^{sB(\eta_t)}\\
&\qquad = a_x a^*(\eta_y)+a^*(\eta_x)a_y+\int_0^s d\tau\, e^{-\tau B(\eta_t)}\left[a_x a^*(\eta_y)+a^*(\eta_x)a_y, B(\eta_t)\right]e^{\tau B(\eta_t)}\\
&\qquad = \eta_t(x;y)+a^*(\eta_y)a_x+a^*(\eta_x)a_y\\
&\qquad\quad +\int_0^s d\tau\, e^{-\tau B(\eta_t)}\Big[2b^*(\eta_x)b^*(\eta_y)+b(\eta_y^{(2)})b_x+b(\eta_x^{(2)})b_y\Big]e^{\tau B(\eta_t)}
\end{aligned}$$

Inserting in (3.204), expanding as in Lemma 3.2.4, and integrating over s,τ, we obtain

$$e^{-B(\eta_t)}\mathcal{L}_{N,t}^{(4)}e^{B(\eta_t)}=\mathcal{V}_N+\mathrm{W}_1+\mathrm{W}_2+\mathrm{W}_3+\mathrm{W}_4 \tag{3.205}$$

where

$$\begin{aligned}
\mathrm{W}_1 &= \frac{1}{2}\sum_{n,k\geq 0}\frac{(-1)^{n+k}}{n!k!(n+k+1)}\int dxdy\, N^2V(N(x-y))\eta_t(x;y)\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y^*)\\
\mathrm{W}_2 &= \sum_{n,k\geq 0}\frac{(-1)^{n+k}}{n!k!(n+k+1)}\int dxdy\, N^2V(N(x-y))\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y^*)a^*(\eta_x)a_y\\
\mathrm{W}_3 &= \sum_{n,k,m,r\geq 0}\frac{(-1)^{n+k+m+r}}{n!k!m!r!(m+r+1)(n+k+m+r+2)}\\
&\qquad\times\int dxdy\, N^2V(N(x-y))\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y^*)\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta_x^{(2)}))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_y)\\
\mathrm{W}_4 &= \sum_{n,k,m,r\geq 0}\frac{(-1)^{n+k+m+r}}{n!k!m!r!(m+r+1)(m+r+n+k+2)}\\
&\qquad\times\int dxdyN^2V(N(x-y))\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y^*)\mathrm{ad}^{(m)}_{B(\eta_t)}(b^*(\eta_x))\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_y))
\end{aligned}$$

Let us now estimate the expectation of W_2. By Cauchy-Schwarz, we have

$$\begin{aligned}
&\left|\int dxdyN^2V(N(x-y))\langle\xi,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y^*)a^*(\eta_x)a_y\xi\rangle\right|\\
&\qquad\leq\int dxdyN^2V(N(x-y))\\
&\qquad\qquad\times\|(\mathcal{N}+1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|\|(\mathcal{N}+1)^{-1/2}a^*(\eta_x)a_y\xi\|
\end{aligned}$$

We bound

$$\|(\mathcal{N}+1)^{-1/2}a^*(\eta_x)a_y\xi\|\leq\|\eta_x\|\|a_y\xi\| \tag{3.206}$$

On the other hand, according to Lemma 3.2.4, $\|(\mathcal{N}+1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ is bounded by the sum of $2^{n+k}n!k!$ contributions having the form

$$\begin{aligned}
\mathrm{T}=\Big\|&(\mathcal{N}+1)^{1/2}\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{y,t,\diamondsuit})\\
&\times\Lambda_1'\dots\Lambda_{i_2}'N^{-k_2}\Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural_1'},\dots,\eta^{(m_{k_2})}_{t,\natural_{k_2}'};\eta^{(\ell_2)}_{x,\diamondsuit'})\xi\Big\|
\end{aligned} \tag{3.207}$$

with $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \geq 0$, $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \geq 0$ and where each Λ_i and Λ_i' operator is either a factor $(N-\mathcal{N})/N$, $(N-\mathcal{N}+1)/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1},\dots,\eta^{(q_p)}_{t,\underline{\natural}_p}) \tag{3.208}$$

According to Lemma 3.4.1, part iv), we have

$$\begin{aligned} \mathrm{T} \leq (n+1)C^{k+n}\|\eta_t\|^{k+n-2}\Big\{ &\|\eta_x\|\|\eta_y\|\|(\mathcal{N}+1)^{3/2}\xi\| \\ &+ \|\eta_t\|\|\eta_x\|\|a_y(\mathcal{N}+1)\xi\| + \|\eta_t\|\|\eta_y\|\|a_x(\mathcal{N}+1)\xi\| \\ &+ \|\eta_t\||\eta_t(x;y)|\|(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|^2\sqrt{N}\|a_xa_y\xi\|\Big\} \end{aligned} \tag{3.209}$$

For $\xi \in \mathcal{F}^{\leq N}$, we obtain

$$\begin{aligned} &\left|\int dxdyN^2V(N(x-y))\eta_t(x;y)\langle\xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y^*)a^*(\eta_x)a_y\xi\rangle\right| \\ &\quad\leq (n+1)!k!\,C^{n+k}\|\eta_t\|^{n+k-2}\int dxdyN^2V(N(x-y))\|\eta_x\|\|a_y\xi\| \\ &\qquad\times\Big\{\,[N\|\eta_x\|\|\eta_y\| + \|\eta_t\||\eta_t(x;y)|]\,\|(\mathcal{N}+1)^{1/2}\xi\| \\ &\qquad\qquad + N\|\eta_t\|\|\eta_y\|\|a_x\xi\| + N\|\eta_t\|\|\eta_x\|\|a_y\xi\| + N^{1/2}\|a_xa_y\xi\|\Big\} \\ &\quad\leq (n+1)!k!\,C^{n+k}\|\eta_t\|^{n+k}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\| \end{aligned}$$

and therefore

$$|\langle\xi, \mathrm{W}_2\xi\rangle| \leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|$$

if $\sup_t\|\eta_t\|$ is small enough.

Now, let us consider the expectation of the term W_3. By Cauchy-Schwarz, we have

$$\begin{aligned} &\left|\int dxdyN^2V(N(x-y))\langle\xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b_x^*)\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y^*)\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta_x^{(2)}))\mathrm{ad}^{(r)}(b_y)\xi\rangle\right| \\ &\qquad\leq \int N^2V(N(x-y))\,\|(\mathcal{N}+1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\| \\ &\qquad\qquad\times\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta_x^{(2)}))\mathrm{ad}^{(r)}(b_y)\xi\| \end{aligned}$$

Expanding $\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta_x^{(2)}))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_y)$ as in Lemma 3.2.3 and using Lemma 3.4.1, we obtain

$$\begin{aligned} &\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta_x^{(2)}))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_y)\xi\| \\ &\qquad\leq m!r!\,C^{m+r}\|\eta_t\|^{m+r}\Big[\|\eta_x\|\|\eta_y\|\|(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|\|\eta_x\|\|a_y\xi\|\Big] \end{aligned} \tag{3.210}$$

As for the norm $\|(\mathcal{N}+1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$, we can estimate it as the sum of $2^{n+k}n!k!$ contributions of the form (3.207). Using (3.209) and integrating over x, y, we conclude

$$|\langle\xi, \mathrm{W}_3\xi\rangle| \leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|$$

if $\sup_t\|\eta_t\|$ is small enough.

Let us now switch to W_4. We proceed analogously as we did for W_3. The only difference is that, instead of (3.210), we need to bound

$$\begin{aligned}\|(\mathcal{N}+1)^{-1/2}&\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta_x))\mathrm{ad}^{(r)}_{B(\eta_t)}(b(\eta_y))\xi\| \\ &\leq m!r!\,C^{m+r}\|\eta_t\|^{m+r}\|\eta_x\|\|\eta_y\|\|(\mathcal{N}+1)^{1/2}\xi\|\end{aligned}$$

We find

$$|\langle\xi, \mathrm{W}_4\xi\rangle| \leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|$$

if $\sup_t\|\eta_t\|$ is small enough.

Finally, we consider the term W_1 in (3.205). We extract from the sum over $n, k \geq 0$ the terms with $(n,k) = (0,0)$ and $(n,k) = (0,1)$. We obtain that

$$\begin{aligned}\mathrm{W}_1 = &\frac{1}{2}\int dxdyN^2V(N(x-y))\eta_t(x;y)b^*_xb^*_y \\ &-\frac{1}{4}\int dxdy\,N^2V(N(x-y))\eta_t(x;y)[B(\eta_t), b^*_x]b^*_y + \widetilde{\mathrm{W}}_1\end{aligned} \tag{3.211}$$

with

$$\widetilde{\mathrm{W}}_1 = \frac{1}{2}\sum^*_{n,k}\frac{(-1)^{n+k}}{n!k!(n+k+1)}\int dxdyN^2V(N(x-y))\eta_t(x;y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y) \tag{3.212}$$

where $\sum^*$ excludes the terms $(n,k) = (0,0), (1,0)$. We bound the expectation of $\widetilde{W}_1$ by

$$\begin{aligned}&\left|\int dxdy\,N^2V(N(x-y))\eta_t(x;y)\langle\xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\xi\rangle\right| \\ &\quad\leq\int dxdyN^2V(N(x-y))|\eta_t(x;y)| \\ &\qquad\times\left\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\right\|\|(\mathcal{N}+1)^{1/2}\xi\|\end{aligned}$$

Following Lemma 3.2.4, we can bound the norm $\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ by the sum of $2^{n+k}n!k!$ terms of the form

$$\begin{aligned}\widetilde{\mathrm{T}} = \Big\|(\mathcal{N}+1)^{-1/2}&\Lambda_1\ldots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{y,t,\diamondsuit}) \\ &\times\Lambda'_1\ldots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural_1},\ldots,\eta^{(m_{k_2})}_{t,\natural_{k_2}};\eta^{(\ell_2)}_{x,\diamondsuit'})\xi\Big\|\end{aligned} \tag{3.213}$$

with $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \geq 0$, $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \geq 0$ and where each Λ_i and Λ_i' operator is either a factor $(N - \mathcal{N})/N$, $(N - \mathcal{N} + 1)/N$ or a $\Pi^{(2)}$-operator of the form (3.208). With Lemma 3.4.1 we find

$$
\begin{aligned}
\widetilde{\mathrm{T}} \leq (n+1) C^{k+n} \|\eta_t\|^{k+n-2} \\
\times \Big\{ \|\eta_x\| \|\eta_y\| \|(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\| \|\eta_x\| \|a_y \xi\| + \|\eta_t\| \eta_y\| \|a_x \xi\| \\
+ \|\eta_t\| N^{-1} |\eta_t(x;y)| \|(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|^2 \|a_x a_y \xi\| \Big\}
\end{aligned}
$$

The important difference with respect to (3.209) is that here, when we consider the cases $\ell_1 = \ell_2 = 0$ and $\ell_1 = 0, \ell_2 = 1$ we can apply (3.108) and (3.110), rather than (3.107) and (3.109), because the assumption $(n,k) \neq (0,0), (1,0)$ implies that $k + n \geq 2$ (the case $(n,k) = (0,1)$ is not compatible with $\ell_2 = 1$). Using $\sup_{x,y} N^{-1} |\eta_t(x;y)| \leq C e^{c|t|}$ from Lemma 3.3.3, we conclude that

$$
|\langle \xi, \widetilde{W_1} \xi \rangle| \leq C e^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2}\xi\|
$$

if $\sup_t \|\eta_t\|$ is small enough.

As for the second term on the r.h.s. of (3.211), we have

$$
[B(\eta_t), b_x^*] = -b(\eta_x) \frac{N - \mathcal{N}}{N} + \frac{1}{N} \int dz dw a_x^* a_z b_w \, \eta_t(z;w)
$$

Hence

$$
\begin{aligned}
- \int & dx dy N^2 V(N(x-y)) \eta_t(x;y) [B(\eta_t), b_x^*] b_y^* \\
&= \int dx dy N^2 V(N(x-y)) \eta_t(x;y) b(\eta_x) b_y^* \frac{N - \mathcal{N} + 1}{N} \\
&\quad - N^{-1} \int dx dy dz dw N^2 V(N(x-y)) \eta_t(x;y) \eta_t(z;w) \, a_x^* a_z b_w b_y^* \\
&= \int dx dy N^2 V(N(x-y)) |\eta_t(x;y)|^2 \frac{N - \mathcal{N}}{N} \frac{N - \mathcal{N} + 1}{N} \\
&\quad - N^{-1} \int dx dy dz N^2 V(N(x-y)) \eta_t(x;y) \eta_t(x;z) a_y^* a_z \frac{N - \mathcal{N} + 1}{N} \\
&\quad - N^{-1} \int dx dy dz dw N^2 V(N(x-y)) \eta_t(x;y) \eta_t(z;w) \, a_x^* a_z b_w b_y^*
\end{aligned}
$$

We conclude that

$$
- \int dx dy N^2 V(N(x-y)) \eta_t(x;y) [B(\eta_t), b_x^*] b_y^* = \int dx dy N^2 V(N(x-y)) |k_t(x;y)|^2 + \mathrm{W}_{12}
$$

where

$$
|\langle \xi, \mathrm{W}_{12} \xi \rangle| \leq C e^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\| \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2}\xi\|
$$

Similarly, the first term on the r.h.s. of (3.211) can be decomposed as

$$\int dxdyN^2V(N(x-y))\eta_t(x;y)b_x^*b_y^* = \int dxdyN^2V(N(x-y))k_t(x;y)b_x^*b_y^* + \mathrm{W}_{11}$$

where

$$\mathrm{W}_{11} = \int dxdyN^2V(N(x-y))\mu_t(x;y)b_x^*b_y^*$$

is such that

$$|\langle \xi, \mathrm{W}_{11}\xi\rangle| \le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|$$

since $|\mu(x;y)| \le Ce^{c|t|}$ uniformly in N. $\square$

3.4.7 Analysis of $(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}$

This subsection is devoted to the study of the first term in the generator $\mathcal{G}_{N,t}$ in (3.88). The properties of $(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}$ are collected in the next proposition.

Proposition 3.4.10. *Under the same assumptions as in Theorem 3.3.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned}
|\langle \xi, (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}\xi\rangle| &\le C\|(\mathcal{N}+1)^{1/2}\xi\|^2 \\
|\langle \xi, \left[\mathcal{N}, (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}\right]\xi\rangle| &\le C\|(\mathcal{N}+1)^{1/2}\xi\|^2 \\
|\langle \xi, \left[a^*(g_1)a(g_2), (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}\right]\xi\rangle| &\le C\|g_1\|\|g_2\|\|(\mathcal{N}+1)^{1/2}\xi\|^2 \\
|\langle \xi, \left[\partial_t(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}\right]\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|^2
\end{aligned} \tag{3.214}$$

for all $\xi \in \mathcal{F}^{\le N}$.

Proof. As in Section 6.5 of [11], we expand $(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}$ as

$$\begin{aligned}
(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} &= -\int_0^1 ds\, e^{-sB(\eta_t)}\left[i\partial_t B(\eta_t)\right]e^{sB(\eta_t)} \\
&= \frac{i}{2}\sum_{k,n\ge 0}\frac{(-1)^{n+k}}{k!n!(n+k+1)}\int dx\, \mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x) + \text{h.c.}
\end{aligned} \tag{3.215}$$

We bound the expectations

$$\begin{aligned}
\Big|\int dx\, \langle \xi, &\mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\rangle\Big| \\
&\le \|(\mathcal{N}+1)^{1/2}\xi\|\int dx\, \|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|
\end{aligned}$$

According to Lemma 3.2.4), the norm $\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ is bounded by the sum of $2^{n+k}n!k!$ terms of the form

$$\begin{aligned} \mathrm{Z} = \|(\mathcal{N}+1)^{-1/2}\Lambda_1\dots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1)}_{t,\diamondsuit}\partial_t\eta_t)_x) \\ \times\Lambda'_1\dots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\diamondsuit'})\xi\| \end{aligned} \tag{3.216}$$

with integers $i_1,k_1,\ell_1,i_2,k_2,\ell_2\geq 0$, $j_1,\dots,j_{k_1},m_1,\dots,m_{k_2}\geq 1$ and where each Λ_i and Λ'_i is either a factor $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1},\dots,\eta^{(q_p)}_{t,\underline{\natural}_p})$$

From Lemma 3.4.1, part iii), we conclude that

$$\mathrm{Z}\leq\begin{cases} C^{n+k}\|\eta_t\|^{n+k-1}\|(\partial_t\eta_t)_x\|\|\eta_x\|\|(\mathcal{N}+1)^{1/2}\xi\| & \text{if } \ell_2>0 \\ C^{n+k}\|\eta_t\|^{n+k}\|(\partial_t\eta_t)_x\|\|a_x\xi\| & \text{if } \ell_2=0\end{cases}$$

With Cauchy-Schwarz, we obtain

$$\Big|\int dx\,\langle\xi,\mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\rangle\Big|\leq n!k!\,C^{n+k}\|\eta_t\|^{n+k}\|\partial_t\eta_t\|\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

From (3.215), we conclude that, if $\sup_t\|\eta_t\|$ is sufficiently small,

$$|\langle\xi,(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}\xi\rangle|\leq C\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

The other bounds in (3.214) can be proven analogously, first expanding $(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}$ as in (3.215), then using Lemma 3.2.4 and Lemma 3.2.3 to write the nested commutators on the r.h.s. of (3.215) as sums of factors like in (3.216), and then commuting each of these factors with $\mathcal{N}$, with $a^*(g_1)a(g_2)$, or taking its time-derivative; we omit the details. □

3.4.8 Proof of Theorem 3.3.4

Combining the results of Subsections 3.4.2-3.4.7 and using the scattering equation (3.62), we conclude that

$$\begin{aligned} \mathcal{G}_{N,t} &= C_{N,t}+\mathcal{H}_N+\widetilde{\mathcal{E}}_{N,t} \\ &\quad + N\int dxdy\left[-\Delta+\frac{1}{2}N^2V(N(x-y))\right](1-w_\ell(N(x-y)))\widetilde{\varphi}_{\xi_t}(x)\widetilde{\varphi}_{\xi_t}(y)b^*_xb^*_y+\text{h.c.} \\ &= C_{N,t}+\mathcal{H}_N+\widetilde{\mathcal{E}}_{N,t}+\mathrm{A} \end{aligned} \tag{3.217}$$

with

$$\mathrm{A}=N^3\lambda_\ell\int dxdy\,f_\ell(N(x-y))\chi(|x-y|\leq\ell)\left[\widetilde{\varphi}_{\xi_t}(x)\widetilde{\varphi}_{\xi_t}(y)b^*_xb^*_y+\text{h.c.}\right]$$

and where $C_{N,t}$ is defined as in (3.85). The error term $\widetilde{\mathcal{E}}_{N,t}$ is such that

$$\begin{aligned}\left|\langle \xi, \widetilde{\mathcal{E}}_{N,t}\xi\rangle\right| &\leq Ce^{c|t|}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi\| \\ \left|\langle \xi, \left[\widetilde{\mathcal{E}}_{N,t}, \mathcal{N}\right]\xi\rangle\right| &\leq Ce^{c|t|}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi\| \\ \left|\langle \xi, \left[\widetilde{\mathcal{E}}_{N,t}, a^*(g_1)a(g_2)\right]\xi\rangle\right| &\leq Ce^{c|t|}\|g_1\|_{H^2}\|g_2\|_{H^2}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi\| \\ \left|\langle \xi, \left[\partial_t\widetilde{\mathcal{E}}_{N,t}\right]\xi\rangle\right| &\leq Ce^{c|t|}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi\|\end{aligned} \tag{3.218}$$

Since $N^3\lambda_\ell \leq C$ and $f_\ell(N(x-y)) \leq 1$, we have , with Lemma 1.2.4,

$$|\langle \xi, \mathrm{A}\xi\rangle| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

and similarly, $\pm[\mathcal{N}, \mathrm{A}], \pm[a^*(g_1)a(g_2), \mathrm{A}], \pm\partial_t\mathrm{A} \leq C(\mathcal{N}+1)$. Setting $\mathcal{E}_{N,t} = \mathrm{A}+\widetilde{\mathcal{E}}_{N,t}$, we conclude that

$$\mathcal{G}_{N,t} = C_{N,t} + \mathcal{H}_N + \mathcal{E}_{N,t}$$

where $\mathcal{E}_{N,t}$ satisfies the same bounds (3.218) as $\widetilde{\mathcal{E}}_{N,t}$. This immediately implies that, in the sense of forms on $\mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}} \times \mathcal{F}^{\leq N}_{\perp\widetilde{\varphi}_{\xi_t}}$,

$$\begin{aligned}\frac{1}{2}\mathcal{H}_N - Ce^{c|t|}(\mathcal{N}+1) \leq \mathcal{G}_{N,t} - C_{N,t} &\leq 2\mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1) \\ \pm i\left[\mathcal{G}_{N,t}, \mathcal{N}\right] &\leq \mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1) \\ \partial_t\left[\mathcal{G}_{N,t} - C_{N,t}\right] &\leq \mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1)\end{aligned}$$

Moreover, since

$$\begin{aligned}[\mathcal{H}_N, a^*(g_1)a(g_2)] = &\int dx \nabla g_1(x)\nabla_x a_x^* a(g_2) - \int dx a^*(g_1)\nabla\bar{g}_2(x)\nabla_x a_x \\ &+ \int dxdy\, N^2V(N(x-y))g_1(y)a_x^*a_y^*a_x a(g_2) \\ &- \int dxdy\, N^2V(N(x-y))\,\bar{g}_2(x)a^*(g_1)a_y^*a_y a_x\end{aligned}$$

we obtain that

$$\begin{aligned}|\langle \xi, &[\mathcal{H}_N, a^*(g_1)a(g_2)]\xi\rangle| \\ &\leq [\|\nabla g_1\|\|g_2\| + \|g_1\|\|\nabla g_2\|]\ \|\mathcal{K}^{1/2}\xi\|\|\mathcal{N}^{1/2}\xi\| \\ &\quad + [\|g_2\|\|g_1\|_\infty + \|g_1\|\|g_2\|_\infty]\left[\int dxdy N^2V(N(x-y))\|a_x a_y\xi\|^2\right]^{1/2} \\ &\qquad\qquad \times\left[\int dxdy N^2V(N(x-y))\|a_y(\mathcal{N}+1)^{1/2}\xi\|^2\right]^{1/2} \\ &\leq \|g_1\|_{H^2}\|g_2\|_{H^2}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}+1)^{1/2}\xi\|\end{aligned}$$

for all $\xi \in \mathcal{F}^{\leq N}$. Combining with (3.218), and choosing $g_1 = \partial_t \widetilde{\varphi_{\xi_t}}$ and $g_2 = \widetilde{\varphi_{\xi_t}}$, we find

$$\pm \text{Re } \left[\mathcal{G}_{N,t}, a^*(\partial_t \widetilde{\varphi_{\xi_t}}) a(\widetilde{\varphi_{\xi_t}}) \right] \leq \mathcal{H}_N + C e^{K|t|} (\mathcal{N} + 1)$$

This concludes the proof of Theorem 3.3.4.

3.5 Bounds on the Growth of Fluctuations

In this section, we are going to complete the proof of Theorem 3.1.1 and of Theorem 3.1.2. The main ingredient to reach this goal is a bound on the growth of the expectation of the number of particles operator with respect to the fluctuation dynamics $\mathcal{W}_{N,t}$, that we prove in the next proposition using the properties of the generator $\mathcal{G}_{N,t}$ established in Theorem 3.1.1.

Proposition 3.5.1. *Under the same assumptions as in Theorem 3.3.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned} \langle \mathcal{W}_{N,t}\, \xi, \mathcal{N} \mathcal{W}_{N,t} \xi \rangle &\leq C \, \langle \xi, (\mathcal{G}_{N,0} - C_{N,0}) + (\mathcal{N} + 1)) \xi \rangle \exp(c \exp(c|t|) \\ \langle \mathcal{W}_{N,t}\, \xi, \mathcal{H}_N \mathcal{W}_{N,t} \xi \rangle &\leq C \, \langle \xi, (\mathcal{G}_{N,0} - C_{N,0}) + (\mathcal{N} + 1)) \xi \rangle \exp(c \exp(c|t|) \end{aligned} \tag{3.219}$$

for all $\xi \in \mathcal{F}^{\leq N}_{\perp \varphi}$. Here the operator $\mathcal{H}_N$ is defined as in (3.87).

Remark: From (3.86), we also have

$$\begin{aligned} \langle \mathcal{W}_{N,t}\, \xi, \mathcal{N} \mathcal{W}_{N,t} \xi \rangle &\leq C \, \langle \xi, (\mathcal{H}_N + \mathcal{N} + 1) \xi \rangle \exp(c \exp(c|t|) \\ \langle \mathcal{W}_{N,t}\, \xi, \mathcal{H}_N \mathcal{W}_{N,t} \xi \rangle &\leq C \, \langle \xi, (\mathcal{H}_N + \mathcal{N} + 1) \xi \rangle \exp(c \exp(c|t|) \end{aligned}$$

Proof. First of all, we observe that, from the first equation in (3.86),

$$\frac{1}{2} \mathcal{H}_N + \mathcal{N} \leq (\mathcal{G}_{N,t} - C_{N,t}) + C e^{K|t|} (\mathcal{N} + 1) \tag{3.220}$$

Hence, it is enough to control the growth of the expectation of the operator on the right hand side. We follow here the approach of [63]. We define $q_t = 1 - |\widetilde{\varphi_{\xi_t}}\rangle\langle\widetilde{\varphi_{\xi_t}}|$ as the orthogonal projection onto $L^2_{\perp \widetilde{\varphi_{\xi_t}}}(\mathbb{R}^3)$. We define moreover $\Gamma_t : \mathcal{F}^{\leq N} \to \mathcal{F}^{\leq N}_{\perp \varphi_t}$ by imposing that $\Gamma_t|_{\mathcal{F}_j} = q_t^{\otimes j}$ for all $j = 1, \dots, N$ ($\mathcal{F}_j$ is the sector of $\mathcal{F}^{\leq N}$ with exactly j particles). We have, restricting our attention to $t \geq 0$ (the case $t < 0$ can be handled very similarly)

$$\begin{aligned} &\Big\langle \mathcal{W}_{N,t}\, \xi, \Big[(\mathcal{G}_{N,t} - C_{N,t}) + C e^{K|t|} (\mathcal{N} + 1) \Big] \mathcal{W}_{N,t}\, \xi \Big\rangle \\ &\qquad\qquad = \left\langle \mathcal{W}_{N,t}\, \xi, \left[(\Gamma_t \mathcal{G}_{N,t} \Gamma_t - C_{N,t}) + C e^{Kt} (\mathcal{N} + 1) \right] \mathcal{W}_{N,t}\, \xi \right\rangle \end{aligned}$$

Hence, since $\mathcal{N}$ commutes with Γ_t,

$$\begin{aligned} i\partial_t &\left\langle \mathcal{W}_{N,t}\, \xi, \left[(\mathcal{G}_{N,t} - C_{N,t}) + C e^{Kt} (\mathcal{N} + 1) \right] \mathcal{W}_{N,t}\, \xi \right\rangle \\ &= \left\langle \mathcal{W}_{N,t}\, \xi, \left[\Gamma_t \mathcal{G}_{N,t} \Gamma_t, (\Gamma_t \mathcal{G}_{N,t} \Gamma_t - C_{N,t}) + C e^{Kt} (\mathcal{N} + 1) \right] \mathcal{W}_{N,t}\, \xi \right\rangle \\ &\quad + \left\langle \mathcal{W}_{N,t}\, \xi, \partial_t \left[(\Gamma_t \mathcal{G}_{N,t} \Gamma_t - C_{N,t}) + C e^{Kt} (\mathcal{N} + 1) \right] \mathcal{W}_{N,t}\, \xi \right\rangle \\ &= C e^{Kt} \left\langle \mathcal{W}_{N,t}\, \xi, [\mathcal{G}_{N,t}, \mathcal{N}] \, \mathcal{W}_{N,t}\, \xi \right\rangle \\ &\quad + \left\langle \mathcal{W}_{N,t}\, \xi, \partial_t \left[(\Gamma_t \mathcal{G}_{N,t} \Gamma_t - C_{N,t}) + C e^{Kt} (\mathcal{N} + 1) \right] \mathcal{W}_{N,t}\, \xi \right\rangle \end{aligned} \tag{3.221}$$

We observe that
$$0 = \partial_t \|\widetilde{\varphi_{\xi_t}}\|_2^2 = \langle \dot{\widetilde{\varphi_{\xi_t}}}, \widetilde{\varphi_{\xi_t}} \rangle + \langle \widetilde{\varphi_{\xi_t}}, \dot{\widetilde{\varphi_{\xi_t}}} \rangle$$
This implies that
$$\dot{q}_t = -|\widetilde{\varphi_{\xi_t}}\rangle\langle \dot{\widetilde{\varphi_{\xi_t}}}| - |\dot{\widetilde{\varphi_{\xi_t}}}\rangle\langle \widetilde{\varphi_{\xi_t}}| = -|\widetilde{\varphi_{\xi_t}}\rangle\langle q_t\dot{\widetilde{\varphi_{\xi_t}}}| - |q_t\dot{\widetilde{\varphi_{\xi_t}}}\rangle\langle \widetilde{\varphi_{\xi_t}}|$$
Therefore
$$\begin{aligned}
\partial_t \Gamma_t^{(j)} &= -\sum_{i=1}^j q_t \otimes \cdots \otimes \left[|\widetilde{\varphi_{\xi_t}}\rangle\langle q_t\dot{\widetilde{\varphi_{\xi_t}}}| q_t + q_t |q_t\dot{\widetilde{\varphi_{\xi_t}}}\rangle\langle \widetilde{\varphi_{\xi_t}}| \right] \otimes \cdots \otimes q_t \\
&= -\sum_{i=1}^j \left[|\widetilde{\varphi_{\xi_t}}\rangle\langle q_t\dot{\widetilde{\varphi_{\xi_t}}}|_i \Gamma_t^{(j)} - \Gamma_t^{(j)} |q_t\dot{\widetilde{\varphi_{\xi_t}}}\rangle\langle \widetilde{\varphi_{\xi_t}}|_i \right]
\end{aligned}$$
We conclude that
$$\partial_t \Gamma_t = -a^*(\widetilde{\varphi_{\xi_t}}) a(q_t\dot{\widetilde{\varphi_{\xi_t}}}) \Gamma_t - \Gamma_t a^*(q_t\dot{\widetilde{\varphi_{\xi_t}}}) a(\widetilde{\varphi_{\xi_t}})$$
Thus
$$\begin{aligned}
&\left\langle \mathcal{W}_{N,t}\,\xi, \partial_t \left[(\Gamma_t \mathcal{G}_{N,t} \Gamma_t - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1) \right] \mathcal{W}_{N,t}\,\xi \right\rangle \\
&\quad = \langle \mathcal{W}_{N,t}\,\xi, [(\partial_t \Gamma_t)(\mathcal{G}_{N,t} - C_{N,t}) + (\mathcal{G}_{N,t} - C_{N,t})(\partial_t \Gamma_t)] \mathcal{W}_{N,t}\,\xi \rangle \\
&\qquad + \left\langle \mathcal{W}_{N,t}\,\xi, \left[\partial_t (\mathcal{G}_{N,t} - C_{N,t}) + CKe^{Kt}(\mathcal{N}+1) \right] \mathcal{W}_{N,t}\,\xi \right\rangle \\
&\quad = 2\mathrm{Re} \left\langle \mathcal{W}_{N,t}\,\xi, \left[a^*(q_t\dot{\widetilde{\varphi_{\xi_t}}}) a(\widetilde{\varphi_{\xi_t}}), \mathcal{G}_{N,t} \right] \mathcal{W}_{N,t}\,\xi \right\rangle \\
&\qquad + \left\langle \mathcal{W}_{N,t}\,\xi, \left[\partial_t (\mathcal{G}_{N,t} - C_{N,t}) + CKe^{Kt}(\mathcal{N}+1) \right] \mathcal{W}_{N,t}\,\xi \right\rangle
\end{aligned}$$
where we used the fact that $a(\widetilde{\varphi_{\xi_t}})\mathcal{W}_{N,t}\xi = 0$, for all $t \in \mathbb{R}$. Together with (3.221), we find
$$\begin{aligned}
&i\partial_t \left\langle \mathcal{W}_{N,t}\,\xi, \left[(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1) \right] \mathcal{W}_{N,t}\,\xi \right\rangle \\
&\quad = Ce^{Kt} \langle \mathcal{W}_{N,t}\,\xi, [\mathcal{G}_{N,t}, \mathcal{N}] \mathcal{W}_{N,t}\,\xi \rangle \\
&\qquad + \left\langle \mathcal{W}_{N,t}\,\xi, \left[\partial_t (\mathcal{G}_{N,t} - C_{N,t}) + CKe^{Kt}(\mathcal{N}+1) \right] \mathcal{W}_{N,t}\,\xi \right\rangle \\
&\qquad + 2\mathrm{Re} \left\langle \mathcal{W}_{N,t}\,\xi, \left[a^*(q_t\dot{\widetilde{\varphi_{\xi_t}}}) a(\widetilde{\varphi_{\xi_t}}), \mathcal{G}_{N,t} \right] \mathcal{W}_{N,t}\,\xi \right\rangle
\end{aligned}$$
From Theorem 3.3.4, we obtain that
$$\begin{aligned}
&\left| \partial_t \left\langle \mathcal{W}_{N,t}\,\xi, \left[(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1) \right] \mathcal{W}_{N,t}\,\xi \right\rangle \right| \\
&\qquad \le \widetilde{C} e^{K|t|} \left\langle \mathcal{W}_{N,t}\,\xi, \left[\mathcal{H}_N + Ce^{Kt}(\mathcal{N}+1) \right] \mathcal{W}_{N,t}\xi \right\rangle \\
&\qquad \le \widetilde{C} e^{K|t|} \left\langle \mathcal{W}_{N,t}\,\xi, \left[(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{K|t|}(\mathcal{N}+1) \right] \mathcal{W}_{N,t}\xi \right\rangle
\end{aligned}$$
Applying Gronwall's inequality, we find a constant $c > 0$ such that
$$\begin{aligned}
&\left\langle \mathcal{W}_{N,t}\,\xi, \left[(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{K|t|}(\mathcal{N}+1) \right] \mathcal{W}_{N,t}\xi \right\rangle \\
&\qquad \le \langle \xi, [(\mathcal{G}_{N,0} - C_{N,0}) + C(\mathcal{N}+1)]\, \xi \rangle \, \exp(c \exp(c|t|))
\end{aligned}$$

With (3.220), we conclude that

$$\begin{aligned}\langle \mathcal{W}_{N,t}\xi, \mathcal{N}\mathcal{W}_{N,t}\xi\rangle &\leq C\,\langle \xi, [(\mathcal{G}_{N,0}-C_{N,0})+(\mathcal{N}+1)]\,\xi\rangle\, \exp(c\exp(c|t|))\\ \langle \mathcal{W}_{N,t}\xi, \mathcal{H}_N\mathcal{W}_{N,t}\xi\rangle &\leq C\,\langle \xi, [(\mathcal{G}_{N,0}-C_{N,0})+(\mathcal{N}+1)]\,\xi\rangle\, \exp(c\exp(c|t|))\end{aligned}$$

as claimed. □

To apply Prop. 3.5.1 to the proof of Theorems 3.1.1 and 3.1.2, we need to control the expectation on the r.h.s. of (3.219) for vectors $\xi \in \mathcal{F}^{\leq N}_{\perp\varphi}$ describing orthogonal excitations around the condensate wave function φ for initial N-particle wave functions ψ_N satisfying (3.10). To this end, we use the next lemma.

Lemma 3.5.2. *As in (3.85), let*

$$\begin{aligned}C_{N,t} = &\frac{1}{2}\left\langle \widetilde{\varphi_\xi}_t, \left([N^3V(N.)(N-1-2Nf_\ell(N.))]*|\widetilde{\varphi_\xi}_t|^2\right)\widetilde{\varphi_\xi}_t\right\rangle\\ &+\int dxdy\;|\nabla_x k_t(x;y)|^2+\frac{1}{2}\int dxdy\,N^2V(N(x-y))|k_t(x;y)|^2\\ &+\mathrm{Re}\int dxdy\,N^3V(N(x-y))\bar{\widetilde{\varphi_\xi}}_t(x)\bar{\widetilde{\varphi_\xi}}_t(y)k_t(x;y).\end{aligned}$$

where $\widetilde{\varphi_\xi}_t$ is the solution of the modified Gross-Pitaevskii equation (3.68), with initial data $\widetilde{\varphi_\xi}_{t=0} = \varphi$ (we assumed in the construction of the fluctuation dynamics that $\varphi \in H^4(\mathbb{R}^3)$; in this lemma, we only need $\varphi \in H^1(\mathbb{R}^3)$). Then there is a constant $C > 0$, independent of N and t, such that

$$\left|\left[C_{N,t}+N\langle i\partial_t\widetilde{\varphi_\xi}_t, \widetilde{\varphi_\xi}_t\rangle\right]-N\mathcal{E}_{GP}(\varphi)\right| \leq C$$

with the translation invariant Gross-Pitaevskii energy functional $\mathcal{E}_{GP}$ defined in (3.14).

Proof. We have

$$N\left\langle i\partial_t\widetilde{\varphi_\xi}_t, \widetilde{\varphi_\xi}_t\right\rangle = N\left\langle \widetilde{\varphi_\xi}_t, -\Delta\widetilde{\varphi_\xi}_t\right\rangle + N\left\langle \widetilde{\varphi_\xi}_t, \left(N^3V(N.)f_\ell(N.)*|\widetilde{\varphi_\xi}_t|^2\right)\widetilde{\varphi_\xi}_t\right\rangle.$$

Therefore

$$\begin{aligned}C_{N,t}&+N\langle i\partial_t\widetilde{\varphi_\xi}_t, \widetilde{\varphi_\xi}_t\rangle\\ &= N\|\nabla\widetilde{\varphi_\xi}_t\|^2+\frac{(N-1)}{2}\langle \widetilde{\varphi_\xi}_t, \left[N^3V(N.)*|\widetilde{\varphi_\xi}_t|^2\right]\widetilde{\varphi_\xi}_t\rangle\\ &\quad+\int dxdy\;|\nabla_x k_t(x;y)|^2+\frac{1}{2}\int dxdy\,N^2V(N(x-y))|k_t(x;y)|^2\\ &\quad+\mathrm{Re}\int dxdy\,N^3V(N(x-y))\bar{\widetilde{\varphi_\xi}}_t(x)\bar{\widetilde{\varphi_\xi}}_t(y)k_t(x;y).\end{aligned} \tag{3.222}$$

Obviously,

$$\frac{(N-1)}{2}\langle \widetilde{\varphi_\xi}_t, \left[N^3V(N.)*|\widetilde{\varphi_\xi}_t|^2\right]\widetilde{\varphi_\xi}_t\rangle = \frac{N}{2}\left\langle \widetilde{\varphi_\xi}_t, \left[N^3V(N.)*|\widetilde{\varphi_\xi}_t|^2\right]\widetilde{\varphi_\xi}_t\right\rangle + \mathcal{O}(1) \tag{3.223}$$

where $\mathcal{O}(1)$ denotes a quantity with absolute value bounded by a constant, independent of N and of t. Furthermore

$$\begin{aligned}\frac{1}{2}\int dxdy\, N^2 V(N(x-y))|k_t(x,y)|^2& \\ = \frac{N}{2}\int dxdy\, N^3 V(N(x-y))w_\ell(N(x-y))^2|\widetilde{\varphi_{\xi_t}}(x)|^2|\widetilde{\varphi_{\xi_t}}(y)|^2&\end{aligned}\tag{3.224}$$

Finally, we consider the third term on the r.h.s. of (3.222), the one with $\nabla_x k_t$. We recall that $k_t(x;y) = -Nw_\ell(N(x-y))\widetilde{\varphi_{\xi_t}}(x)\widetilde{\varphi_{\xi_t}}(y)$. Hence, we find

$$\begin{aligned}-\Delta_x k_t(x;y) = N^3(\Delta w_\ell)(N(x-y))\widetilde{\varphi_{\xi_t}}(x)\widetilde{\varphi_{\xi_t}}(y) + Nw_\ell(N(x-y))\Delta\widetilde{\varphi_{\xi_t}}(x)\widetilde{\varphi_{\xi_t}}(y)\\ + 2N^2(\nabla w_\ell)(N(x-y))\cdot\nabla\widetilde{\varphi_{\xi_t}}(x)\widetilde{\varphi_{\xi_t}}(y).\end{aligned}\tag{3.225}$$

Since, by (3.61), $\Delta w_\ell = -\Delta f_\ell = -(1/2)Vf_\ell + \lambda_\ell f_\ell$ we have

$$\begin{aligned}&\int dxdy\,\bar{k}_t(x;y)(-\Delta_x k_t)(x;y)\\ &= -\frac{N}{2}\int dxdy\, N^3V(N(y-x))(w_\ell(N(x-y))-1)w_\ell(N(x-y))|\widetilde{\varphi_{\xi_t}}(x)|^2|\widetilde{\varphi_{\xi_t}}(y)|^2\\ &\quad - N^3\lambda_\ell\int dxdy\, f_\ell(N(x-y))Nw_\ell(N(x-y))\;|\widetilde{\varphi_{\xi_t}}(x)|^2|\widetilde{\varphi_{\xi_t}}(y)|^2\\ &\quad + 2\int dxdy\, Nw_\ell(N(y-x))N^2(\nabla w_\ell)(N(y-x))\cdot\nabla\bar{\widetilde{\varphi}}_{\xi_t}(x)\widetilde{\varphi_{\xi_t}}(x)|\widetilde{\varphi_{\xi_t}}(y)|^2\\ &\quad - \int dxdy\, N^2w_\ell^2(N(x-y))(\Delta\widetilde{\varphi_{\xi_t}})(x)\widetilde{\varphi_{\xi_t}}(x)|\widetilde{\varphi_{\xi_t}}(y)|^2\\ &= \frac{N}{2}\int dxdy\, N^3V(N(y-x))(1-w_\ell(N(x-y)))w_\ell(N(x-y))|\widetilde{\varphi_{\xi_t}}(x)|^2|\widetilde{\varphi_{\xi_t}}(y)|^2\\ &\quad + 2\int dxdy\, Nw_\ell(N(y-x))N^2(\nabla w_\ell)(N(y-x))\cdot\nabla\bar{\widetilde{\varphi}}_{\xi_t}(x)\widetilde{\varphi_{\xi_t}}(x)|\widetilde{\varphi_{\xi_t}}(y)|^2 + \mathcal{O}(1).\end{aligned}\tag{3.226}$$

In the last step, we used the bounds $N^3\lambda_\ell = \mathcal{O}(1)$, $Nw_\ell(N(x-y)) \le C|x-y|^{-1}$ and $0 \le f_\ell(N(x-y)) \le 1$. Integrating by parts in the last term, we find

$$\begin{aligned}&2\int dxdy\, N^2(\nabla w_\ell)(N(y-x))\cdot\nabla\bar{\widetilde{\varphi}}_{\xi_t}(x)Nw_\ell(N(y-x))\widetilde{\varphi_{\xi_t}}(x)|\widetilde{\varphi_{\xi_t}}(y)|^2\\ &= -\int dxdy\,\nabla_x(N^2w_\ell(N(y-x))^2)\cdot\nabla\bar{\widetilde{\varphi}}_{\xi_t}(x)\widetilde{\varphi_{\xi_t}}(x)|\widetilde{\varphi_{\xi_t}}(y)|^2\\ &= \int dxdy\, N^2w_\ell(N(x-y))^2\Delta\bar{\widetilde{\varphi}}_{\xi_t}(x)\widetilde{\varphi_{\xi_t}}(x)|\widetilde{\varphi_{\xi_t}}(y)|^2\\ &\quad + \int dxdy\, N^2w_\ell(N(x-y))^2\nabla\bar{\widetilde{\varphi}}_{\xi_t}(x)\cdot\nabla\widetilde{\varphi_{\xi_t}}(x)|\widetilde{\varphi_{\xi_t}}(y)|^2\end{aligned}$$

With (3.226), this leads us (using again the bound $Nw_\ell(N(x-y)) \leq C|x-y|^{-1}$) to

$$\begin{aligned}
&\int dxdy\, \bar{k}_t(x;y)(-\Delta_x k_t)(x;y) \\
&\quad = \frac{N}{2}\int dxdy\, N^3 V(N(y-x))(1-w_\ell(N(x-y)))w_\ell(N(x-y))|\widetilde{\varphi_{\xi_t}}(x)|^2|\widetilde{\varphi_{\xi_t}}(y)|^2 \\
&\qquad + \mathcal{O}(1)
\end{aligned}$$

Combining this bound with (3.223) and (3.224), we find

$$\begin{aligned}
&C_{N,t} + N\langle i\partial_t\widetilde{\varphi_{\xi_t}}, \widetilde{\varphi_{\xi_t}}\rangle \\
&\quad = N\left[\int |\nabla\widetilde{\varphi_{\xi_t}}(x)|^2 dx + \frac{1}{2}\int dxdy N^3 V(N(x-y))f_\ell(N(x-y))|\widetilde{\varphi_{\xi_t}}(x)|^2|\widetilde{\varphi_{\xi_t}}(y)|^2\right] \\
&\qquad + \mathcal{O}(1)
\end{aligned}$$

The expression in the parenthesis on the r.h.s. is exactly the energy functional associated with the time-dependent modified Gross-Pitaevskii equation (3.68). By energy conservation, we conclude that

$$\begin{aligned}
&C_{N,t} + N\langle i\partial_t\widetilde{\varphi_{\xi_t}}, \widetilde{\varphi_{\xi_t}}\rangle \\
&\quad = N\left[\int |\nabla\varphi(x)|^2 dx + \frac{1}{2}\int dxdy N^3 V(N(x-y))f_\ell(N(x-y))|\varphi(x)|^2|\varphi(y)|^2\right] \\
&\qquad + \mathcal{O}(1)
\end{aligned} \tag{3.227}$$

Observe that, with (3.63),

$$\begin{aligned}
&\int dxdy\, N^3 V(N(x-y))f_\ell(N(x-y))|\varphi(x)|^2|\varphi(y)|^2 \\
&\quad = \int dxdy V(y)f_\ell(y)|\varphi(x)|^2|\varphi(x+y/N)|^2 \\
&\quad = \left[8\pi\mathfrak{a}_0 + \mathcal{O}(N^{-1})\right]\int |\varphi(x)|^4 dx \\
&\qquad + \int dxdy V(y)f_\ell(y)|\varphi(x)|^2\left[|\varphi(x+y/N)|^2 - |\varphi(x)|^2\right]
\end{aligned} \tag{3.228}$$

where

$$\begin{aligned}
&\left|\int dxdy\, V(y)f(y)|\varphi(x)|^2\left[|\varphi(x+y/N)|^2 - |\varphi(x)|^2\right]\right| \\
&\quad \leq N^{-1}\int_0^1 ds\int dxdy\, V(y)f(y)|\varphi(x)|^2|\nabla\varphi(x+sy/N)||\varphi(x+y/N)||y| \\
&\quad \leq CN^{-1}
\end{aligned}$$

for a constant $C > 0$ depending only on the H^1-norm of φ. Inserting the last bound and (3.228) in (3.227), we conclude that

$$C_{N,t} + N\langle i\partial_t\widetilde{\varphi_{\xi_t}}, \widetilde{\varphi_{\xi_t}}\rangle = N\mathcal{E}_{GP}(\varphi) + \mathcal{O}(1)$$

as claimed. □

With Proposition 3.5.1 and Lemma 3.5.2, we can now conclude the proof of our main theorem.

Proof of Theorems 3.1.1 and 3.1.2. We observe, first of all, that, by Proposition 3.3.2,

$$\left|\langle\varphi_t,\gamma^{(1)}_{N,t}\varphi_t\rangle-\langle\widetilde{\varphi}_{\xi_t},\gamma^{(1)}_{N,t}\widetilde{\varphi}_{\xi_t}\rangle\right|\le 2\|\varphi_t-\widetilde{\varphi}_{\xi_t}\|\le CN^{-1}\exp(c\exp(c|t|)) \tag{3.229}$$

Hence, it is enough to compute

$$\begin{aligned}\langle\widetilde{\varphi}_{\xi_t},\gamma^{(1)}_{N,t}\widetilde{\varphi}_{\xi_t}\rangle&=\frac{1}{N}\langle e^{-iH_Nt}\psi_N,a^*(\widetilde{\varphi}_{\xi_t})a(\widetilde{\varphi}_{\xi_t})e^{-iH_Nt}\psi_N\rangle\\&=\frac{1}{N}\langle U_{N,t}e^{-iH_Nt}\psi_N,(N-\mathcal{N})U_{N,t}e^{-iH_Nt}\psi_N\rangle\\&=1-\frac{1}{N}\langle U_{N,t}e^{-iH_Nt}\psi_N,\mathcal{N}U_{N,t}e^{-iH_Nt}\psi_N\rangle\end{aligned}$$

We define $\xi=e^{-B(\eta_t\mathfrak{a}_0)}U_{N,0}\psi_N\in\mathcal{F}^{\le N}_{\perp\varphi}$. Then we have $\psi_N=U^*_{N,0}e^{B(\eta_t\mathfrak{a}_0)}\xi$ and therefore

$$1-\langle\widetilde{\varphi}_{\xi_t},\gamma^{(1)}_{N,t}\widetilde{\varphi}_{\xi_t}\rangle=\frac{1}{N}\langle\mathcal{W}_{N,t}\xi,e^{-B(\eta_t)}\mathcal{N}e^{-B(\eta_t)}\mathcal{W}_{N,t}\xi\rangle\le\frac{C}{N}\langle\mathcal{W}_{N,t}\xi,\mathcal{N}\mathcal{W}_{N,t}\xi\rangle$$

where we applied Lemma 3.2.2. By Prop. 3.5.1, we conclude that

$$1-\langle\widetilde{\varphi}_{\xi_t},\gamma^{(1)}_{N,t}\widetilde{\varphi}_{\xi_t}\rangle\le N^{-1}\exp(c\exp(c|t|))\,\langle\xi,[(\mathcal{G}_{N,0}-C_{N,0})+C(\mathcal{N}+1)]\,\xi\rangle \tag{3.230}$$

In order to apply Prop. 3.5.1, we used here the assumption (valid both, in the proof of Theorem 3.1.1 and Theorem 3.1.2) that $\widetilde{\varphi}_{t=0}=\varphi\in H^4(\mathbb{R}^3)$.

Recalling from (3.10) the definition $a_N=1-\langle\varphi,\gamma^{(1)}_N\varphi\rangle$, we bound, with the above definition of ξ,

$$\begin{aligned}\langle\xi,\mathcal{N}\xi\rangle&=\langle U_{N,0}\psi_N,e^{B(\eta_t\mathfrak{a}_0)}\mathcal{N}e^{-B(\eta_t\mathfrak{a}_0)}U_{N,0}\psi_N\rangle\\&\le C\langle U_{N,0}\psi_N,\mathcal{N}U_{N,0}\psi_N\rangle\\&=C\langle\psi_N,(N-a^*(\varphi)a(\varphi))\psi_N\rangle\\&=CN(1-\langle\varphi,\gamma^{(1)}_N\varphi\rangle)=CNa_N\end{aligned}$$

We still have to bound the expectation of $(\mathcal{G}_{N,0}-C_{N,0})$ in the state ξ. We have

$$\mathcal{G}_{N,0}=i\partial_te^{-B(\eta_t)}|_{t=0}e^{B(\eta_t\mathfrak{a}_0)}+e^{-B(\eta_t\mathfrak{a}_0)}\left[(i\partial_tU_{N,t})|_{t=0}U^*_{N,0}+U_{N,0}H_NU^*_{N,0}\right]e^{B(\eta_t\mathfrak{a}_0)}$$

With Proposition 3.4.10, we find

$$\left|\langle\xi,i\partial_te^{-B(\eta_t)}|_{t=0}e^{B(\eta_t\mathfrak{a}_0)}\xi\rangle\right|\le C\langle\xi,(\mathcal{N}+1)\xi\rangle\le CNa_N+C \tag{3.231}$$

From Eq. (3.89), we obtain

$$\begin{aligned}&\langle e^{B(\eta_t\mathfrak{a}_0)}\xi,(i\partial_tU_{N,t})|_{t=0}\,U^*_{N,0}\,e^{B(\eta_t\mathfrak{a}_0)}\xi\rangle\\&\quad=-\langle(i\partial_t\widetilde{\varphi}_{\xi_t})|_{t=0},\varphi\rangle\langle U_{N,0}\psi_N,(N-\mathcal{N})U_{N,0}\psi_N\rangle\\&\qquad-2\mathrm{Re}\langle U_{N,0}\psi_N,\sqrt{N-\mathcal{N}}a(q_0(i\partial_t\widetilde{\varphi}_{\xi_t})|_{t=0})U_{N,0}\psi_N\rangle\\&\quad=-N\langle(i\partial_t\widetilde{\varphi}_{\xi_t})|_{t=0},\varphi\rangle+N\langle(i\partial_t\widetilde{\varphi}_{\xi_t})|_{t=0},\varphi\rangle(1-\langle\varphi,\gamma^{(1)}_N\varphi\rangle)\\&\qquad-2N\mathrm{Re}\,\langle\varphi,\gamma^{(1)}_Nq_0(i\partial_t\widetilde{\varphi}_{\xi_t})|_{t=0}\rangle\end{aligned}$$

Combining this identity with the bound (3.231) and with the observation that, by definition of ξ,

$$\langle \xi, e^{-B(\eta_t \mathfrak{a}_0)} U_{N,0} H_N U_{N,0}^* e^{B(\eta_t \mathfrak{a}_0)} \xi \rangle = \langle \psi_N, H_N \psi_N \rangle$$

we conclude that

$$\begin{aligned} \langle \xi, (\mathcal{G}_{N,0} - C_{N,0})\xi \rangle \leq & \left[\langle \psi_N, H_N \psi_N \rangle - (C_{N,0} + N \langle (i\partial_t \widetilde{\varphi}_t)|_{t=0}, \varphi \rangle) \right] \\ & - 2N\mathrm{Re}\, \langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \widetilde{\varphi}_t)|_{t=0} \rangle + CNa_N + C \end{aligned}$$

Hence, with Lemma 3.5.2, we get

$$\begin{aligned} \langle \xi, (\mathcal{G}_{N,0} - C_{N,0})\xi \rangle \leq & \left[\langle \psi_N, H_N \psi_N \rangle - N\mathcal{E}_{\mathrm{GP}}(\varphi) \right] - 2N\mathrm{Re}\, \langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \widetilde{\varphi}_t)|_{t=0} \rangle \\ & + CNa_N + C \end{aligned} \tag{3.232}$$

where $\mathcal{E}_{\mathrm{GP}}$ denotes the translation invariant Gross-Pitaevskii functional defined in (3.14).

To bound the second term on the r.h.s. of the last equation, we proceed differently depending on whether we want to show Theorem 3.1.1 or Theorem 3.1.2. To prove Theorem 3.1.2, we notice that

$$\begin{aligned} \langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \widetilde{\varphi_{\xi_t}})|_{t=0} \rangle &= \langle \varphi, \gamma_N^{(1)} (i\partial_t \widetilde{\varphi_{\xi_t}})|_{t=0} \rangle - \langle \varphi, \gamma_N^{(1)} \varphi \rangle \langle \varphi, (i\partial_t \widetilde{\varphi_{\xi_t}})|_{t=0} \rangle \\ &= \langle \varphi, (i\partial_t \widetilde{\varphi_{\xi_t}})|_{t=0} \rangle (1 - \langle \varphi, \gamma_N^{(1)} \varphi \rangle) + \langle \varphi, (\gamma^{(1)} - |\varphi\rangle\langle\varphi|)(i\partial_t \widetilde{\varphi_{\xi_t}})|_{t=0} \rangle \end{aligned}$$

With $\widetilde{a}_N = \mathrm{tr}\, |\gamma_N^{(1)} - |\varphi\rangle\langle\varphi||$, we obtain that

$$|\langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \widetilde{\varphi_{\xi_t}})|_{t=0} \rangle| \leq C(a_N + \widetilde{a}_N)$$

Since $a_N \leq \widetilde{a}_N$, we conclude from (3.232) that

$$\langle \xi, (\mathcal{G}_{N,0} - C_{N,0})\xi \rangle \leq C \left[N\widetilde{a}_N + N\widetilde{b}_N + 1 \right]$$

Inserting in (3.230) and using (3.229), we arrive at

$$1 - \langle \varphi_t, \gamma_N^{(1)} \varphi_t \rangle \leq C \left[\widetilde{a}_N + \widetilde{b}_N + N^{-1} \right] \exp(c \exp(c|t|))\,.$$

This concludes the proof of Theorem 3.1.2.

To show Theorem 3.1.1, we use instead the fact that

$$i\partial_t \widetilde{\varphi}_t|_{t=0} = -\Delta\varphi + (N^3 V(N.) f_\ell(N.) * |\varphi|^2)\varphi$$

Since here we assume that the initial data $\varphi = \phi_{\mathrm{GP}}$ is the minimizer of the Gross-Pitaevskii energy functional (3.6), it must satisfy the Euler-Lagrange equation

$$-\Delta\varphi + V_{\mathrm{ext}}\varphi + 8\pi\mathfrak{a}_0|\varphi|^2\varphi = \mu\varphi$$

for some $\mu \in \mathbb{R}$. We find

$$i\partial_t \widetilde{\varphi}_t|_{t=0} = \mu\varphi - V_{\text{ext}}\varphi + \big[(N^3 V(N.) f_\ell(N.) * |\varphi|^2) - 8\pi\mathfrak{a}_0|\varphi|^2\big]\varphi$$

Using (3.63) the fact that the minimizer φ of (3.6) is continuously differentiable and vanishes at infinity (see [71, Theorem 2.1]), we obtain

$$\Big\| \big[(N^3 V(N.) f_\ell(N.) * |\varphi|^2) - 8\pi\mathfrak{a}_0|\varphi|^2\big]\varphi \Big\|_2 \leq C N^{-1}$$

and therefore

$$-2N\text{Re}\, \langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \widetilde{\varphi_{\xi_t}})|_{t=0}\rangle \leq 2N\text{Re}\, \langle \varphi, \gamma_N^{(1)} q_0 (V_{\text{ext}} + \kappa)\varphi\rangle + C$$

for any constant $\kappa \in \mathbb{R}$. Choosing $\kappa \geq 0$ so that $V_{\text{ext}} + \kappa \geq 0$ (from the assumptions, V_{ext} is bounded below), we find

$$\begin{aligned} -2N\text{Re}\, &\langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \widetilde{\varphi_{\xi_t}})|_{t=0}\rangle \\ &\leq 2N\text{Re}\, \langle \varphi, \gamma_N^{(1)} (V_{\text{ext}} + \kappa)\varphi\rangle - 2N\langle \varphi, \gamma_N^{(1)}\varphi\rangle\langle \varphi, (V_{\text{ext}} + \kappa)\varphi\rangle + C \\ &\leq 2N\text{Re}\, \langle \varphi, \gamma_N^{(1)} (V_{\text{ext}} + \kappa)\varphi\rangle - 2N\langle \varphi, (V_{\text{ext}} + \kappa)\varphi\rangle + C(Na_N + 1) \end{aligned}$$

With Cauchy-Schwarz and since $0 \leq \gamma_N^{(1)} \leq 1$ implies that $(\gamma_N^{(1)})^2 \leq \gamma_N^{(1)}$, we get

$$\begin{aligned} -2N\text{Re}\, &\langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \widetilde{\varphi_{\xi_t}})|_{t=0}\rangle \\ &\leq N\langle \varphi, \gamma_N^{(1)} (V_{\text{ext}} + \kappa)\gamma_N^{(1)}\varphi\rangle - N\langle \varphi, (V_{\text{ext}} + \kappa)\varphi\rangle + C(Na_N + 1) \\ &\leq N\text{tr}\, \gamma_N^{(1)} V_{\text{ext}} - N\langle \varphi, V_{\text{ext}}\varphi\rangle + C(Na_N + 1) \end{aligned}$$

Inserting back in (3.232) we conclude that, under the assumptions of Theorem 3.1.1,

$$\langle \xi, (\mathcal{G}_{N,0} - C_{N,0})\xi\rangle \leq \big[\langle \psi_N, H_N^{\text{trap}}\psi_N\rangle - N\mathcal{E}_{\text{GP}}^{\text{trap}}(\varphi)\big] + CNa_N + C \leq C\big[Na_N + Nb_N + 1\big]$$

With (3.230) and (3.229), we find now

$$1 - \langle \varphi_t, \gamma_{N,t}^{(1)}\varphi_t\rangle \leq C\big[a_N + b_N + N^{-1}\big]\exp(c\exp(c|t|))$$

This concludes the proof of Theorem 3.1.1. □

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Chapter 4

Bogoliubov Theory in the Gross-Pitaevskii Limit

In this chapter, we provide the details for the proof of Theorem 1.6.1. As discussed in Section 1.6, our result rigorously confirms Bogoliubov's predictions on the low-energy spectrum of the weakly interacting Bose gas in the Gross-Pitaevskii limit. Our main result is proved in the article [15].

The following manuscript is a slightly modified version of the paper [15]. Sections 4.1 and 4.2 are shortened and slightly rephrased versions of the introduction [15, Section 1] and of [15, Sections 2], since we already introduced the Fock space setting in which we work and related standard results in Section 1.2. Apart from these changes and up to the notational modifications already mentioned in Section 1.A, the following sections appear as in the paper [15].

4.1 Main result

Let us recall from Section 1.6 that we consider systems of N bosons in the unit torus $\Lambda = \mathbb{T}^3$ with periodic boundary conditions. The Hamiltonian has the form

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \kappa \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) \tag{4.1}$$

The non-negative coupling constant κ will be assumed to be small enough (but fixed, independent of N). Furthermore, we require $V \in L^3(\mathbb{R}^3)$ to be non-negative, radial, compactly supported and to have scattering length $\mathfrak{a}_0$.

Let us also recall that the scattering length of the interaction potential is defined through the zero-energy scattering equation

$$\left[-\Delta + \frac{\kappa}{2} V(x)\right] f(x) = 0 \tag{4.2}$$

with the boundary condition $f(x) \to 1$, as $|x| \to \infty$. For $|x|$ large enough (outside the support of V), we have

$$f(x) = 1 - \frac{\mathfrak{a}_0}{|x|}$$

for an appropriate constant $\mathfrak{a}_0$, which is known as the scattering length of κV. It can be computed by

$$8\pi\mathfrak{a}_0 = \kappa \int V(x) f(x) dx \tag{4.3}$$

By scaling, we obtain the scattering length of $\kappa N^2 V(Nx)$ is given by $\mathfrak{a}_0/N$.

For sufficiently small values of the coupling constant $\kappa > 0$, the results of [13] prove that the ground state energy E_N of H_N satisfies

$$E_N = 4\pi\mathfrak{a}_0 N + \mathcal{O}(1) \tag{4.4}$$

and that the one-particle reduced density $\gamma_N^{(1)}$ associated with the ground state of (4.1) is such that

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq C N^{-1} \tag{4.5}$$

for a constant $C > 0$. These results improve previously known results from [66, 71, 79].

In this paper, we go beyond the first order ground state approximation (4.4), computing the ground state energy and the low-lying excitation spectrum of (4.1), up to errors vanishing in the limit $N \to \infty$. This is the content of our main theorem.

Theorem 4.1.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, spherically symmetric, compactly supported and assume that the coupling constant $\kappa > 0$ is small enough. Then, in the limit $N \to \infty$, the ground state energy E_N of the Hamilton operator H_N defined in (4.1) is given by*

$$\begin{aligned} E_N =& \, 4\pi(N-1)\mathfrak{a}_N \\ & - \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2} \right] + \mathcal{O}(N^{-1/4}) \end{aligned} \tag{4.6}$$

Here we introduced the notation $\Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$ and we defined

$$8\pi\mathfrak{a}_N = \kappa\widehat{V}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k \kappa^{k+1}}{(2N)^k} \sum_{p_1,\dots,p_k \in \Lambda_+^*} \frac{\widehat{V}(p_1/N)}{p_1^2} \left(\prod_{i=1}^{k-1} \frac{\widehat{V}((p_i - p_{i+1})/N)}{p_{i+1}^2} \right) \widehat{V}(p_k/N) \tag{4.7}$$

Moreover, the spectrum of $H_N - E_N$ below a threshold ζ consists of eigenvalues given, in the limit $N \to \infty$, by

$$\sum_{p \in \Lambda_+^*} n_p \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} + \mathcal{O}(N^{-1/4}(1 + \zeta^3)) \tag{4.8}$$

Here $n_p \in \mathbb{N}$ for all $p \in \Lambda_+^*$ and $n_p \neq 0$ for finitely many $p \in \Lambda_+^*$ only.

Theorem 4.1.1 determines preceisely the low-lying eigenvalues of (4.1). In (4.138) and (4.139) we also provide a norm approximation of eigenvectors associated with the low-energy spectrum of (4.1) (we postpone the precise statement of this result, because it requires additional notation that will be introduced in the next sections). As mentioned already in Section 1.6, this enables us to compute the condensate depletion in the ground state ψ_N of (4.1). Denoting by $\gamma_N^{(1)}$ the one-particle reduced density of ψ_N, we find

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle = \frac{1}{N} \sum_{p \in \Lambda_+^*} \left[\frac{p^2 + 8\pi \mathfrak{a}_0 - \sqrt{p^4 + 16\pi \mathfrak{a}_0 p^2}}{2\sqrt{p^4 + 16\pi \mathfrak{a}_0 p^2}} \right] + \mathcal{O}(N^{-9/8}) . \tag{4.9}$$

The proof of (4.9), which is based on the approximation (4.139) of the ground state vector and on some additional bounds from Section 4.7, is deferred to Appendix 4.A.

As already mentioned in Section 1.6, in the last years, rigorous versions of Bogoliubov's approach have been used to establish the ground state energy and excitation spectrum for mean-field models describing systems of N trapped bosons interacting weakly through a potential whose range is comparable with the size of the trap. This has been done in the works [96, 46, 64, 30, 89, 90, 91]. Recently, we extended these results by considering more singular interaction regimes in [14].

Let us now quickly recall our strategy before explaining the structure of this paper. In our approach we follow [14] where we considered systems with Hamilton operator

$$H_N^\beta = \sum_{i=1}^N -\Delta_{x_i} + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} N^{3\beta} V(N^\beta (x_i - x_j)) \tag{4.10}$$

for $\beta \in (0;1)$. Our goal here is to extend the results of [14] to the physically more interesting and mathematically more challenging Gross-Pitaevskii regime, where $\beta = 1$. The first part of our analysis follows [13], where we proved that low-energy states of the Hamiltonian (4.1) exhibit complete Bose-Einstein condensation in the zero-momentum mode $\varphi_0(x) = 1$ for all $x \in \Lambda$, with only a finite number of orthogonal excitations. We start with the map $U_N = U_N(\varphi_0)$, introduced in Section 1.2, and construct the excitation Hamiltonian $\mathcal{L}_N = U_N H_N U_N^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, from now on denoting by $\mathcal{F}_+^{\leq N} = \mathcal{F}_{\perp \varphi_0}^{\leq N}$ the Fock space of excited particles. As we will discuss in Section 4.3, conjugation with U_N is reminiscent of the Bogoliubov approximation described in Section 1.3; it produces constant contributions and also terms that are quadratic, cubic and quartic in creation and annihilation operators a_p^*, a_p associated with momenta $p \in \Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$. In contrast with what Bogoliubov did and in contrast with what was done in [64] in the mean-field regime, here we cannot neglect cubic and quartic terms resulting from conjugation with U_N; they are large and they have to be taken into account to obtain a rigorous proof of Theorem 4.1.1.

The reason why, in the Gross-Pitaevskii regime, cubic and quartic terms are still important is that conjugation with U_N factors out products of the condensate wave function φ_0, while it does not affect correlations. Hence, the correlation structure that carries an important contribution to the energy and characterizes all low-energy states

$\psi_N \in L^2_s(\Lambda^N)$ is left in the corresponding excitation vector $U_N\psi_N \in \mathcal{F}_+^{\leq N}$. To extract the large contributions to the energy that are still hidden in cubic and quartic terms, we have to conjugate the excitation Hamiltonian $\mathcal{L}_N$ with a unitary map generating the correct correlation structure. To reach this goal, we make use of generalized Bogoliubov transformations having the form

$$T = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*} \eta_p \left(b_p^* b_{-p}^* - b_p b_{-p}\right)\right] \tag{4.11}$$

With an appropriate choice of the coefficients η_p in the definition of T (related with a modification of the solution of the zero-energy scattering equation (4.2)), we can define a new, renormalized, excitation Hamiltonian $\mathcal{G}_N = T^*\mathcal{L}_N T = T^* U_N H_N U_N^* T : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$. We find that

$$\mathcal{G}_N = 4\pi\mathfrak{a}_0 N + \mathcal{H}_N + \Delta_N \tag{4.12}$$

where $\mathcal{H}_N$ is the Hamiltonian H_N restricted on the excitation space $\mathcal{F}_+^{\leq N}$, while Δ_N is an error term with the property that, for every $\delta > 0$ there exists $C > 0$ with

$$\pm\, \Delta_N \leq \delta\mathcal{H}_N + C\kappa(\mathcal{N}_+ + 1) \tag{4.13}$$

where $\mathcal{N}_+$ is the number of particles operator on $\mathcal{F}_+^{\leq N}$. To determine the low-energy spectrum of H_N we need to go one step further: Combining (4.13) with similar bounds for the commutator of $\mathcal{G}_N$ and $\mathcal{N}_+$, we can show that, if $\psi_N \in L^2_s(\Lambda^N)$ is such that $\psi_N = \chi(H_N - E_N \leq \zeta)\psi_N$ (i.e. if ψ_N belongs to a low-energy spectral subspace of H_N), the corresponding excitation vector $\xi_N = T^* U_N \psi_N$ satisfies the stronger a-priori bound

$$\left\langle \xi_N, \left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3\right] \xi_N \right\rangle \leq C(1 + \zeta^3) \tag{4.14}$$

uniformly in N.

Armed with this estimate, we can have a second look at the renormalized excitation Hamiltonian $\mathcal{G}_N$ and we can prove that several terms contributing to $\mathcal{G}_N$ are negligible on low-energy states. We find that

$$\mathcal{G}_N = C_{\mathcal{G}_N} + \mathcal{Q}_{\mathcal{G}_N} + \mathcal{C}_N + \mathcal{H}_N + \mathcal{E}_{\mathcal{G}_N} \tag{4.15}$$

where $C_{\mathcal{G}_N}$ is a constant, $\mathcal{Q}_{\mathcal{G}_N}$ is quadratic in (generalized) creation and annihilation operators, $\mathcal{C}_N$ is the cubic term

$$\mathcal{C}_N = \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^* : q\neq -p} \widehat{V}(p/N) \left[b_{p+q}^* b_{-p}^* (b_q \cosh(\eta_q) + b_{-q}^* \sinh(\eta_q)) + \text{h.c.}\right] \tag{4.16}$$

and $\mathcal{E}_{\mathcal{G}_N}$ is an error term that can be estimated by

$$\pm\, \mathcal{E}_{\mathcal{G}_N} \leq C N^{-1/2} \Big[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3\Big] \tag{4.17}$$

and thus, by (4.14), is negligible on low-energy states.

Up to (4.15), our analysis is similar to that of [14], where we determined the ground state energy and low-energy excitation spectrum for the Hamiltonian (4.10), for $0 < \beta < 1$. The main new challenge that we have to face in the Gross-Pitaevskii regime, i.e. for $\beta = 1$, is the appearance, on the r.h.s. of (4.15), of the cubic term $\mathcal{C}_N$ and of the potential energy operator restricted on $\mathcal{F}_+^{\leq N}$ that will be denoted by $\mathcal{V}_N$, which is quartic in creation and annihilation operators (with this notation, $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$). For $\beta < 1$, these terms were small (on low-energy states) and they could be included in the error $\mathcal{E}_{\mathcal{G}_N}$. For $\beta = 1$, this is no longer the case; it is easy to find normalized $\xi_N \in \mathcal{F}_+^{\leq N}$ satisfying (4.14), and with $\langle \xi_N, \mathcal{C}_N \xi_N \rangle$ and $\langle \xi_N, \mathcal{V}_N \xi_N \rangle$ of order one (not vanishing in the limit $N \to \infty$).

It is important to notice that cubic and quartic terms do not improve with different choices of the coefficients η_p. This is related with the observation, going back to the work of Erdős-Schlein-Yau in [38] and more recently to the papers [80, 81] of Napiorkowski-Reuvers-Solovej that quasi-free states can only approximate the ground state energy of a dilute Bose gas in the Gross-Pitaevskii regime up to errors of order one (to be more precise, [38, 80, 81] study the ground state energy of an extended dilute Bose gas in the thermodynamic limit, but it is clear how to translate those results to the Gross-Pitaevskii regime).

To extract the missing energy from the cubic and quartic terms in (4.15), we are going to conjugate the excitation Hamiltonian $\mathcal{G}_N$ with a unitary operator of the form $S = e^A$, where A is an antisymmetric operator, cubic in (generalized) creation and annihilation operators. Observe that a similar idea, formulated however with a different language and in a different setting, was used by Yau-Yin in [99] to find an upper bound to the ground state energy of a dilute Bose gas in the thermodynamic limit matching the Lee-Huang-Yang prediction up to second order.

With S, we define yet another (cubically renormalized) excitation Hamiltonian

$$\mathcal{J}_N = S^* \mathcal{G}_N S = S^* T^* U_N H_N U_N^* T S : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

With the appropriate choice of A, we show that

$$\mathcal{J}_N = C_{\mathcal{J}_N} + \mathcal{Q}_{\mathcal{J}_N} + \mathcal{V}_N + \mathcal{E}_{\mathcal{J}_N} \tag{4.18}$$

where $C_{\mathcal{J}_N}$ and $\mathcal{Q}_{\mathcal{J}_N}$ are new constant and quadratic terms, while $\mathcal{E}_{\mathcal{J}_N}$ is an error term, satisfying an estimate similar to (4.17) and thus negligible on low-energy states. The important difference with respect to (4.15) is that now, on the r.h.s. of (4.18), there is no cubic term! There is still the quartic interaction term $\mathcal{V}_N$, but this is a positive operator and therefore it can be ignored, at least for proving lower bounds.

Conjugating $\mathcal{J}_N$ with a last generalized Bogoliubov transformation R to diagonalize the quadratic operator $\mathcal{Q}_{\mathcal{J}_N}$, we obtain a final excitation Hamiltonian

$$\mathcal{M}_N = R^* \mathcal{J}_N R = R^* S^* T^* U_N H_N U_N^* T S R : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

which can be written as

$$\begin{aligned}\mathcal{M}_N = 4\pi\mathfrak{a}_N(N-1) - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] \\ + \sum_{p\in\Lambda_+^*}\sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}\, a_p^* a_p + \mathcal{V}_N + \mathcal{E}_{\mathcal{M}_N}\end{aligned} \tag{4.19}$$

with an error term $\mathcal{E}_{\mathcal{M}_N}$ which satisfies

$$\pm\mathcal{E}_{\mathcal{M}_N} \leq CN^{-1/4}\Big[(\mathcal{H}_N+1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3\Big]$$

and is therefore negligible on low-energy states. With (4.19), Theorem 4.1.1 follows comparing the eigenvalues of $\mathcal{M}_N$ with those of its quadratic part, by means of the min-max principle. To prove lower bounds, we can ignore the quartic interaction $\mathcal{V}_N$. To prove upper bounds, on the other hand, it is enough to control the values of $\mathcal{V}_N$ on low-energy eigenspaces of the quadratic operator; they turn out to be negligible.

The plan of the paper is as follows. In Section 4.2, we recall the definition of generalized Bogoliubov transformations that play a very important role in our analysis and we review their properties. In Section 4.3 we introduce the excitation Hamiltonian $\mathcal{L}_N$, the renormalized excitation Hamiltonian $\mathcal{G}_N$ and the excitation Hamiltonian $\mathcal{J}_N$ with renormalized cubic term, and we study their properties. In particular, Prop. 4.3.2 provides important bounds on $\mathcal{G}_N$ while Prop. 4.3.3 gives a precise description of $\mathcal{J}_N$. In Section 4.4, we prove estimates for the excitation vectors associated with low-energy many-body wave functions. Section 4.5 is devoted to the diagonalization of the quadratic part of $\mathcal{J}_N$ and Section 4.6 applies the min-max principle to conclude the proof of Theorem 4.1.1. Finally, Section 4.7 and Section 4.8 contain the proof of Prop. 4.3.2 and, respectively, of Prop. 4.3.3.

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4.2 Further Properties of Generalized Bogoliubov Transformations

In this section, we quickly recall the definition of generalized Bogoliubov transformations for translation invariant systems, already discussed in Section 2.2, and discuss some additional properties. For $\eta \in \ell^2(\Lambda_+^*)$ with $\eta_{-p} = \eta_p$ for all $p \in \Lambda_+^*$, we define

$$B(\eta) = \frac{1}{2}\sum_{p\in\Lambda_+^*}\left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_p b_{-p}\right) \tag{4.20}$$

and we consider

$$e^{B(\eta)} = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*}\left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_p b_{-p}\right)\right] \tag{4.21}$$

We refer to unitary operators of the form (4.21) as generalized Bogoliubov transformations, in analogy with the standard Bogoliubov transformations

$$e^{\widetilde{B}(\eta)} = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*}\left(\eta_p a_p^* a_{-p}^* - \bar{\eta}_p a_p a_{-p}\right)\right] \tag{4.22}$$

defined by means of the standard creation and annihilation operators. In this paper, we will work with (4.21), rather than (4.22), because the generalized Bogoliubov transformations, in contrast with the standard transformations, leave the truncated Fock space $\mathcal{F}_+^{\leq N}$ invariant. The price we will have to pay is the fact that, while the action of standard Bogoliubov transformation on creation and annihilation operators is explicitly given by

$$e^{-\widetilde{B}(\eta)} a_p e^{\widetilde{B}(\eta)} = \cosh(\eta_p) a_p + \sinh(\eta_p) a_{-p}^* \tag{4.23}$$

there is no such formula describing the action of generalized Bogoliubov transformations. An important part of our analysis is therefore devoted to the control of the action of (4.21). A first important observation in this direction is the following lemma, whose proof can be found in [20, Lemma 3.1] (a similar result has been previously established in [96]).

Lemma 4.2.1. *Let $\eta \in \ell^2(\Lambda^*)$ and $B(\eta)$ be defined as in (4.20). Then, for every $n_1, n_2 \in \mathbb{Z}$, there exists a constant $C > 0$ (depending on $\|\eta\|$) such that, on $\mathcal{F}_+^{\leq N}$,*

$$e^{-B(\eta)}(\mathcal{N}_+ + 1)^{n_1}(N + 1 - \mathcal{N}_+)^{n_2} e^{B(\eta)} \leq C(\mathcal{N}_+ + 1)^{n_1}(N + 1 - \mathcal{N}_+)^{n_2}$$

Unfortunately, controlling the change of the number of particles operator is not enough for our purposes. To obtain more precise information we expand, for any $p \in \Lambda_+^*$,

$$\begin{aligned} e^{-B(\eta)}\, b_p\, e^{B(\eta)} &= b_p + \int_0^1 ds\, \frac{d}{ds} e^{-sB(\eta)} b_p e^{sB(\eta)} \\ &= b_p - \int_0^1 ds\, e^{-sB(\eta)}[B(\eta), b_p] e^{sB(\eta)} \\ &= b_p - [B(\eta), b_p] + \int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{-s_2 B(\eta)}[B(\eta),[B(\eta), b_p]] e^{s_2 B(\eta)} \end{aligned}$$

Iterating m times, we find

$$\begin{aligned} e^{-B(\eta)} b_p e^{B(\eta)} = &\sum_{n=1}^{m-1} (-1)^n \frac{\mathrm{ad}_{B(\eta)}^{(n)}(b_p)}{n!} \\ &+ \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m\, e^{-s_m B(\eta)} \mathrm{ad}_{B(\eta)}^{(m)}(b_p) e^{s_m B(\eta)} \end{aligned} \tag{4.24}$$

where we recursively defined

$$\mathrm{ad}^{(0)}_{B(\eta)}(A) = A \quad \text{and} \quad \mathrm{ad}^{(n)}_{B(\eta)}(A) = [B(\eta), \mathrm{ad}^{(n-1)}_{B(\eta)}(A)]$$

We are going to expand the nested commutators $\mathrm{ad}^{(n)}_{B(\eta)}(b_p)$ and $\mathrm{ad}^{(n)}_{B(\eta)}(b^*_p)$. To this end, we need to introduce some additional notation. We follow here [20, 13, 14]. For $f_1, \dots, f_n \in \ell_2(\Lambda^*_+)$, $\sharp = (\sharp_1, \dots, \sharp_n), \flat = (\flat_0, \dots, \flat_{n-1}) \in \{\cdot, *\}^n$, we set

$$\begin{aligned}\Pi^{(2)}_{\sharp,\flat}&(f_1, \dots, f_n) \\ &= \sum_{p_1,\dots,p_n \in \Lambda^*} b^{\flat_0}_{\alpha_0 p_1} a^{\sharp_1}_{\beta_1 p_1} a^{\flat_1}_{\alpha_1 p_2} a^{\sharp_2}_{\beta_2 p_2} a^{\flat_2}_{\alpha_2 p_3} \dots a^{\sharp_{n-1}}_{\beta_{n-1} p_{n-1}} a^{\flat_{n-1}}_{\alpha_{n-1} p_n} b^{\sharp_n}_{\beta_n p_n} \prod_{\ell=1}^n f_\ell(p_\ell)\end{aligned} \tag{4.25}$$

where, for $\ell = 0, 1, \dots, n$, we define $\alpha_\ell = 1$ if $\flat_\ell = *$, $\alpha_\ell = -1$ if $\flat_\ell = \cdot$, $\beta_\ell = 1$ if $\sharp_\ell = \cdot$ and $\beta_\ell = -1$ if $\sharp_\ell = *$. In (4.25), we require that, for every $j = 1, \dots, n-1$, we have either $\sharp_j = \cdot$ and $\flat_j = *$ or $\sharp_j = *$ and $\flat_j = \cdot$ (so that the product $a^{\sharp_\ell}_{\beta_\ell p_\ell} a^{\flat_\ell}_{\alpha_\ell p_{\ell+1}}$ always preserves the number of particles, for all $\ell = 1, \dots, n-1$). With this assumption, we find that the operator $\Pi^{(2)}_{\sharp,\flat}(f_1, \dots, f_n)$ maps $\mathcal{F}^{\leq N}_+$ into itself. If, for some $\ell = 1, \dots, n$, $\flat_{\ell-1} = \cdot$ and $\sharp_\ell = *$ (i.e. if the product $a^{\flat_{\ell-1}}_{\alpha_{\ell-1} p_\ell} a^{\sharp_\ell}_{\beta_\ell p_\ell}$ for $\ell = 2, \dots, n$, or the product $b^{\flat_0}_{\alpha_0 p_1} a^{\sharp_1}_{\beta_1 p_1}$ for $\ell = 1$, is not normally ordered) we require additionally that $f_\ell \in \ell^1(\Lambda^*_+)$. In position space, the same operator can be written as

$$\Pi^{(2)}_{\sharp,\flat}(f_1, \dots, f_n) = \int \check{b}^{\flat_0}_{x_1} \check{a}^{\sharp_1}_{y_1} \check{a}^{\flat_1}_{x_2} \check{a}^{\sharp_2}_{y_2} \check{a}^{\flat_2}_{x_3} \dots \check{a}^{\sharp_{n-1}}_{y_{n-1}} \check{a}^{\flat_{n-1}}_{x_n} \check{b}^{\sharp_n}_{y_n} \prod_{\ell=1}^n \check{f}_\ell(x_\ell - y_\ell)\, dx_\ell dy_\ell \tag{4.26}$$

An operator of the form (4.25), (4.26) with all the properties listed above, will be called a $\Pi^{(2)}$-operator of order n.

For $g, f_1, \dots, f_n \in \ell_2(\Lambda^*_+)$, $\sharp = (\sharp_1, \dots, \sharp_n) \in \{\cdot, *\}^n$, $\flat = (\flat_0, \dots, \flat_n) \in \{\cdot, *\}^{n+1}$, we also define the operator

$$\begin{aligned}\Pi^{(1)}_{\sharp,\flat}&(f_1, \dots, f_n; g) \\ &= \sum_{p_1,\dots,p_n \in \Lambda^*} b^{\flat_0}_{\alpha_0, p_1} a^{\sharp_1}_{\beta_1 p_1} a^{\flat_1}_{\alpha_1 p_2} a^{\sharp_2}_{\beta_2 p_2} a^{\flat_2}_{\alpha_2 p_3} \dots a^{\sharp_{n-1}}_{\beta_{n-1} p_{n-1}} a^{\flat_{n-1}}_{\alpha_{n-1} p_n} a^{\sharp_n}_{\beta_n p_n} a^{\flat_n}(g) \prod_{\ell=1}^n f_\ell(p_\ell)\end{aligned} \tag{4.27}$$

where α_ℓ and β_ℓ are defined as above. Also here, we impose the condition that, for all $\ell = 1, \dots, n$, either $\sharp_\ell = \cdot$ and $\flat_\ell = *$ or $\sharp_\ell = *$ and $\flat_\ell = \cdot$. This implies that $\Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g)$ maps $\mathcal{F}^{\leq N}_+$ back into $\mathcal{F}^{\leq N}_+$. Additionally, we assume that $f_\ell \in \ell^1(\Lambda^*_+)$ if $\flat_{\ell-1} = \cdot$ and $\sharp_\ell = *$ for some $\ell = 1, \dots, n$ (i.e. if the pair $a^{\flat_{\ell-1}}_{\alpha_{\ell-1} p_\ell} a^{\sharp_\ell}_{\beta_\ell p_\ell}$ is not normally ordered). In position space, the same operator can be written as

$$\Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g) = \int \check{b}^{\flat_0}_{x_1} \check{a}^{\sharp_1}_{y_1} \check{a}^{\flat_1}_{x_2} \check{a}^{\sharp_2}_{y_2} \check{a}^{\flat_2}_{x_3} \dots \check{a}^{\sharp_{n-1}}_{y_{n-1}} \check{a}^{\flat_{n-1}}_{x_n} \check{a}^{\sharp_n}_{y_n} \check{a}^{\flat_n}(g) \prod_{\ell=1}^n \check{f}_\ell(x_\ell - y_\ell)\, dx_\ell dy_\ell \tag{4.28}$$

An operator of the form (4.27), (4.28) will be called a $\Pi^{(1)}$-operator of order n. Operators of the form $b(f)$, $b^*(f)$, for a $f \in \ell^2(\Lambda_+^*)$, will be called $\Pi^{(1)}$-operators of order zero.

The next lemma gives a detailed analysis of the nested commutators $\mathrm{ad}^{(n)}_{B(\eta)}(b_p)$ and $\mathrm{ad}^{(n)}_{B(\eta)}(b_p^*)$ for $n \in \mathbb{N}$; the proof can be found in [13, Lemma 2.5](it is a translation to momentum space of [20, Lemma 3.2]).

Lemma 4.2.2. *Let $\eta \in \ell^2(\Lambda_+^*)$ be such that $\eta_p = \eta_{-p}$ for all $p \in \ell^2(\Lambda^*)$. To simplify the notation, assume also η to be real-valued (as it will be in applications). Let $B(\eta)$ be defined as in (4.20), $n \in \mathbb{N}$ and $p \in \Lambda^*$. Then the nested commutator $\mathrm{ad}^{(n)}_{B(\eta)}(b_p)$ can be written as the sum of exactly $2^n n!$ terms, with the following properties.*

i) Possibly up to a sign, each term has the form

$$\Lambda_1 \Lambda_2 \dots \Lambda_i \, N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta_p^s \varphi_{\alpha p}) \tag{4.29}$$

for some $i, k, s \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\backslash\{0\}$, $\sharp \in \{\cdot, *\}^k$, $\flat \in \{\cdot, *\}^{k+1}$ and $\alpha \in \{\pm 1\}$ chosen so that $\alpha = 1$ if $\flat_k = \cdot$ and $\alpha = -1$ if $\flat_k = *$ (recall here that $\varphi_p(x) = e^{-ip\cdot x}$). In (4.29), each operator $\Lambda_w : \mathcal{F}^{\leq N} \to \mathcal{F}^{\leq N}$, $w = 1, \dots, i$, is either a factor $(N - \mathcal{N}_+)/N$, a factor $(N - (\mathcal{N}_+ - 1))/N$ or an operator of the form

$$N^{-h} \Pi^{(2)}_{\sharp',\flat'}(\eta^{z_1}, \eta^{z_2}, \dots, \eta^{z_h}) \tag{4.30}$$

for some $h, z_1, \dots, z_h \in \mathbb{N}\backslash\{0\}$, $\sharp, \flat \in \{\cdot, *\}^h$.

ii) If a term of the form (4.29) contains $m \in \mathbb{N}$ factors $(N - \mathcal{N}_+)/N$ or $(N - (\mathcal{N}_+ - 1))/N$ and $j \in \mathbb{N}$ factors of the form (4.30) with $\Pi^{(2)}$-operators of order $h_1, \dots, h_j \in \mathbb{N}\backslash\{0\}$, then we have

$$m + (h_1 + 1) + \dots + (h_j + 1) + (k + 1) = n + 1$$

iii) If a term of the form (4.29) contains (considering all Λ-operators and the $\Pi^{(1)}$-operator) the arguments $\eta^{i_1}, \dots, \eta^{i_m}$ and the factor η_p^s for some $m, s \in \mathbb{N}$, and $i_1, \dots, i_m \in \mathbb{N}\backslash\{0\}$, then

$$i_1 + \dots + i_m + s = n.$$

iv) There is exactly one term having of the form (4.29) with $k = 0$ and such that all Λ-operators are factors of $(N - \mathcal{N}_+)/N$ or of $(N + 1 - \mathcal{N}_+)/N$. It is given by

$$\left(\frac{N - \mathcal{N}_+}{N}\right)^{n/2} \left(\frac{N + 1 - \mathcal{N}_+}{N}\right)^{n/2} \eta_p^n b_p$$

if n is even, and by

$$-\left(\frac{N - \mathcal{N}_+}{N}\right)^{(n+1)/2} \left(\frac{N + 1 - \mathcal{N}_+}{N}\right)^{(n-1)/2} \eta_p^n b_{-p}^*$$

if n is odd.

v) If the $\Pi^{(1)}$-operator in (4.29) is of order $k \in \mathbb{N}\backslash\{0\}$, it has either the form

$$\sum_{p_1,\dots,p_k} b^{\flat_0}_{\alpha_0 p_1} \prod_{i=1}^{k-1} a^{\sharp_i}_{\beta_i p_i} a^{\flat_i}_{\alpha_i p_{i+1}} a^*_{-p_k} \eta_p^{2r} a_p \prod_{i=1}^{k} \eta_{p_i}^{j_i}$$

or the form

$$\sum_{p_1,\dots,p_k} b^{\flat_0}_{\alpha_0 p_1} \prod_{i=1}^{k-1} a^{\sharp_i}_{\beta_i p_i} a^{\flat_i}_{\alpha_i p_{i+1}} a_{p_k} \eta_p^{2r+1} a^*_p \prod_{i=1}^{k} \eta_{p_i}^{j_i}$$

for some $r \in \mathbb{N}$, $j_1,\dots,j_k \in \mathbb{N}\backslash\{0\}$. If it is of order $k=0$, then it is either given by $\eta_p^{2r} b_p$ or by $\eta_p^{2r+1} b^*_{-p}$, for some $r \in \mathbb{N}$.

vi) For every non-normally ordered term of the form

$$\sum_{q\in\Lambda^*} \eta_q^i a_q a_q^*, \quad \sum_{q\in\Lambda^*} \eta_q^i b_q a_q^*$$
$$\sum_{q\in\Lambda^*} \eta_q^i a_q b_q^*, \quad or \quad \sum_{q\in\Lambda^*} \eta_q^i b_q b_q^*$$

appearing either in the Λ-operators or in the $\Pi^{(1)}$-operator in (4.29), we have $i \geq 2$.

With Lemma 4.2.2, it follows from (4.24) that, if $\|\eta\|$ is sufficiently small,

$$\begin{aligned} e^{-B(\eta)} b_p e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{ad}^{(n)}_{B(\eta)}(b_p) \\ e^{-B(\eta)} b^*_p e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{ad}^{(n)}_{B(\eta)}(b^*_p) \end{aligned} \tag{4.31}$$

where the series converge absolutely (the proof is a translation to momentum space of [20, Lemma 3.3]).

In our analysis, we will use the fact that, on states with $\mathcal{N}_+ \ll N$, the action of the generalized Bogoliubov transformation (4.21) can be approximated by the action of the standard Bogoliubov transformation (4.22), which is explicitly given by (4.23) (from the definition (1.22), we expect that $b_p \simeq a_p$ and $b^*_p \simeq a^*_p$ on states with $\mathcal{N}_+ \ll N$). To make this statement more precise we define, under the assumption that $\kappa > 0$ is small enough, the remainder operators

$$d_q = \sum_{m\geq 0} \frac{1}{m!} \Big[\mathrm{ad}^{(m)}_{-B(\eta)}(b_q) - \eta_q^m b^{\sharp_m}_{\alpha_m q} \Big], \quad d^*_q = \sum_{m\geq 0} \frac{1}{m!} \Big[\mathrm{ad}^{(m)}_{-B(\eta)}(b^*_q) - \eta_q^m b^{\sharp_{m+1}}_{\alpha_m q} \Big] \tag{4.32}$$

where $q \in \Lambda^*_+$, $(\sharp_m, \alpha_m) = (\cdot, +1)$ if m is even and $(\sharp_m, \alpha_m) = (*, -1)$ if m is odd. It follows then from (4.31) that

$$e^{-B(\eta)} b_q e^{B(\eta)} = \gamma_q b_q + \sigma_q b^*_{-q} + d_q, \qquad e^{-B(\eta)} b^*_q e^{B(\eta)} = \gamma_q b^*_q + \sigma_q b_{-q} + d^*_q \tag{4.33}$$

where we introduced the notation $\gamma_q = \cosh(\eta_q)$ and $\sigma_q = \sinh(\eta_q)$. It will also be useful to introduce remainder operators in position space. For $x \in \Lambda$, we define the operator valued distributions $\check{d}_x, \check{d}_x^*$ through

$$e^{-B(\eta)} \check{b}_x e^{B(\eta)} = b(\check{\gamma}_x) + b^*(\check{\sigma}_x) + \check{d}_x, \qquad e^{-B(\eta)} \check{b}_x^* e^{B(\eta)} = b^*(\check{\gamma}_x) + b(\check{\sigma}_x) + \check{d}_x^*$$

where $\check{\gamma}_x(y) = \sum_{q\in\Lambda^*} \cosh(\eta_q) e^{iq\cdot(x-y)}$ and $\check{\sigma}_x(y) = \sum_{q\in\Lambda^*} \sinh(\eta_q) e^{iq\cdot(x-y)}$.

The next lemma confirms the intuition that remainder operators are small, on states with $\mathcal{N}_+ \ll N$. This Lemma is the result that will be used in the rest of the paper (in particular in Section 4.7) to control the action of generalized Bogoliubov transformations.

Lemma 4.2.3. *Let $\eta \in \ell^2(\Lambda_+^*)$, $n \in \mathbb{Z}$. Let the remainder operators be defined as in (4.32). Then, if $\kappa > 0$ is small enough, there exists $C > 0$ such that*

$$\begin{aligned} \|(\mathcal{N}_+ + 1)^{n/2} d_p \xi\| &\leq \frac{C}{N} \Big[|\eta_p| \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|b_p (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \Big], \\ \|(\mathcal{N}_+ + 1)^{n/2} d_p^* \xi\| &\leq \frac{C}{N} \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| \end{aligned} \tag{4.34}$$

for all $p \in \Lambda_+^*$ and, in position space, such that

$$\begin{aligned} \|(\mathcal{N}_+ + 1)^{n/2} \check{d}_x \xi\| &\leq \frac{C}{N} \Big[\|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|\check{a}_x (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \Big] \\ \|(\mathcal{N}_+ + 1)^{n/2} \check{a}_y \check{d}_x \xi\| &\leq \frac{C}{N} \Big[\|\check{a}_x (\mathcal{N}_+ + 1)^{(n+1)/2} \xi\| + (1 + |\check{\eta}(x-y)|) \|(\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \\ &\qquad + \|\check{a}_y (\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \Big] \\ \|(\mathcal{N}_+ + 1)^{n/2} \check{d}_x \check{d}_y \xi\| &\leq \frac{C}{N^2} \Big[\|(\mathcal{N}_+ + 1)^{(n+6)/2} \xi\| + |\check{\eta}(x-y)| \|(\mathcal{N}_+ + 1)^{(n+4)/2} \xi\| \\ &\qquad + \|\check{a}_x (\mathcal{N}_+ + 1)^{(n+5)/2} \xi\| + \|\check{a}_y (\mathcal{N}_+ + 1)^{(n+5)/2} \xi\| \\ &\qquad + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1)^{(n+4)/2} \xi\| \Big] \end{aligned} \tag{4.35}$$

for all $x, y \in \Lambda$, in the sense of distributions.

Proof. To prove the first bound in (4.34), we notice that, from (4.32) and from the triangle inequality (for simplicity, we focus on $n = 0$, powers of $\mathcal{N}_+$ can be easily commuted through the operators d_p),

$$\|d_q \xi\| \leq \sum_{m \geq 0} \frac{1}{m!} \left\| \left[\mathrm{ad}^{(m)}_{-B(\eta)}(b_q) - \eta_q^m b^{\sharp_m}_{\alpha_m p} \right] \xi \right\| \tag{4.36}$$

From Lemma 4.2.2, we can bound the norm $\|[\mathrm{ad}^{(m)}_{-B(\eta)}(b_q) - \eta_q^m b^{\sharp_m}_{\alpha_m p}]\xi\|$ by the sum of one term of the form

$$\left\| \left[\left(\frac{N - \mathcal{N}_+}{N} \right)^{\frac{m + (1-\alpha_m)/2}{2}} \left(\frac{N + 1 - \mathcal{N}_+}{N} \right)^{\frac{m - (1-\alpha_m)/2}{2}} - 1 \right] \eta_p^m b^{\sharp_m}_{\alpha_m p} \xi \right\| \tag{4.37}$$

and of exactly $2^m m! - 1$ terms of the form

$$\Big\| \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_{\alpha_{\ell_1} p}) \xi \Big\| \tag{4.38}$$

where $i_1, k_1, \ell_1 \in \mathbb{N}$, $j_1, \dots, j_{k_1} \in \mathbb{N}\backslash\{0\}$ and where each Λ_r-operator is either a factor $(N - \mathcal{N}_+)/N$, a factor $(N + 1 - \mathcal{N}_+)/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-h} \Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{z_1}, \dots, \eta^{z_h}) \tag{4.39}$$

with $h, z_1, \dots, z_h \in \mathbb{N}\backslash\{0\}$. Furthermore, since we are considering the term (4.37) separately, each term of the form (4.38) must have either $k_1 > 0$ or it must contain at least one Λ-operator having the form (4.39). Since (4.37) vanishes for $m = 0$, it is easy to bound

$$\begin{aligned} \Bigg\| \Bigg[\left(\frac{N - \mathcal{N}_+}{N} \right)^{\frac{m + (1-\alpha_m)/2}{2}} \left(\frac{N + 1 - \mathcal{N}_+}{N} \right)^{\frac{m - (1-\alpha_m)/2}{2}} - 1 \Bigg] \eta_p^m b^{\sharp_m}_{\alpha_m p} \xi \Bigg\| \\ \leq C^m \kappa^{m-1} N^{-1} |\eta_p| \|(\mathcal{N}_+ + 1)^{3/2} \xi\| \end{aligned}$$

On the other hand, distinguishing the cases $\ell_1 = 0$ and $\ell_1 > 0$, we can bound

$$\begin{aligned} &\Big\| \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_{\alpha_{\ell_1} p}) \xi \Big\| \\ &\qquad \leq C^m \kappa^{m-1} N^{-1} \Big[|\eta_p| \|(\mathcal{N}_+ + 1)^{3/2} \xi\| + \|b_p (\mathcal{N}_+ + 1) \xi\| \Big] \end{aligned}$$

Inserting the last two bounds in (4.36) and summing over m under the assumption that $\kappa > 0$ is small enough, we arrive at the first estimate (4.34). The second estimate in (4.34) can be proven similarly (notice that, when dealing with the second estimate in (4.34), contributions of the form (4.38) with $\ell_1 = 0$, can only be bounded by $\|b_p^*(\mathcal{N}_+ + 1)\xi\| \leq \|(\mathcal{N}_+ + 1)^{3/2} \xi\|$). Also the bounds in (4.35) can be shown analogously, using [14, Lemma 7.2]. $\square$

4.3 Excitation Hamiltonians

Recall the definition (1.15) of the unitary operator $U_N : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N}$, first introduced in [64]. In terms of creation and annihilation operators, U_N is given by

$$U_N \, \psi_N = \bigoplus_{n=0}^{N} (1 - |\varphi_0\rangle\langle\varphi_0|)^{\otimes n} \frac{a(\varphi_0)^{N-n}}{\sqrt{(N-n)!}} \, \psi_N$$

for all $\psi_N \in L^2_s(\Lambda^N)$ (on the r.h.s. we identify $\psi_N \in L^2_s(\Lambda^N)$ with $\{0, \dots, 0, \psi_N, 0, \dots\} \in \mathcal{F}$). The map $U_N^* : \mathcal{F}_+^{\leq N} \to L^2_s(\Lambda^N)$ is given, on the other hand, by

$$U_N^* \, \{\alpha^{(0)}, \dots, \alpha^{(N)}\} = \sum_{n=0}^{N} \frac{a^*(\varphi_0)^{N-n}}{\sqrt{(N-n)!}} \, \alpha^{(n)}$$

It is instructive to compute the action of U_N on products of a creation and an annihilation operator (products of the form $a_p^* a_q$ can be thought of as operators mapping $L^2_s(\Lambda^N)$ to itself). For any $p, q \in \Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$, we find (see [64]):

$$\begin{aligned} U_N\, a_0^* a_0\, U_N^* &= N - \mathcal{N}_+ \\ U_N\, a_p^* a_0\, U_N^* &= a_p^* \sqrt{N - \mathcal{N}_+} \\ U_N\, a_0^* a_p\, U_N^* &= \sqrt{N - \mathcal{N}_+}\, a_p \\ U_N\, a_p^* a_q\, U_N^* &= a_p^* a_q \end{aligned} \tag{4.40}$$

Writing (4.1) in momentum space and using the formalism of second quantization, we find

$$H_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{2N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r} \tag{4.41}$$

where

$$\widehat{V}(k) = \int_{\mathbb{R}^3} V(x) e^{-ik \cdot x} dx$$

is the Fourier transform of V, defined for all $k \in \mathbb{R}^3$. With (4.40), we can compute the excitation Hamiltonian $\mathcal{L}_N = U_N H_N U_N^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$. We obtain

$$\mathcal{L}_N = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)} \tag{4.42}$$

with

$$\begin{aligned} \mathcal{L}_N^{(0)} &= \frac{N-1}{2N} \kappa \widehat{V}(0)(N - \mathcal{N}_+) + \frac{\kappa \widehat{V}(0)}{2N} \mathcal{N}_+ (N - \mathcal{N}_+) \\ \mathcal{L}_N^{(2)} &= \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p + \sum_{p \in \Lambda_+^*} \kappa \widehat{V}(p/N) \left[b_p^* b_p - \frac{1}{N} a_p^* a_p \right] \\ &\quad + \frac{\kappa}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[b_p^* b_{-p}^* + b_p b_{-p} \right] \\ \mathcal{L}_N^{(3)} &= \frac{\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) \left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right] \\ \mathcal{L}_N^{(4)} &= \frac{\kappa}{2N} \sum_{\substack{p,q \in \Lambda_+^*, r \in \Lambda^* : \\ r \neq -p, -q}} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r} \end{aligned} \tag{4.43}$$

Conjugation with U_N extracts, from the original quartic interaction, some constant and quadratic contributions, collected in $\mathcal{L}_N^{(0)}$ and $\mathcal{L}_N^{(2)}$. In the Gross-Pitevskii regime, however, this is not enough; there are still important (order N) contributions to the ground state energy and to the energy of low-lying excitations that are hidden in the cubic and quartic terms. In other words, in contrast with the mean-field regime, here we cannot expect $\mathcal{L}_N^{(3)}$ and $\mathcal{L}_N^{(4)}$ to be small. To extract the relevant contributions from $\mathcal{L}^{(3)}$

and $\mathcal{L}^{(4)}$, we are going to conjugate $\mathcal{L}_N$ with a generalized Bogoliubov transformation of the form (4.21).

To choose the function $\eta \in \ell^2(\Lambda_+^*)$ entering the generalized Bogoliubov transformation (4.21), we consider the solution of the Neumann problem

$$\left[-\Delta + \frac{\kappa}{2} V\right] f_\ell = \lambda_\ell f_\ell \tag{4.44}$$

on the ball $|x| \leq N\ell$ (we omit the N-dependence in the notation for f_ℓ and for λ_ℓ; notice that λ_ℓ scales as N^{-3}), with the normalization $f_\ell(x) = 1$ if $|x| = N\ell$. It is also useful to define $w_\ell = 1 - f_\ell$ (so that $w_\ell(x) = 0$ if $|x| > N\ell$). By scaling, we observe that $f_\ell(N.)$ satisfies the equation

$$\left[-\Delta + \frac{\kappa N^2}{2} V(Nx)\right] f_\ell(Nx) = N^2 \lambda_\ell f_\ell(Nx)$$

on the ball $|x| \leq \ell$. We choose $0 < \ell < 1/2$, so that the ball of radius ℓ is contained in the box $\Lambda = [-1/2; 1/2]^3$. We extend then $f_\ell(N.)$ to Λ, by choosing $f_\ell(Nx) = 1$ for all $|x| > \ell$. Then

$$\left(-\Delta + \frac{\kappa N^2}{2} V(Nx)\right) f_\ell(Nx) = N^2 \lambda_\ell f_\ell(Nx) \chi_\ell(x) \tag{4.45}$$

where χ_ℓ is the characteristic function of the ball of radius ℓ. It follows that the functions $x \to f_\ell(Nx)$ and also $x \to w_\ell(Nx) = 1 - f_\ell(Nx)$ can be extended as periodic functions on the torus Λ. The Fourier coefficients of the function $x \to w_\ell(Nx)$ are given by

$$\int_\Lambda w_\ell(Nx) e^{-ip\cdot x} dx = \frac{1}{N^3} \widehat{w}_\ell(p/N)$$

where

$$\widehat{w}_\ell(p) = \int_{\mathbb{R}^3} w_\ell(x) e^{-ip\cdot x} dx$$

is the Fourier transform of the (compactly supported) function w_ℓ. The Fourier coefficients of $x \to f_\ell(Nx)$ are then given by

$$\widehat{f}_{\ell,N}(p) := \int_\Lambda f_\ell(Nx) e^{-ip\cdot x} dx = \delta_{p,0} - \frac{1}{N^3} \widehat{w}_\ell(p/N) \tag{4.46}$$

for all $p \in \Lambda^*$. From (4.45), we derive

$$-p^2 \widehat{w}_\ell(p/N) + \frac{\kappa N^2}{2} \sum_{q \in \Lambda^*} \widehat{V}((p-q)/N) \widehat{f}_{\ell,N}(q) = N^5 \lambda_\ell \sum_{q \in \Lambda^*} \widehat{\chi}_\ell(p-q) \widehat{f}_{\ell,N}(q) \tag{4.47}$$

In the next lemma we collect some important properties of w_ℓ, f_ℓ. The proof of the lemma can be found in [36, Lemma A.1] and in [20, Lemma 4.1].

Lemma 4.3.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric. Fix $\ell > 0$ and let f_ℓ denote the solution of* (4.44).

i) We have

$$\lambda_\ell = \frac{3\mathfrak{a}_0}{N^3\ell^3}\left(1+\mathcal{O}(\mathfrak{a}_0/N\ell)\right)$$

ii) We have $0 \leq f_\ell, w_\ell \leq 1$ *and*

$$\left|\kappa\int V(x)f_\ell(x)dx - 8\pi\mathfrak{a}_0\right| \leq \frac{C\kappa}{N}\,. \tag{4.48}$$

iii) There exists a constant $C>0$ *such that*

$$w_\ell(x) \leq \frac{C\kappa}{|x|+1} \qquad \text{and} \qquad |\nabla w_\ell(x)| \leq \frac{C\kappa}{x^2+1}. \tag{4.49}$$

for all $|x| \leq N\ell$.

iv) There exists a constant $C>0$ *such that*

$$|\widehat{w}_\ell(p)| \leq \frac{C\kappa}{p^2}$$

for all $p\in\Lambda^*_+$.

We define $\eta : \Lambda^* \to \mathbb{R}$ through

$$\eta_p = -\frac{1}{N^2}\widehat{w}_\ell(p/N) \tag{4.50}$$

From (4.47), we find these coefficients satisfy the relation

$$p^2\eta_p + \frac{\kappa}{2}\widehat{V}(p/N) + \frac{\kappa}{2N}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N)\eta_q = N^3\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\widehat{f}_{\ell,N}(q) \tag{4.51}$$

or equivalently, expressing also the r.h.s. through the coefficients η_p,

$$\begin{aligned} p^2\eta_p + \frac{\kappa}{2}\widehat{V}(p/N) &+ \frac{\kappa}{2N}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N)\eta_q \\ &= N^3\lambda_\ell\widehat{\chi}_\ell(p) + N^2\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q \end{aligned} \tag{4.52}$$

With Lemma 4.3.1, we can bound

$$|\eta_p| \leq \frac{C\kappa}{p^2} \tag{4.53}$$

for all $p\in\Lambda^*_+ = 2\pi\mathbb{Z}^3\backslash\{0\}$. Eq. (4.53) implies that $\eta\in\ell^2(\Lambda^*_+)$, with norm bounded uniformly in N. On the other hand, the H^1-norm of η diverges, as $N\to\infty$. From Lemma 4.3.1, part iii), we find

$$\sum_{p\in\Lambda^*_+} p^2|\eta_p|^2 = \|\nabla\check{\eta}\|_2^2 \leq CN\kappa^2 \tag{4.54}$$

We will mostly use the coefficients η_p with $p \neq 0$. Sometimes, however, it will also be useful to have an estimate for η_0 (because the equation (4.52) involves η_0). From Lemma 4.3.1, part iii), we find

$$|\eta_0| \leq N^{-2} \int_{\mathbb{R}^3} w_\ell(x)dx \leq C\kappa$$

Sometimes, it will also be useful to switch to position space. Defining $\check{\eta}(x) = \sum_{q\in\Lambda^*} \eta_q e^{iq\cdot x}$ we find by Plancherel that $\|\check{\eta}\|_2 \leq C$ uniformly in N and, from (4.49) that

$$\|\check{\eta}\|_\infty \leq C\kappa N. \tag{4.55}$$

Because of (4.33), it will also be useful to have bounds for the quantities $\sigma_q = \sinh(\eta_q)$ and $\gamma_q = \cosh(\eta_q)$, and, in position space, for $\check{\sigma}(x) = \sum_{q\in\Lambda^*} \sinh(\eta_q)e^{iq\cdot x}$ and $\check{\gamma}(x) = \sum_{q\in\Lambda^*} \cosh(\eta_q)e^{iq\cdot x} = \delta(x) + \check{r}(x)$, with $\check{r}(x) = \sum_{q\in\Lambda^*} [\cosh(\eta_q) - 1]\, e^{iq\cdot x}$. In momentum space, we find the pointwise bounds

$$|\sigma_q| \leq C\kappa|q|^{-2}, \quad |\sigma_q - \eta_q| \leq C\kappa^3|q|^{-6}, \quad |\gamma_q| \leq C, \quad |\gamma_q - 1| \leq C\kappa^2|q|^{-4} \tag{4.56}$$

for all $q \in \Lambda^*_+$. In position space, we obtain from (4.55) the estimates

$$\|\check{\sigma}\|_2 \leq C\kappa, \quad \|\check{\sigma}\|_\infty \leq C\kappa N, \quad \|\check{\sigma} * \check{\gamma}\|_\infty \leq C\kappa N \tag{4.57}$$

With $\eta \in \ell^2(\Lambda^*_+)$, we construct the generalized Bogoliubov transformation $e^{B(\eta)} : \mathcal{F}^{\leq N}_+ \to \mathcal{F}^{\leq N}_+$, defined as in (4.21). Furthermore, we define a new, renormalized, excitation Hamiltonian $\mathcal{G}_N : \mathcal{F}^{\leq N}_+ \to \mathcal{F}^{\leq N}_+$ by setting

$$\mathcal{G}_N = e^{-B(\eta)}\mathcal{L}_N e^{B(\eta)} = e^{-B(\eta)}U_N H_N U_N^* e^{B(\eta)} : \mathcal{F}^{\leq N}_+ \to \mathcal{F}^{\leq N}_+ \tag{4.58}$$

In the next proposition, we collect some important properties of the renormalized excitation Hamiltonian $\mathcal{G}_N$. Here and in the following, we will use the notation

$$\mathcal{K} = \sum_{p\in\Lambda^*_+} p^2 a_p^* a_p \qquad \text{and} \qquad \mathcal{V}_N = \frac{\kappa}{2N} \sum_{p,q\in\Lambda^*_+, r\in\Lambda^*: r\neq -p,-q} \widehat{V}(r/N) a^*_{p+r} a^*_q a_{q+r} a_p \tag{4.59}$$

for the kinetic and potential energy operators, restricted on $\mathcal{F}^{\leq N}_+$. Furthermore, we will write $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$.

Proposition 4.3.2. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Let $\mathcal{G}_N$ be defined as in (4.58) and E_N be the ground state energy of the Hamilton operator (4.41).*

a) We have

$$\mathcal{G}_N - E_N = \mathcal{H}_N + \Delta_N$$

where the error term Δ_N is such that for every $\delta > 0$, there exists $C > 0$ with

$$\pm\, \Delta_N \leq \delta \mathcal{H}_N + C\kappa(\mathcal{N}_+ + 1) \tag{4.60}$$

Furthermore, for every $k \in \mathbb{N}$ there exists a $C > 0$ such that

$$\pm\, ad^{(k)}_{i\mathcal{N}_+}(\mathcal{G}_N) = \pm ad^{(k)}_{i\mathcal{N}_+}(\Delta_N) = \pm\big[i\mathcal{N}_+, \dots \big[i\mathcal{N}_+, \Delta_N\big] \dots\big] \leq C(\mathcal{H}_N + 1) \tag{4.61}$$

b) For $p \in \Lambda_+^*$, we use the notation, already introduced in (4.33), $\sigma_p = \sinh \eta_p$ and $\gamma_p = \cosh \eta_p$. Let

$$\begin{aligned} C_{\mathcal{G}_N} = & \frac{(N-1)}{2} \kappa \widehat{V}(0) + \sum_{p \in \Lambda_+^*} \Big[p^2 \sigma_p^2 + \kappa \widehat{V}(p/N) \left(\sigma_p \gamma_p + \sigma_p^2 \right) \Big] \\ & + \frac{\kappa}{2N} \sum_{p,q \in \Lambda_+^*} \widehat{V}((p-q)/N) \sigma_q \gamma_q \sigma_p \gamma_p + \frac{1}{N} \sum_{p \in \Lambda^*} \Big[p^2 \eta_p^2 + \frac{\kappa}{2N} \big(\widehat{V}(\cdot/N) * \eta \big)_p \eta_p \Big] \\ & - \frac{1}{N} \sum_{q \in \Lambda^*} \kappa \hat{V}(q/N) \eta_q \sum_{p \in \Lambda_+^*} \sigma_p^2 \end{aligned} \tag{4.62}$$

For every $p \in \Lambda_+^*$, let

$$\begin{aligned} \Phi_p = & \, 2p^2 \sigma_p^2 + \kappa \widehat{V}(p/N) \left(\gamma_p + \sigma_p \right)^2 + \frac{2\kappa}{N} \gamma_p \sigma_p \sum_{q \in \Lambda^*} \widehat{V}((p-q)/N) \eta_q \\ & - (\gamma_p^2 + \sigma_p^2) \frac{\kappa}{N} \sum_{q \in \Lambda^*} \widehat{V}(q/N) \widetilde{\eta}_q \end{aligned} \tag{4.63}$$

and

$$\begin{aligned} \Gamma_p = & \, 2p^2 \sigma_p \gamma_p + \kappa \widehat{V}(p/N) (\gamma_p + \sigma_p)^2 + (\gamma_p^2 + \sigma_p^2) \frac{\kappa}{N} \sum_{q \in \Lambda^*} \widehat{V}((p-q)/N) \eta_q \\ & - 2 \gamma_p \sigma_p \frac{\kappa}{N} \sum_{q \in \Lambda^*} \widehat{V}(q/N) \eta_q \end{aligned} \tag{4.64}$$

Using $\Phi_p, \mathcal{G}_p$ we construct the operator

$$\mathcal{Q}_{\mathcal{G}_N} = \sum_{p \in \Lambda_+^*} \Phi_p b_p^* b_p + \frac{1}{2} \sum_{p \in \Lambda_+^*} \Gamma_p (b_p^* b_{-p}^* + b_p^* b_{-p}^*) \tag{4.65}$$

Moreover, we define

$$\mathcal{C}_N = \frac{\kappa}{\sqrt{N}} \sum_{\substack{p,q \in \Lambda_+^* \\ q \neq -p}} \widehat{V}(p/N) \Big[b_{p+q}^* b_{-p}^* \big(\gamma_q b_q + \sigma_q b_{-q}^* \big) + h.c. \Big] \tag{4.66}$$

Then, we have

$$\mathcal{G}_N = C_{\mathcal{G}_N} + \mathcal{Q}_{\mathcal{G}_N} + \mathcal{H}_N + \mathcal{C}_N + \mathcal{E}_{\mathcal{G}_N} \tag{4.67}$$

with an error term $\mathcal{E}_{\mathcal{G}_N}$ satisfying, on $\mathcal{F}_+^{\leq N}$, the bound

$$\pm \, \mathcal{E}_{\mathcal{G}_N} \leq \frac{C}{\sqrt{N}} (\mathcal{H}_N + \mathcal{N}_+^2 + 1)(\mathcal{N}_+ + 1) \tag{4.68}$$

For the Hamilton operator (4.10) with parameter $\beta \in (0;1)$, a result similar to Proposition 4.3.2 has been recently established in Theorem 3.2 of [14]. The main difference between Prop. 4.3.2 and previous results for $\beta < 1$ is the emergence, in (4.67), of a cubic and a quartic term in the generalized creation and annihilation operators (the quartic term $\mathcal{V}_N$ is included in the Hamiltonian $\mathcal{H}_N$). As explained in the introduction, for $\beta < 1$, the cubic and the quartic parts of $\mathcal{G}_N$ were negligible and could be absorbed in the error $\mathcal{E}_{\mathcal{G}_N}$. In the Gross-Pitaevskii regime, on the other hand, this is not possible. It is easy to find normalized $\xi \in \mathcal{F}_+^{\leq N}$ with bounded expectation of $(\mathcal{N}_+ + 1)(\mathcal{H}_N + \mathcal{N}_+^2 + 1)$ such that $\langle \xi, \mathcal{C}_N \xi\rangle$ and $\langle \xi, \mathcal{V}_N \xi\rangle$ are of order one and do not tend to zero, as $N \to \infty$.

To extract the important contributions that are still hidden in the cubic and in the quartic terms on the r.h.s. of (4.67), we conjugate the renormalized excitation Hamiltonian $\mathcal{G}_N$ with a unitary operator obtained by exponentiating a cubic expression in creation and annihilation operators.

More precisely, we define the skew-symmetric operator $A : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ by

$$\begin{aligned} A &= \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r \left[\sigma_v b^*_{r+v} b^*_{-r} b^*_{-v} + \gamma_v b^*_{r+v} b^*_{-r} b_v - \text{h.c.} \right] \\ &=: A_\sigma + A_\gamma - \text{h.c.} \end{aligned} \tag{4.69}$$

where $P_L = \{ p \in \Lambda_+^* : |p| \leq N^{1/2} \}$ corresponds to low momenta and $P_H = \Lambda^*_+ \setminus P_L$ to high momenta (by definition $r + v \neq 0$). The coefficients η_p are defined in (4.50); they are the same as those used in the definition of the generalized Bogoliubov transformation $\exp(B(\eta))$ appearing in $\mathcal{G}_N$. We then define the cubically renormalized excitation Hamiltonian

$$\mathcal{J}_N := e^{-A} e^{-B(\eta)} U_N H_N U_N^* e^{B(\eta)} e^{A} = e^{-A} \mathcal{G}_N e^{A} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}. \tag{4.70}$$

In the next proposition, we collect important properties of $\mathcal{J}_N$.

Proposition 4.3.3. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Let $\mathcal{J}_N$ be defined as in (4.70). For $p \in \Lambda_+^*$, we use again the notation $\sigma_p = \sinh(\eta_p)$, $\gamma_p = \cosh(\eta_p)$ and we recall the notation $\widehat{f}_{\ell,N}$ from (4.46). Let*

$$\begin{aligned} C_{\mathcal{J}_N} := \frac{(N-1)}{2} \kappa \widehat{V}(0) &+ \sum_{p \in \Lambda_+^*} \left[p^2 \sigma_p^2 + \kappa \widehat{V}(p/N) \sigma_p \gamma_p + \kappa \big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N} \big)_p \sigma_p^2 \right] \\ &+ \frac{\kappa}{2N} \sum_{p,q \in \Lambda_+^*} \widehat{V}((p-q)/N) \sigma_q \gamma_q \sigma_p \gamma_p + \frac{1}{N} \sum_{p \in \Lambda^*} \left[p^2 {\eta_p}^2 + \frac{\kappa}{2N} \big(\widehat{V}(\cdot/N) * \eta \big)_p \eta_p \right] \end{aligned} \tag{4.71}$$

Moreover, for every $p \in \Lambda_+^*$ we define

$$\begin{aligned} F_p &:= p^2 (\sigma_p^2 + \gamma_p^2) + \kappa \big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N} \big)_p (\gamma_p + \sigma_p)^2; \\ G_p &:= 2p^2 \sigma_p \gamma_p + \kappa \big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N} \big)_p (\gamma_p + \sigma_p)^2 \end{aligned} \tag{4.72}$$

With the coefficients F_p and G_p, we construct the operator

$$\mathcal{Q}_{\mathcal{J}_N} := \sum_{p\in\Lambda_+^*}\Big[F_p b_p^* b_p + \frac{1}{2}G_p\big(b_p^* b_{-p}^* 1 + b_p b_{-p}\big)\Big]$$

quadratic in the b, b^*-fields. Then, we have

$$\mathcal{J}_N = C_{\mathcal{J}_N} + \mathcal{Q}_{\mathcal{J}_N} + \mathcal{V}_N + \mathcal{E}_{\mathcal{J}_N}$$

for an error term $\mathcal{E}_{\mathcal{J}_N}$ satisfying, on $\mathcal{F}_+^{\leq N}$,

$$\pm\,\mathcal{E}_{\mathcal{J}_N} \leq CN^{-1/4}\Big[(\mathcal{H}_N+1)(\mathcal{N}_++1) + (\mathcal{N}_++1)^3\Big]\,. \tag{4.73}$$

The proof of Proposition 4.3.2 is deferred to Section 4.7. Proposition 4.3.3 will then be proved in Section 4.8. In the next three sections, on the other hand, we show how to use these two propositions to complete the proof of Theorem 4.1.1.

4.4 Bounds on excitations vectors

To make use of the bounds (4.68) and (4.73), we need to prove that excitation vectors associated with many-body wave functions $\psi_N \in L_s^2(\Lambda^N)$ with small excitation energy, defined either as $e^{B(\eta)}U_N\,\psi_N$ (if we want to apply (4.68)) or as $e^A e^{B(\eta)}U_N\,\psi_N$ (if we want to apply (4.73)) have finite expectations of the operator $(\mathcal{H}_N+1)(\mathcal{N}_++1)+(\mathcal{N}_++1)^3$. This is the goal of this section.

We start with estimates on the excitation vector $\xi_N = e^{B(\eta)}U_N\,\psi_N$, that are relevant to bound errors arising before conjugation with the cubic exponential $\exp(A)$.

Proposition 4.4.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa \geq 0$ is small enough. Let E_N be the ground state energy of the Hamiltonian H_N defined in (4.41) (or, equivalently, in (4.1)). Let $\psi_N \in L_s^2(\Lambda^N)$ with $\|\psi_N\| = 1$ belong to the spectral subspace of H_N with energies below $E_N+\zeta$, for some $\zeta > 0$, i.e.*

$$\psi_N = \mathbf{1}_{(-\infty;E_N+\zeta]}(H_N)\psi_N \tag{4.74}$$

Let $\xi_N = e^{-B(\eta)}U_N\psi_N$ be the renormalized excitation vector associated with ψ_N. Then, for any $k\in\mathbb{N}$ there exists a constant $C>0$ such that

$$\langle \xi_N, (\mathcal{N}_++1)^k(\mathcal{H}_N+1)\xi_N\rangle \leq C(1+\zeta^{k+1}) \tag{4.75}$$

Remark: since $\mathcal{N}_+ \leq C\mathcal{H}_N$, (4.75) immediately implies bounds on all moments of $\mathcal{N}_+$. In particular, taking $k=0$, (4.75) implies that $\langle\xi_N, \mathcal{N}_+\xi_N\rangle \leq C(1+\zeta)$ and therefore that many-body wave functions $\psi_N \in L_s^2(\Lambda^N)$ satisfying (4.74) exhibit complete Bose-Einstein condensation in the zero-momentum mode φ_0 with optimal rate. In other words, (4.75) with $k = 0$ implies already that the one-particle reduced density $\gamma_N^{(1)}$ associated with ψ_N is such that

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq \frac{C(\zeta+1)}{N} \tag{4.76}$$

Notice that the proof of (4.75) for $k = 0$ (and also of its consequence (4.76)) has already been given in [13]; here we extend it to all $k \in \mathbb{N}$ (the case $k = 1$ has been discussed in [14, Prop. 4.2], for Hamilton operators of the form (4.10) and $\beta \in (0;1)$).

Proof of Prop. 4.4.1. The proof goes by induction and is similar to the proof of [14, Prop. 4.2]. For $k = 0$, we use (4.60) from Prop. 4.3.2 above, choosing for example $\delta = 1/2$, to show that

$$E_N + \zeta \geq \langle \psi_N, H_N \psi_N \rangle = \langle \xi_N, \mathcal{G}_N \xi_N \rangle \geq E_N + \frac{1}{2} \langle \xi_N, \mathcal{H}_N \xi_N \rangle - C\kappa \langle \xi_N, (\mathcal{N}_+ + 1)\xi_N \rangle$$

Since $\mathcal{N}_+ \leq (2\pi)^2 \mathcal{K} \leq (2\pi)^2 \mathcal{H}_N$ we obtain, for $\kappa > 0$ sufficiently small, that

$$\langle \xi_N, \mathcal{H}_N \xi_N \rangle \leq C(1+\zeta)$$

Let us now consider the induction step. We assume (4.75) holds true for a $k \in \mathbb{N}$, we show it for k replaced by $(k+1)$. Let $\mathcal{G}'_N = \mathcal{G}_N - E_N$. By assumption, $\xi_N = \mathbf{1}_{(-\infty;\zeta]}(\mathcal{G}'_N)\xi_N$. From (4.60), we find

$$\begin{aligned} \langle \xi_N, &(\mathcal{N}_+ + 1)^{k+1} (\mathcal{H}_N + 1) \xi_N \rangle \\ &= \langle \xi_N, (\mathcal{N}_+ + 1)^{(k+1)/2} (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)^{(k+1)/2} \xi_N \rangle \\ &\leq 2 \langle \xi_N, (\mathcal{N}_+ + 1)^{(k+1)/2} (\mathcal{G}'_N + C)(\mathcal{N}_+ + 1)^{(k+1)/2} \xi_N \rangle \end{aligned} \tag{4.77}$$

We write

$$\begin{aligned} (\mathcal{N}_+ + 1)^{(k+1)/2} &(\mathcal{G}'_N + C)(\mathcal{N}_+ + 1)^{(k+1)/2} \\ &= (\mathcal{N}_+ + 1)^{(k+1)} (\mathcal{G}'_N + C) + (\mathcal{N}_+ + 1)^{(k+1)/2} \Big[\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{(k+1)/2} \Big] \end{aligned} \tag{4.78}$$

Using the induction assumption and the fact that $\xi_N = \mathbf{1}_{(-\infty;\zeta]}(\mathcal{G}'_N)\xi_N$, the expectation of the first operator in (4.78) can be controlled by

$$\begin{aligned} &\big| \langle \xi_N, (\mathcal{N}_+ + 1)^{(k+1)} (\mathcal{G}'_N + C) \xi_N \rangle \big| \\ &\quad \leq \langle \xi_N, (\mathcal{N}_+ + 1)^{(k+1)} (\mathcal{G}'_N + C)^{-k-1} (\mathcal{N}_+ + 1)^{(k+1)} \xi \rangle^{1/2} \, \langle \xi_N, (\mathcal{G}'_N + C)^{k+3} \xi_N \rangle^{1/2} \\ &\quad \leq \langle \xi_N, (\mathcal{N}_+ + 1)^{(k+1)} \xi_N \rangle^{1/2} \langle \xi_N, (\mathcal{G}'_N + C)^{k+3} \xi_N \rangle^{1/2} \\ &\quad \leq C(1 + \zeta^{k+2}) \end{aligned} \tag{4.79}$$

To bound the expectation of the second term on the r.h.s. of (4.78), we use the identity

$$\frac{1}{\sqrt{z}} = \frac{1}{\pi} \int_0^\infty \frac{dt}{\sqrt{t}} \frac{1}{t+z}$$

to write

$$
\begin{aligned}
&(\mathcal{N}_+ + 1)^{(k+1)/2}\Big[\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{(k+1)/2}\Big] \\
&\quad = \frac{1}{\pi}\int_0^\infty dt\,\sqrt{t}\,\frac{(\mathcal{N}_+ + 1)^{(k+1)/2}}{t + (\mathcal{N}_+ + 1)^{k+1}}\,\Big[\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{k+1}\Big]\,\frac{1}{t + (\mathcal{N}_+ + 1)^{k+1}}
\end{aligned}
$$

With the identity

$$
\Big[\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{k+1}\Big] = -\sum_{j=1}^{k+1}\binom{k+1}{j}\mathrm{ad}^{(j)}_{\mathcal{N}_+}(\mathcal{G}'_N)\,(\mathcal{N}_+ + 1)^{k+1-j}
$$

which can be proven by induction over k, we obtain

$$
\begin{aligned}
&(\mathcal{N}_+ + 1)^{(k+1)/2}\Big[\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{(k+1)/2}\Big] \\
&\quad = \frac{1}{\pi}\sum_{j=1}^{k+1}(-i)^{j+1}\binom{k+1}{j}\int_0^\infty dt\,\sqrt{t}\,\frac{(\mathcal{N}_+ + 1)^{(k+1)/2}}{t + (\mathcal{N}_+ + 1)^{k+1}}\,\mathrm{ad}^{(j)}_{i\mathcal{N}_+}(\mathcal{G}'_N)\,\frac{(\mathcal{N}_+ + 1)^{k+1-j}}{t + (\mathcal{N}_+ + 1)^{k+1}}
\end{aligned}
$$

From (4.61) in Prop. 4.3.2 we know that $\mathcal{A}_j := (\mathcal{H}_N + 1)^{-1/2}\,\mathrm{ad}^{(j)}_{i\mathcal{N}_+}(\mathcal{G}'_N)(\mathcal{H}_N + 1)^{-1/2}$ is a self-adjoint operator on $\mathcal{F}_+^{\leq N}$, with norm bounded uniformly in N. Hence, we find

$$
\begin{aligned}
&\Big|\langle \xi_N, (\mathcal{N}_+ + 1)^{(k+1)/2}\Big[\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{(k+1)/2}\Big]\xi_N\rangle\Big| \\
&\quad\leq C\sum_{j=1}^{k+1}\int_0^\infty \frac{dt}{(t+1)^{1/2}}\,\Big\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^{(k+1)/2}\xi_N\Big\| \\
&\qquad\qquad\qquad\times\Big\|(\mathcal{H}_N + 1)^{1/2}\frac{(\mathcal{N}_+ + 1)^{k+1-j}}{t + (\mathcal{N}_+ + 1)^{k+1}}\xi_N\Big\| \\
&\quad\leq C\int_0^\infty \frac{dt}{(t+1)^{\frac{2k+3}{2k+2}}}\,\Big\|(\mathcal{N}_+ + 1)^{(k+1)/2}(\mathcal{H}_N + 1)^{1/2}\xi_N\Big\|\Big\|(\mathcal{N}_+ + 1)^{k/2}(\mathcal{H}_N + 1)^{1/2}\xi_N\Big\| \\
&\quad\leq C\Big\|(\mathcal{N}_+ + 1)^{(k+1)/2}(\mathcal{H}_N + 1)^{1/2}\xi_N\Big\|\Big\|(\mathcal{N}_+ + 1)^{k/2}(\mathcal{H}_N + 1)^{1/2}\xi_N\Big\|
\end{aligned}
$$

for a constant $C > 0$ depending on k. With the induction assumption we conclude that, for any $\delta > 0$, there exists $C > 0$ such that

$$
\begin{aligned}
&\big|\langle \xi_N, (\mathcal{N}_+ + 1)^{(k+1)/2}\Big[\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{(k+1)/2}\Big]\xi_N\rangle\big| \\
&\quad\leq \delta\langle \xi_N, (\mathcal{N}_+ + 1)^{k+1}(\mathcal{H}_N + 1)\xi_N\rangle + C(1 + \zeta^{k+1})
\end{aligned} \tag{4.80}
$$

Choosing $\delta > 0$ sufficiently small and combining (4.77), (4.78), (4.79) and (4.80) yields

$$
\langle \xi_N, (\mathcal{N}_+ + 1)^{k+1}(\mathcal{H}_N + 1)\xi_N \leq C(1 + \zeta^{k+2})
$$

which completes the proof. $\square$

Next, we control the growth of powers of $\mathcal{N}_+$ and of the product $(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1)$ under conjugation with the operator $\exp(A)$. These bounds are needed to apply Prop. 4.3.3. First, we focus on the growth of powers of the number of particles operator.

Proposition 4.4.2. *Suppose that A is defined as in (4.69). For any $k \in \mathbb{N}$, there exists $C > 0$ such that, on $\mathcal{F}_+^{\leq N}$, we have the operator inequality*

$$e^{-A}(\mathcal{N}_+ + 1)^k e^A \leq C(\mathcal{N}_+ + 1)^k$$

Proof. Let $\xi \in \mathcal{F}_+^{\leq N}$ and define $\varphi_\xi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\xi(s) := \langle \xi, e^{-sA}(\mathcal{N}_+ + 1)^k e^{sA}\xi\rangle$$

Then we have, using the decomposition $A = A_\sigma + A_\gamma - \text{h.c.}$ from (4.69),

$$\partial_s \varphi_\xi(s) = 2\text{Re}\ \langle \xi, e^{-sA}\big[(\mathcal{N}_+ + 1)^k, A_\sigma\big]e^{sA}\xi\rangle + 2\text{Re}\ \langle \xi, e^{-sA}\big[(\mathcal{N}_+ + 1)^k, A_\gamma\big]e^{sA}\xi\rangle$$

We start by controlling the commutator with A_σ. We find

$$\begin{aligned}
&\langle \xi, e^{-sA}\big[(\mathcal{N}_+ + 1)^k, A_\sigma\big]e^{sA}\xi\rangle \\
&\qquad = \frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} \eta_r\sigma_v \langle e^{sA}\xi, b^*_{r+v}b^*_{-r}b^*_{-v}\big[(\mathcal{N}_+ + 4)^k - (\mathcal{N}_+ + 1)^k\big]e^{sA}\xi\rangle
\end{aligned}$$

With the mean value theorem, we find a function $\theta : \mathbb{N} \to (0;3)$ such that

$$(\mathcal{N}_+ + 4)^k - (\mathcal{N}_+ + 1)^k = k(\mathcal{N}_+ + \theta(\mathcal{N}_+) + 1)^{k-1}$$

Since $b_p\mathcal{N}_+ = (\mathcal{N}_+ + 1)b_p$ and $b^*_p\mathcal{N}_+ = (\mathcal{N}_+ - 1)b^*_p$, we obtain, using Cauchy-Schwarz and the boundedness of θ,

$$\begin{aligned}
&\Big|\langle \xi, e^{-sA}\big[(\mathcal{N}_+ + 1)^k, A_\sigma\big]e^{sA}\xi\rangle\Big| \\
&\leq \frac{C}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} |\eta_r||\sigma_v| \big\|(\mathcal{N}_+ + 1)^{-3/4+(k-1)/2} b_{r+v}b_{-r}b_{-v}e^{sA}\xi\big\| \\
&\qquad\qquad\qquad\qquad \times \big\|(\mathcal{N}_+ + 1)^{3/4+(k-1)/2}e^{sA}\xi\big\| \\
&\leq \frac{C}{\sqrt{N}} \|\eta\|_2\|\sigma\|_2 \big\|(\mathcal{N}_+ + 1)^{3/4+(k-1)/2}e^{sA}\xi\big\|^2 \\
&\leq \frac{C}{\sqrt{N}} \langle e^{sA}\xi, (\mathcal{N}_+ + 1)^{k+1/2}e^{sA}\xi\rangle \\
&\leq C\langle e^{sA}\xi, (\mathcal{N}_+ + 1)^k e^{sA}\rangle
\end{aligned} \tag{4.81}$$

for a constant $C > 0$ depending on k. Similarly, the commutator with A_γ is bounded by

$$\begin{aligned}
&\Big|\langle \xi, e^{-sA}\big[(\mathcal{N}_+ + 1)^k, A_\gamma\big] e^{sA}\xi\rangle\Big| \\
&\leq \frac{C}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} |\eta_r| \big\|(\mathcal{N}_+ + 1)^{-1/4+(k-1)/2} b_{r+v} b_{-r} e^{sA}\xi\big\| \\
&\hspace{6cm} \times \big\|(\mathcal{N}_+ + 1)^{1/4+(k-1)/2} b_{-v} e^{sA}\xi\big\| \qquad (4.82)\\
&\leq \frac{C}{\sqrt{N}} \|\eta\|_2 \big\|(\mathcal{N}_+ + 1)^{3/4+(k-1)/2} e^{sA}\xi\big\|^2 \\
&\leq C\langle \xi, e^{-sA}(\mathcal{N}_+ + 1)^k e^{sA}\xi\rangle
\end{aligned}$$

This proves that

$$\partial_s \varphi_\xi(s) \leq C\varphi_\xi(s)$$

so that, by Gronwall's lemma, we find a constant C (depending on k) with

$$\langle \xi, e^{-A}(\mathcal{N}_+ + 1)^k e^{A}\xi\rangle = C\langle \xi, (\mathcal{N}_+ + 1)^k \xi\rangle \,.$$

□

To control the growth of the product $(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)$ with respect to conjugation by e^A, we will use the following lemma.

Lemma 4.4.3. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Let A and $\mathcal{H}_N$ be defined as in* (4.69) *and, respectively, after (4.59). Then,*

$$[\mathcal{H}_N, A] = \sum_{j=0}^{9} \Theta_j + \text{h.c.} \qquad (4.83)$$

where

$$\begin{aligned}
\Theta_0 &= \Theta_0^{(1)} + \Theta_0^{(2)} = -\frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \kappa \widehat{V}(r/N)\, b^*_{r+v} b^*_{-r}\big(\gamma_v b_v + \sigma_v b^*_{-v}\big) \\
\Theta_1 &= \Theta_1^{(1)} + \Theta_1^{(2)} = \frac{2}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} \eta_r b^*_{r+v} b^*_{-r}\ \big[r\cdot v\,\gamma_v\, b_v + (v^2 + r\cdot v)\,\sigma_v b^*_{-v}\big] \\
\Theta_2 &= \Theta_2^{(1)} + \Theta_2^{(2)} = \frac{1}{N^{3/2}} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{q\in\Lambda^*_+, u\in\Lambda^*:\\ u\neq -q, -r-v}} \widehat{V}(u/N)\eta_r b^*_{r+v+u} b^*_{-r} a^*_q a_{q+u}(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Theta_3 &= \Theta_3^{(1)} + \Theta_3^{(2)} = \frac{1}{N^{3/2}} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{q\in\Lambda^*_+, u\in\Lambda^*:\\ u\neq -q, r}} \widehat{V}(u/N)\eta_r b^*_{r+v} b^*_{-r+u} a^*_q a_{q+u}(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Theta_4 &= \Theta_4^{(1)} + \Theta_4^{(2)} = \frac{1}{N^{3/2}} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{q\in\Lambda^*_+, u\in\Lambda^*:\\ u\neq -q, r}} \widehat{V}(u/N)\eta_r b^*_{r+v} b^*_{-r} \\
&\hspace{6cm} \times \big(-\gamma_v a^*_q a_{q+u} b_{-u+v} + \sigma_v b^*_{-v+u} a^*_q a_{q+u}\big)
\end{aligned}$$

and

$$\begin{aligned}
\Theta_5 &= \Theta_5^{(1)} + \Theta_5^{(2)} = -\frac{1}{N^{3/2}} \sum_{p\in P_H, v\in P_L} \sum_{r\in P_L} \widehat{V}((p-r)/N)\eta_r b^*_{p+v} b^*_{-p}(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Theta_6 &= \Theta_6^{(1)} + \Theta_6^{(2)} = -\frac{1}{N^{3/2}} \sum_{p\in P_H, v\in P_L} \widehat{V}(p/N)\,\eta_0\, b^*_{p+v} b^*_{-p}(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Theta_7 &= \Theta_7^{(1)} + \Theta_7^{(2)} = \frac{1}{N^{3/2}} \sum_{r\in P_H, v\in P_L} \sum_{p\in P_L : p\neq -v} \widehat{V}((p-r)/N)\eta_r b^*_{p+v} b^*_{-p}(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Theta_8 &= \Theta_8^{(1)} + \Theta_8^{(2)} = 2N^2\sqrt{N}\lambda_\ell \sum_{r\in P_H, v\in P_L} \widehat{\chi}_\ell(r)\, b^*_{r+v} b^*_{-r}\big(\gamma_v b_v + \sigma_v b^*_{-v}\big) \\
\Theta_9 &= \Theta_9^{(1)} + \Theta_9^{(2)} = 2N\sqrt{N}\lambda_\ell \sum_{\substack{r\in P_H,\\ v\in P_L}} \sum_{q\in\Lambda^*} \widehat{\chi}_\ell(r-q)\eta_q\, b^*_{r+v} b^*_{-r}\big(\gamma_v b_v + \sigma_v b^*_{-v}\big)
\end{aligned}$$

We have

$$|\langle \xi_1, \Theta_j^{(i)} \xi_2\rangle| \leq C\Big[\, \big\langle \xi_1, \big(\mathcal{H}_N + (\mathcal{N}_+ + 1)^2\big)\xi_1\big\rangle + \big\langle \xi_2, \big(\mathcal{H}_N + (\mathcal{N}_+ + 1)^2\big)\xi_2\big\rangle \,\Big] \tag{4.84}$$

for a constant $C > 0$, all $\xi_1, \xi_2 \in \mathcal{F}_+^{\leq N}$, $i = 1, 2$ and all $j = 0, 1, \dots, 9$, and

$$\pm\big(\Theta_j^{(i)} + h.c.\big) \leq CN^{-1/4}\Big[(\mathcal{N}_+ + 1)(\mathcal{K} + 1) + (\mathcal{N}_+ + 1)^3\Big] \tag{4.85}$$

for $i = 1, 2$ and all $j = 1, \dots, 9$ (but not for $j = 0$).

Proof. We use the formulas

$$[a_p^* a_q, b_r^*] = \delta_{qr} b_p^*, \qquad [a_p^* a_q, b_r] = -\delta_{pr} b_q \tag{4.86}$$

to compute

$$\begin{aligned}
\big[\mathcal{K}, A\big] &= \frac{1}{\sqrt{N}} \sum_{p\in\Lambda_+^*} \sum_{r\in P_H, v\in P_L} \eta_r p^2 \left\{\delta_{p,r+v} b_p^* b_{-r}^*(\gamma_v b_v + \sigma_v b^*_{-v})\right. \\
&\qquad \left. + \delta_{p,-r} b^*_{r+v} b^*_p(\gamma_v b_v + \sigma_v b^*_{-v}) - \gamma_v \delta_{v,p} b^*_{r+v} b^*_{-r} b_p + \sigma_v \delta_{-v,p} b^*_{r+v} b^*_{-r} b^*_p\right\} + \text{h.c.} \\
&= \frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} 2r^2 \eta_r b^*_{r+v} b^*_{-r}(\gamma_v b_v + \sigma_v b^*_{-v}) + \Theta_1 + \text{h.c.}
\end{aligned} \tag{4.87}$$

Writing

$$a^*_{p+u} a^*_q a_{q+u} a_p = a^*_{p+u} a_p a^*_q a_{q+u} - \delta_{p,q} a^*_{p+u} a_{p+u}\,, \tag{4.88}$$

using (4.86) to commute the r.h.s. of (4.88) with b^*_{r+v}, b^*_{-r}, b_v and, respectively, with b^*_{-v}, and normal ordering the operators appearing to the left of the factor $(\gamma_v b_v + \sigma_v b^*_{-v})$

leads to

$$\begin{aligned}\left[\mathcal{V}_N, A\right] = \frac{1}{N^{3/2}} &\sum_{\substack{r\in P_H, v\in P_L, u\in\Lambda^*:\\ u\neq -r,-r-v}} \widehat{V}(u/N)\eta_r b^*_{r+v+u} b^*_{-r-u}(\gamma_v b_v + \sigma_v b^*_{-v}) \\ &+ \sum_{j=2}^{4}\Theta_j + \text{h.c.}\end{aligned} \tag{4.89}$$

The first term on the r.h.s. of the last equation can be further decomposed as

$$\begin{aligned}&\frac{1}{N^{3/2}} \sum_{\substack{r\in P_H, v\in P_L, u\in\Lambda^*:\\ u\neq -r,-r-v}} \widehat{V}(u/N)\eta_r b^*_{r+v+u} b^*_{-r-u}(\gamma_v b_v + \sigma_v b^*_{-v}) \\ &\quad= \frac{1}{N^{3/2}} \sum_{r\in P_H, v\in P_L} \sum_{p\in\Lambda^*_+ : p\neq -v} \widehat{V}((p-r)/N)\eta_r b^*_{p+v} b^*_{-p}(\gamma_v b_v + \sigma_v b^*_{-v}) \\ &\quad= \frac{1}{N^{3/2}} \sum_{r\in P_H, v\in P_L} \sum_{p\in P_H} \widehat{V}((p-r)/N)\eta_r b^*_{p+v} b^*_{-p}(\gamma_v b_v + \sigma_v b^*_{-v}) \\ &\qquad+ \frac{1}{N^{3/2}} \sum_{r\in P_H, v\in P_L} \sum_{p\in P_L : p\neq -v} \widehat{V}((p-r)/N)\eta_r b^*_{p+v} b^*_{-p}(\gamma_v b_v + \sigma_v b^*_{-v}) \\ &\quad= \frac{1}{N^{3/2}} \sum_{p\in P_H, v\in P_L} \sum_{r\in\Lambda^*} \widehat{V}((p-r)/N)\eta_r b^*_{p+v} b^*_{-p}(\gamma_v b_v + \sigma_v b^*_{-v}) + \sum_{j=5}^{7}\Theta_j\end{aligned} \tag{4.90}$$

The first term on the r.h.s. of (4.90) can be combined with the first term on the r.h.s. of (4.87); with the relation (4.52), we obtain

$$\begin{aligned}&\frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} 2r^2\eta_r b^*_{r+v} b^*_{-r}(\gamma_v b_v + \sigma_v b^*_{-v}) \\ &\quad+ \frac{1}{N^{3/2}} \sum_{p\in P_H, v\in P_L} \sum_{r\in\Lambda^*} \widehat{V}((p-r)/N)\eta_r b^*_{p+v} b^*_{-p}(\gamma_v b_v + \sigma_v b^*_{-v}) = \Theta_0 + \Theta_8 + \Theta_9\end{aligned}$$

Combining (4.87), (4.89) and (4.90) with the last equation, we obtain the decomposition (4.83). Now, we prove the bounds (4.84), (4.85). First of all, using (4.53), we observe that

$$\begin{aligned}|\langle \xi_1, \Theta_1^{(1)}\, \xi_2\rangle| &\leq \frac{2}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} |\eta_r||r||v| \|b_{-r} b_{r+v}\xi_1\| \|b_v\xi_2\| \\ &\leq CN^{-1/2}\|(\mathcal{K}+1)^{1/2}(\mathcal{N}_+ +1)^{1/2}\xi_1\| \|(\mathcal{K}+1)^{1/2}\xi_2\|\end{aligned}$$

The term $\Theta_1^{(2)}$ can be estimated similarly as

$$
\begin{aligned}
|\langle \xi_1, \Theta_1^{(2)} \xi_2\rangle| &\leq \frac{2}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} |\eta_r||\sigma_v||v||r+v| \|b_{-r}b_{r+v}b_v(\mathcal{N}_+ +1)^{-1}\xi_1\| \|(\mathcal{N}_+ +1)\xi_2\| \\
&\leq CN^{-1/2}\Big(\sum_{r\in P_H, v\in P_L} |\eta_r|^2|\sigma_v|^2|v|^2\Big)^{1/2} \|(\mathcal{K}+1)^{1/2}\xi_1\| \|(\mathcal{N}+1)\xi_2\| \\
&\leq CN^{-1/2}\|(\mathcal{K}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)\xi_2\|
\end{aligned}
$$

This implies, on the one hand, that

$$
|\langle \xi_1, \big(\Theta_1^{(1)}+\Theta_1^{(2)}\big)\xi_2\rangle| \leq C\left[\langle \xi_1,(\mathcal{K}+1)\xi_1\rangle + \langle \xi_2, (\mathcal{K}+1)\xi_2\rangle\right]
$$

and, on the other hand, taking $\xi_1 = \xi_2$, that

$$
\pm\Big(\Theta_1^{(1)}+\Theta_1^{(2)}+\text{h.c.}\Big) \leq CN^{-1/2}(\mathcal{K}+1)(\mathcal{N}_+ +1)
$$

Next, we consider the quintic terms $\Theta_2, \Theta_3, \Theta_4$. Switching to position space, we find

$$
\langle \xi_1, \Theta_2^{(i)}\xi_2\rangle = \int dxdy\, N^{3/2}V(N(x-y))\langle \xi_1, \check{b}_x^* b^*(\check{\eta}_{H,x})\check{a}_y^*\check{a}_y b^{\sharp_i}(\check{\mu}_{L,x})\xi_2\rangle \tag{4.91}
$$

where $\check{\eta}_{H,x}(z) = \check{\eta}_H(z-x)$ with $\check{\eta}_H$ being the function with Fourier coefficients $\eta_H(p) = \eta_p\chi(p\in P_H)$ and where $\mu=\gamma$ and $\sharp_i = \cdot$, if $i=1$, and $\mu = \sigma$ and $\sharp_i = *$ if $i=2$, with $\check{\gamma}_L$, $\check{\sigma}_L$ defined similarly as $\check{\eta}_H$ (but in this case, with the characteristic function of the set P_L). From (4.91), and using that, by definition of the sets P_H, P_L, $\|\eta_H\|_2 \leq CN^{-1/4}$, $\|\gamma_L\|_2 \leq CN^{3/4}$, $\|\sigma_L\|_2 \leq \|\sigma\|_2 \leq C$, we obtain that

$$
\begin{aligned}
|\langle \xi_1, \Theta_2^{(i)}\xi_2\rangle| &\leq C\int dxdy N^2 V(N(x-y))\|\check{a}_x\check{a}_y\xi_1\|\|\check{a}_y(\mathcal{N}_+ +1)\xi_2\| \\
&\leq C\delta\langle \xi_1, \mathcal{V}_N\xi_1\rangle + C\delta^{-1}N^{-1}\langle \xi_2, (\mathcal{N}_+ +1)^3\xi_2\rangle
\end{aligned}
$$

for all $\delta>0$ and for $i=1,2$. Choosing $\delta=1$ and $\delta = N^{-1/2}$ we obtain (4.84) and, respectively, (4.85), with $j=2$ and $i=1,2$. The bounds (4.84), (4.85) for $j=3,4$ can be proven analogously.

As for the terms $\Theta_5, \Theta_6, \Theta_7$, we can proceed as follows:

$$
\begin{aligned}
|\langle \xi_1, \Theta_5^{(1)}\xi_2\rangle| &\leq \frac{1}{N^{3/2}} \sum_{p\in P_H, r,v\in P_L} |\widehat{V}((p-r)/N)||\eta_r|\|b_{p+v}b_{-p}\xi_1\|\|b_v\xi_2\| \\
&\leq \frac{1}{N^{3/2}}\Big(\sum_{p\in P_H, r,v\in P_L} |\eta_r|\, p^2 \|b_{-p}b_{p+v}\xi_1\|^2\Big)^{1/2} \\
&\qquad \times \Big(\sum_{p\in P_H, r,v\in P_L} \frac{|\widehat{V}((p-r)/N)|^2|\eta_r|}{p^2}\|b_v\xi_2\|^2\Big)^{1/2} \\
&\leq \frac{1}{\sqrt{N}}\|(\mathcal{K}+1)^{1/2}(\mathcal{N}_+ +1)^{1/2}\xi_1\|\, \|(\mathcal{N}_+ +1)^{1/2}\xi_2\|
\end{aligned}
$$

which immediately implies (4.84), (4.85) for $j = 5$ and $i = 1$. The contribution $\Theta_5^{(2)}$ can be bounded analogously, replacing $\|b_v \xi_2\|$ by $|\sigma_v| \|(\mathcal{N}_+ + 1)^{1/2}\xi_2\|$. The term $\Theta_6^{(i)}$ can be bounded similarly. As for $\Theta_7^{(1)}$ (a similar bound holds for $\Theta_7^{(2)}$) we find:

$$\begin{aligned}
|\langle \xi_1, \Theta_7^{(1)}\, \xi_2\rangle| &\leq \frac{1}{N^{3/2}} \Big(\sum_{r\in P_H, p,v \in P_L} |\widehat{V}((p-r)/N)||\eta_r| \, p^2 \|b_{-p} b_{p+v} \xi_1\|^2 \Big)^{1/2} \\
&\quad \times \Big(\sum_{r\in P_H, p,v\in P_L} \frac{|\widehat{V}((p-r)/N)||\eta_r|}{p^2} \|b_v \xi_2\|^2 \Big)^{1/2} \\
&\leq N^{-1/4} \|(\mathcal{K}+1)^{1/2}(\mathcal{N}_+ +1)^{1/2}\xi_1\| \, \|(\mathcal{N}_+ +1)^{1/2}\xi_2\|
\end{aligned}$$

Finally, let us consider the terms Θ_8, Θ_9. Since $\|\widehat{\chi}_\ell\|_2 \leq C$ (for a constant C depending only on ℓ), we have

$$\begin{aligned}
|\langle \xi_1, \Theta_8^{(1)} \xi_2\rangle| &\leq \frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} |\widehat{\chi}_\ell(r)| \|b_{r+v} b_{-r} \xi_1\| \|b_v \xi_2\| \\
&\leq \frac{1}{\sqrt{N}} \|(\mathcal{N}_+ +1)\xi_1\| \|(\mathcal{N}_+ +1)^{1/2}\xi_2\|
\end{aligned}$$

which implies (4.84) and (4.85) for $j = 8$ and $i = 1$. The bounds for $j = 8$ and $i = 2$ follow as usual replacing $\|b_v\xi_2\|$ by $|\sigma_v|\|(\mathcal{N}_+ + 1)^{1/2}\xi_2\|$, and using the boundedness of $\|\sigma\|_2$. Also the estimates for $j = 9$ can be proven analogously, since also $\|\widehat{\chi}_\ell * \eta\|_2 = \|\chi_\ell \check{\eta}\|_2 \leq \|\check{\eta}\|_2 = \|\eta\|_2$ is finite, uniformly in N.

To conclude the proof of the lemma, we still have to show that $\Theta_0^{(i)}$ satisfies (4.84), for $i = 1, 2$. To this end, we observe that

$$\begin{aligned}
|\langle \xi_1, \Theta_0^{(1)} \xi_2\rangle| &\leq \frac{\kappa}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} |\widehat{V}(r/N)| \|b_{r+v} b_{-r}\xi_1\| \|b_v \xi_2\| \\
&\leq C \left[\sum_{r\in P_H, v\in P_L} r^2 \|b_{r+v} b_{-r} \xi_1\|^2 \right]^{1/2} \left[\frac{1}{N} \sum_{r\in P_H, v\in P_L} \frac{|\widehat{V}(r/N)|^2}{r^2} \|b_v \xi_2\|^2 \right]^{1/2} \\
&\leq \|(\mathcal{K}+1)^{1/2}(\mathcal{N}_+ +1)^{1/2}\xi_1\| \|(\mathcal{N}_+ +1)^{1/2}\xi_2\|
\end{aligned}$$

and that a similar estimate holds for $\Theta_0^{(2)}$. Here, we used the fact that

$$\frac{1}{N} \sum_{r\in P_H} \frac{|\widehat{V}(r/N)|^2}{r^2} \leq \frac{1}{N} \sum_{r \in \Lambda_+^*} \frac{|\widehat{V}(r/N)|^2}{r^2} \leq C$$

uniformly in N. □

With the bounds on the commutator $[\mathcal{H}_N, A]$ established in Lemma 4.4.3, we can now control the growth of $(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)$ under the action of the e^A.

Proposition 4.4.4. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Let A and $\mathcal{H}_N$ be defined as in (4.69) and, respectively, after (4.59). Then there exists a constant $C > 0$ such that for all $s \in [0;1]$ we have on $\mathcal{F}_+^{\leq N}$ the operator inequality*

$$e^{-sA}(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1)e^{sA} \leq C(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1) + C(\mathcal{N}_+ + 1)^3$$

Proof. For a fixed $\xi \in \mathcal{F}_+^{\leq N}$ we define $\varphi_\xi : \mathbb{R} \to \mathbb{R}$ through

$$\varphi_\xi(s) := \langle \xi, e^{-sA}(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1)e^{sA}\xi\rangle$$

Then, we have

$$\begin{aligned}\partial_s \varphi_\xi(s) &= \langle \xi, e^{-sA}\big[(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1), A\big]e^{sA}\xi\rangle \\ &= \langle \xi, e^{-sA}(\mathcal{N}_+ + 1)\big[\mathcal{H}_N, A\big]e^{sA}\xi\rangle + \langle \xi, e^{-sA}\big[\mathcal{N}_+, A\big](\mathcal{H}_N + 1)e^{sA}\xi\rangle \\ &=: \mathrm{P}_1 + \mathrm{P}_2\end{aligned} \tag{4.92}$$

We start by analysing P_1. From Lemma 4.4.3, we have

$$\begin{aligned}\mathrm{P}_1 &= \sum_{j=0}^{9}\sum_{i=1}^{2}\langle e^{sA}\xi, (\mathcal{N}_+ + 1)\Theta_j^{(i)} e^{sA}\xi\rangle \\ &= \sum_{j=0}^{9}\sum_{i=1}^{2}\langle e^{sA}\xi, (\mathcal{N}_+ + 1)^{1/2}\Theta_j^{(i)}(\mathcal{N}_+ + 1 + \ell_{ij})^{1/2} e^{sA}\xi\rangle\end{aligned}$$

for appropriate $\ell_{ij} \in \{\pm 1, \pm 2, \pm 3\}$. With (4.84) and with Proposition 4.4.2, we conclude that

$$\begin{aligned}\big|\mathrm{P}_1\big| &\leq C\langle \xi, e^{-sA}(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1)e^{sA}\xi\rangle + C\langle \xi, e^{-sA}(\mathcal{N}_+ + 1)^3 e^{sA}\xi\rangle \\ &\leq C\langle \xi, e^{-sA}(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1)e^{sA}\xi\rangle + C\langle \xi, (\mathcal{N}_+ + 1)^3\xi\rangle\end{aligned} \tag{4.93}$$

Next we analyse P_2. From (4.81) and (4.82), we have

$$|\mathrm{P}_2| \leq C\big\langle \xi, (\mathcal{N}_+ + 1)\,\xi\big\rangle + |\langle e^{sA}\xi, [\mathcal{N}_+, A]\mathcal{H}_N e^{sA}\xi\rangle| \tag{4.94}$$

With

$$\begin{aligned}[\mathcal{N}_+, A] &= \frac{1}{\sqrt{N}}\sum_{r \in P_H,\, v \in P_L} \eta_r\big(3\sigma_v b^*_{r+v}b^*_{-r}b^*_{-v} + \gamma_v b^*_{r+v}b^*_{-r}b_v + \text{h.c.}\big) \\ &= 3A_\sigma + A_\gamma + \text{h.c.}\end{aligned}$$

we write

$$\begin{aligned}\big[\mathcal{N}_+, A\big]\mathcal{H}_N &= 3A_\sigma\mathcal{H}_N + A_\gamma\mathcal{H}_N + 3A^*_\sigma\mathcal{H}_N + A^*_\gamma\mathcal{H}_N \\ &= \big(3A_\sigma\mathcal{H}_N + \text{h.c.}\big) + \big(A_\gamma\mathcal{H}_N + \text{h.c.}\big) + \big[A^*_\gamma, \mathcal{H}_N\big] + 3\big[A^*_\sigma, \mathcal{H}_N\big] \\ &=: \mathrm{P}_{21} + \mathrm{P}_{22} + \big[A^*_\gamma, \mathcal{H}_N\big] + 3\big[A^*_\sigma, \mathcal{H}_N\big]\end{aligned} \tag{4.95}$$

Here, we introduced the normally ordered operators $\mathrm{P}_{21} = \mathrm{P}_{211}+\mathrm{P}_{212}$, $\mathrm{P}_{22} = \mathrm{P}_{221}+\mathrm{P}_{222}$, where

$$\begin{aligned}
\mathrm{P}_{211} &:= \frac{1}{\sqrt{N}} \sum_{\substack{p\in\Lambda_+^* \\ r\in P_H, v\in P_L}} p^2 \eta_r \sigma_v b^*_{r+v} b^*_{-r} b^*_{-v} a^*_p a_p + \text{h.c.}; \\
\mathrm{P}_{221} &:= \frac{1}{\sqrt{N}} \sum_{\substack{p\in\Lambda_+^* \\ r\in P_H, v\in P_L}} p^2 \eta_r \gamma_v b^*_{r+v} b^*_{-r} a^*_p a_p b_v + \frac{1}{\sqrt{N}} \sum_{\substack{r\in P_H, \\ v\in P_L}} v^2 \eta_r \gamma_v b^*_{r+v} b^*_{-r} b_v + \text{h.c.}
\end{aligned} \tag{4.96}$$

and, switching to position space,

$$\begin{aligned}
\mathrm{P}_{212} :=& \frac{1}{2N^{3/2}} \sum_{\substack{r\in P_H, \\ v\in P_L}} \sum_{\substack{p,q,u\in\Lambda_+^*, \\ u\neq -p,-q}} \widehat{V}(u/N) \eta_r \sigma_v b^*_{r+v} b^*_{-r} b^*_{-v} a^*_{p+u} a^*_q a_p a_{q+u} + \text{h.c.} \\
=& \frac{1}{2} \int_{\Lambda^3} dxdydz\, N^{3/2} V(N(x-y)) \check{b}^*_x \check{b}^*_y \check{b}^*_z a^*(\check{\eta}_{H,z}) a^*(\check{\sigma}_{L,z}) \check{a}_x \check{a}_y + \text{h.c.}; \\
\mathrm{P}_{222} :=& \frac{1}{2N^{3/2}} \sum_{\substack{r\in P_H, \\ v\in P_L}} \sum_{\substack{p,q,u\in\Lambda_+^*, \\ u\neq -p,-q}} \widehat{V}(u/N) \eta_r \gamma_v b^*_{r+v} b^*_{-r} a^*_{p+u} a^*_q a_p a_{q+u} b_v \\
&+ \frac{1}{2N^{3/2}} \sum_{\substack{r\in P_H, \\ v\in P_L}} \sum_{\substack{p,u\in\Lambda_+^*, \\ u\neq -p,-v}} \widehat{V}(u/N) \eta_r \gamma_v b^*_{r+v} b^*_{-r} \big(a^*_{p+u} a_p b_{v+u} + a^*_{-p} a_{v+u} b_{-p-u} \big) + \text{h.c.} \\
=& \frac{1}{N^{1/2}} \sum_{\substack{r\in P_H, \\ v\in P_L}} \eta_r \gamma_v b^*_{r+v} b^*_{-r} \mathcal{V}_N b_v \\
&+ \int_{\Lambda^3} dxdydz\, N^{3/2} V(N(x-y)) \check{\gamma}_L(x-z) \check{b}^*_z b^*(\check{\eta}_{H,z}) \check{a}^*_y \check{a}_x \check{b}_y + \text{h.c.}
\end{aligned} \tag{4.97}$$

where, as we did in (4.91) in the proof of Lemma 4.4.3, we introduced the notation $\check{\eta}_H, \check{\gamma}_L$ to indicate functions on Λ, with Fourier coefficients given by $\eta\chi_H$ and, respectively, by $\gamma\chi_L$, with χ_H and χ_L being characteristic functions of high ($|p| > N^{1/2}$) and low ($|p| < N^{1/2}$) momenta. Since, with the notation introduced in Lemma 4.4.3 after (4.83),

$$\big[A^*_\gamma, \mathcal{H}_N\big] = \sum_{j=0}^{9} (\Theta_j^{(1)})^*, \qquad \big[A^*_\sigma, \mathcal{H}_N\big] = \sum_{j=0}^{9} (\Theta_j^{(2)})^*,$$

it follows from (4.84) that

$$\begin{aligned}
|\langle e^{sA}\xi, \big[A^*_\gamma, \mathcal{H}_N\big] e^{sA}\xi\rangle| &\leq C \langle e^{sA}\xi, \big(\mathcal{H}_N + (\mathcal{N}_+ + 1)^2\big) e^{sA}\xi\rangle \\
|\langle e^{sA}\xi, \big[A^*_\sigma, \mathcal{H}_N\big] e^{sA}\xi\rangle| &\leq C \langle e^{sA}\xi, \big(\mathcal{H}_N + (\mathcal{N}_+ + 1)^2\big) e^{sA}\xi\rangle
\end{aligned} \tag{4.98}$$

Finally, we estimate the expectations of the operators (4.96), (4.97). The term P_{211} defined in (4.96) is bounded by

$$\begin{aligned}
\big|\langle e^{sA}\xi, \mathrm{P}_{211} e^{sA}\xi\rangle\big| &\leq \frac{1}{\sqrt{N}}\left[\sum_{p,r,v\in\Lambda_+^*} p^2\,\|b_{r+v}b_{-r}b_{-v}a_p(\mathcal{N}_++1)^{-1}e^{sA}\xi\|^2\right]^{1/2} \\
&\qquad\qquad \times \left[\sum_{p,r,v\in\Lambda_+^*} \eta_r^2\sigma_v^2 p^2\|a_p\,(\mathcal{N}_++1)e^{sA}\xi\|^2\right]^{1/2} \\
&\leq C\langle e^{sA}\xi, (\mathcal{N}_++1)(\mathcal{K}+1)e^{sA}\xi\rangle
\end{aligned} \tag{4.99}$$

because $\|\eta\|_2, \|\sigma\|_2$ are finite, uniformly in N. Similarly (using that $v^2 \leq 2(r+v)^2+2r^2$), we find

$$\big|\langle e^{sA}\xi, \mathrm{P}_{221} e^{sA}\xi\rangle\big| \leq C\langle e^{sA}\xi, (\mathcal{N}_++1)(\mathcal{K}+1)e^{sA}\xi\rangle \tag{4.100}$$

The expectation of the operator P_{212} in (4.97) can be bounded using its expression in position space by

$$\begin{aligned}
&\big|\langle e^{sA}\xi, \mathrm{P}_{212} e^{sA}\xi\rangle\big| \\
&\quad\leq \int_{\Lambda^3} dxdydz\, N^{3/2}V(N(x-y))\big\|a\big(\check{\eta}_{H,z}\big)\check{a}_z\check{a}_x\check{a}_y e^{sA}\xi\big\|\big\|a^*(\check{\sigma}_{L,z})\check{a}_x\check{a}_y e^{sA}\xi\big\| \\
&\quad\leq \int_{\Lambda^3} dxdydz\, N^{5/4}V(N(x-y))\,\big\|\check{a}_z\check{a}_x\check{a}_y(\mathcal{N}_++1)^{1/2}e^{sA}\xi\big\|\big\|\check{a}_x\check{a}_y(\mathcal{N}_++1)^{1/2}e^{sA}\xi\big\| \\
&\quad\leq\; CN^{-1/4}\langle e^{sA}\xi, (\mathcal{N}_++1)\mathcal{V}_N e^{sA}\xi\rangle
\end{aligned} \tag{4.101}$$

where we used the estimates $\|\check{\eta}_{H,z}\|_2 \leq CN^{-1/4}$, $\|\check{\sigma}_{L,z}\|_2 \leq C$. As for the operator P_{222} in (4.97), the expectation of the first term is controlled by

$$\begin{aligned}
&\Big|\frac{1}{N^{1/2}}\sum_{\substack{r\in P_H,\\ v\in P_L}} \eta_r\gamma_v\big\langle e^{sA}\xi, b^*_{r+v}b^*_{-r}\mathcal{V}_N b_v e^{sA}\xi\big\rangle\Big| \\
&\qquad\qquad\leq\; \frac{C}{N}\sum_{r,v\in\Lambda_+^*}\big\|\mathcal{V}_N^{1/2}a_{r+v}a_{-r}e^{sA}\xi\big\|^2 + \sum_{r,v\in\Lambda_+^*}|\eta_r|^2\big\|\mathcal{V}_N^{1/2}a_v e^{sA}\xi\big\|^2 \\
&\qquad\qquad\leq\; \frac{C}{N}\int_{\Lambda^2} dxdy \sum_{r,v\in\Lambda_+^*} N^2V(N(x-y))\big\|a_v a_r\check{a}_x\check{a}_y e^{sA}\xi\big\|^2 \\
&\qquad\qquad\qquad + C\int_{\Lambda^2} dxdy \sum_{v\in\Lambda_+^*} N^2V(N(x-y))\big\|a_v\check{a}_x\check{a}_y e^{sA}\xi\big\|^2 \\
&\qquad\qquad\leq\; C\langle \xi, e^{-sA}(\mathcal{N}_++1)(\mathcal{V}_N+1)e^{sA}\xi\rangle
\end{aligned} \tag{4.102}$$

while the expectation of the second term is bounded in position space by

$$\begin{aligned}
&\Big| \int_{\Lambda^3} dxdydz\, N^{3/2} V(N(x-y)) \check{\gamma}_L(x-z) \big\langle e^{sA}\xi, \check{b}^*_z b^*(\check{\eta}_{H,z}) \check{a}^*_y \check{a}_x \check{b}_y e^{sA}\xi \big\rangle \Big| \\
&\quad\leq \Big(\int_{\Lambda^3} dxdydz\, N^{3/2} V(N(x-y)) \|\check{\eta}_{H,z}\|_2^2 \, \big\| \check{a}_y \check{a}_z (\mathcal{N}_+ + 1)^{1/2} e^{sA}\xi \big\|^2 \Big)^{1/2} \\
&\qquad\qquad \times \Big(\int_{\Lambda^3} dxdydz\, N^{3/2} V(N(x-y)) \, |\check{\gamma}_L(x-z)|^2 \, \big\| \check{a}_x \check{a}_y e^{sA}\xi \big\|^2 \Big)^{1/2} \\
&\quad\leq CN^{-1} \|\check{\gamma}_L\|_2 \|\check{\eta}_H\|_2 \langle \xi, e^{-sA} (\mathcal{N}_+ + 1)^3 e^{sA}\xi\rangle^{1/2} \langle \xi, e^{-sA} \mathcal{V}_N e^{sA}\xi\rangle^{1/2} \\
&\quad\leq C\langle \xi, e^{-sA} \mathcal{V}_N e^{sA}\xi\rangle + C\langle \xi, (\mathcal{N}_+ + 1)^2 \xi\rangle
\end{aligned} \tag{4.103}$$

because $\|\check{\gamma}_L\|_2 \leq CN^{3/4}$ and $\|\check{\eta}_{H,z}\|_2 = \|\check{\eta}_H\|_2 \leq CN^{-1/4}$ for all $z \in \Lambda$. From (4.102) and (4.103), we obtain that

$$\langle e^{sA}\xi, \mathrm{P}_{222} e^{sA}\xi\rangle \leq C\langle e^{sA}\xi, (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) e^{sA}\xi\rangle \tag{4.104}$$

Combining (4.94) with (4.95), (4.96), (4.97), (4.98), (4.99), (4.100), (4.101) and (4.104), we conclude that

$$\big|\mathrm{P}_2\big| \leq C\langle \xi, e^{-sA} (\mathcal{N}_+ + 1)(\mathcal{H}_N + 1) e^{sA}\xi\rangle + C\langle \xi, (\mathcal{N}_+ + 1)^3 \xi\rangle$$

Applying (4.93) and the last bound on the r.h.s. of (4.92), we arrive at

$$\partial_s \varphi_\xi(s) \leq C\varphi_\xi(s) + C\langle \xi, (\mathcal{N}_+ + 1)^3 \xi\rangle$$

for some constant $C > 0$, independent of $\xi \in \mathcal{F}_+^{\leq N}$. By Gronwall's lemma, we conclude that there exists another constant $C > 0$ such that, for all $s \in [0;1]$,

$$\begin{aligned}
\langle e^{sA}\xi, (\mathcal{N}_+ + 1)(\mathcal{H}_N + 1) e^{sA}\xi\rangle &= \varphi_\xi(s) \\
&\leq C\varphi_\xi(0) + C\langle \xi, (\mathcal{N}_+ + 1)^3 \xi\rangle \\
&= C\langle \xi, (\mathcal{N}_+ + 1)(\mathcal{H}_N + 1)\xi\rangle + C\langle \xi, (\mathcal{N}_+ + 1)^3 \xi\rangle \,.
\end{aligned}$$

This concludes the proof of the proposition. □

We summarize the results of this section in the following corollary, which is a simple consequence of Prop. 4.4.1, Prop. 4.4.2 and Prop. 4.4.4.

Corollary 4.4.5. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa \geq 0$ is small enough. Let E_N be the ground state energy of H_N, defined in (4.1). Let $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ belong to the spectral subspace of H_N with energies below $E_N + \zeta$, for some $\zeta > 0$, i.e.*

$$\psi_N = \mathbf{1}_{(-\infty; E_N + \zeta]}(H_N)\psi_N$$

Let $\xi_N = e^{-A} e^{-B(\eta)} U_N \psi_N$ be the cubically renormalized excitation vector associated with ψ_N. Then, there exists a constant $C > 0$ such that

$$\big\langle \xi_N, \big[(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1) + (\mathcal{N}_+ + 1)^3\big]\, \xi_N \big\rangle \leq C(1 + \zeta^3)\,.$$

4.5 Diagonalization of the Quadratic Hamiltonian

From Proposition 4.3.3 we can decompose the cubically renormalized excitation Hamiltonian $\mathcal{J}_N$ defined in (4.70) as

$$\mathcal{J}_N = C_{\mathcal{J}_N} + \mathcal{Q}_{\mathcal{J}_N} + \mathcal{V}_N + \mathcal{E}_{\mathcal{J}_N} \tag{4.105}$$

with the constant $C_{\mathcal{J}_N}$ given in (4.71), the quadratic part

$$\mathcal{Q}_{\mathcal{J}_N} = \sum_{p\in\Lambda_+^*} \Big[F_p b_p^* b_p + \frac{1}{2} G_p \big(b_p^* b_{-p}^* + b_p b_{-p}\big)\Big] \tag{4.106}$$

with the coefficients F_p, G_p as in (4.72) and the error term $\mathcal{E}_{\mathcal{J}_N}$ satisfying

$$\pm\mathcal{E}_{\mathcal{J}_N} \le CN^{-1/4}\big[\mathcal{H}_N + (\mathcal{N}_+ + 1)^2\big](\mathcal{N}_+ + 1)$$

as an operator inequality on $\mathcal{F}_+^{\le N}$.

Our goal in this section is to diagonalise the quadratic operator $\mathcal{Q}_{\mathcal{J}_N}$. To reach this goal, we need first to establish some bounds for the coefficients F_p, G_p in (4.106).

Lemma 4.5.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Let F_p, G_p be defined as in* (4.72)*. Then there exists a constant $C > 0$ such that*

$$p^2/2 \le F_p \le C(1+p^2)\ ; \qquad |G_p| \le \frac{C\kappa}{p^2}$$

and

$$\frac{|G_p|}{F_p} \le \frac{C\kappa}{|p|^4} \le \frac{1}{2}$$

for all $p \in \Lambda_+^*$.

Proof. The proof is essentially the same as the proof of [14, Lemma 5.1]; the bound for G_p makes use of the relation (4.52) for the coefficients η_p. □

Lemma 4.5.1 shows that, if $\kappa > 0$ is small enough, $|G_p|/F_p \le 1/2$ for all $p \in \Lambda_+^*$. Hence, we can introduce coefficients $\tau_p \in \mathbb{R}$ such that

$$\tanh(2\tau_p) = -\frac{G_p}{F_p} \tag{4.107}$$

for all $p \in \Lambda_+^*$. Equivalently,

$$\tau_p = \frac{1}{4}\log\frac{1 - G_p/F_p}{1 + G_p/F_p}.$$

Using these coefficients, we define the generalized Bogoliubov transformation $e^{B(\tau)} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ with

$$B(\tau) := \frac{1}{2} \sum_{p \in \Lambda_+^*} \tau_p \big(b^*_{-p} b^*_p - b_{-p} b_p \big)$$

The next lemma, whose proof can be found in [14, Lemma 5.2], shows that the generalized Bogoliubov transformation $e^{B(\tau)}$ diagonalizes the quadratic operator $\mathcal{Q}_{\mathcal{J}_N}$, up to errors that are negligible in the limit of large N.

Lemma 4.5.2. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Let $\mathcal{Q}_{\mathcal{J}_N}$ be defined as in (4.106) and τ_p as in (4.107) with the coefficients F_p, G_p as in* (4.72)*. Then*

$$e^{-B(\tau)} \mathcal{Q}_{\mathcal{J}_N} e^{B(\tau)} = \frac{1}{2} \sum_{p \in \Lambda_+^*} \Big[-F_p + \sqrt{F_p^2 - G_p^2} \Big] + \sum_{p \in \Lambda_+^*} \sqrt{F_p^2 - G_p^2}\, a_p^* a_p + \delta_N$$

where the operator δ_N is such that, on $\mathcal{F}_+^{\leq N}$,

$$\pm \delta_N \leq C N^{-1} (\mathcal{K} + 1)(\mathcal{N}_+ + 1)$$

Apart from diagonalizing $\mathcal{Q}_{\mathcal{J}_N}$, conjugation with the Bogoliubov transformation $e^{B(\tau)}$ also acts on the other terms on the r.h.s. of (4.105). The resulting contributions are controlled by the next lemma (and by Lemma 4.2.1). Here, we use the fact that, from Lemma 4.5.1, $|\tau_p| \leq C|p|^{-4}$ for some constant $C > 0$ and all $p \in \Lambda_+^*$.

Lemma 4.5.3. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Let τ_p be defined as in (4.107), with F_p, G_p as in* (4.72) *and $\mathcal{V}_N, \mathcal{H}_N$ be as defined in (4.59). Then, there exists a constant $C > 0$ such that*

$$e^{-B(\tau)} (\mathcal{N}_+ + 1)(\mathcal{H}_N + 1) e^{B(\tau)} \leq C (\mathcal{N}_+ + 1)(\mathcal{H}_N + 1) \tag{4.108}$$

and

$$\pm \left[e^{-B(\tau)} \mathcal{V}_N e^{B(\tau)} - \mathcal{V}_N \right] \leq C N^{-1/2} (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) \tag{4.109}$$

Proof. The proof of (4.108) is similar to the one of [14, Lemma 5.4]; the only difference is the fact that, here, the potential energy $\mathcal{V}_N$ scales differently with N. We review therefore the main steps of the proof, focussing on terms involving $\mathcal{V}_N$.

We are going to apply Gronwall's lemma. For $\xi \in \mathcal{F}_+^{\leq N}$ and $s \in \mathbb{R}$, we compute

$$\partial_s \langle \xi, e^{-sB(\tau)} (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) e^{sB(\tau)} \xi \rangle = - \langle \xi, e^{-sB(\tau)} [B(\tau), (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)] e^{sB(\tau)} \xi \rangle$$

By the product rule, we have

$$\begin{aligned} &[B(\tau), (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)] \\ &\quad = (\mathcal{H}_N + 1)[B(\tau), \mathcal{N}_+] + [B(\tau), \mathcal{K}](\mathcal{N}_+ + 1) + [B(\tau), \mathcal{V}_N](\mathcal{N}_+ + 1) \end{aligned} \tag{4.110}$$

The first term on the r.h.s. of (4.110) can be written as

$$\begin{aligned}
&\langle \xi, e^{-sB(\tau)}(\mathcal{H}_N+1)[B(\tau),\mathcal{N}_+]e^{sB(\tau)}\xi\rangle \\
&\qquad\qquad = \sum_{p,q\in\Lambda_+^*} \tau_p q^2 \langle \xi, e^{-sB(\tau)} a_q^* a_q (b_p b_{-p} + b_p^* b_{-p}^*) e^{sB(\tau)}\xi\rangle \\
&\qquad\qquad\quad + \sum_{p\in\Lambda_+^*} \tau_p \langle \xi, e^{-sB(\tau)} \mathcal{V}_N (b_p b_{-p} + b_p^* b_{-p}^*) e^{sB(\tau)}\xi\rangle \\
&\qquad\qquad =: \mathrm{I} + \mathrm{II}
\end{aligned} \tag{4.111}$$

From the proof of [14, Lemma 5.4], we have

$$|\mathrm{I}| \leq C\langle e^{sB(\tau)}\xi, (\mathcal{N}_+ +1)(\mathcal{K}+1)e^{sB(\tau)}\xi\rangle$$

To estimate II, we switch to position space. We find

$$\begin{aligned}
|\mathrm{II}| &\leq \sum_{p\in\Lambda_+^*} |\tau_p| \int dxdy\, N^2 V(N(x-y)) \Big| \langle \check{a}_x\check{a}_y e^{sB(\tau)}\xi, \check{a}_x\check{a}_y (b_p b_{-p} + b_p^* b_{-p}^*) e^{sB(\tau)}\xi\rangle \Big| \\
&\leq \sum_{p\in\Lambda_+^*} |\tau_p| \int dxdy\, N^2 V(N(x-y)) \|\check{a}_x\check{a}_y (\mathcal{N}_+ +1)^{1/2} e^{sB(\tau)}\xi\| \\
&\quad \times \Big[\|(b_p b_{-p} + b_p^* b_{-p}^*)(\mathcal{N}_+ +1)^{-1/2}\check{a}_x\check{a}_y e^{sB(\tau)}\xi\| + \|\check{a}_y e^{sB(\tau)}\xi\| + \|\check{a}_x e^{sB(\tau)}\xi\| + \|\xi\| \Big] \\
&\leq C\langle \xi, e^{-sB(\tau)}(\mathcal{V}_N+1)(\mathcal{N}_+ +1)e^{sB(\tau)}\xi\rangle
\end{aligned}$$

since $(\tau_p)_{p\in\Lambda_+^*} \in \ell^1(\Lambda_+^*)$, uniformly in N. From (4.111), we obtain that

$$\Big| \langle \xi, e^{-sB(\tau)}(\mathcal{H}_N+1)[B(\tau),\mathcal{N}_+]e^{sB(\tau)}\xi\rangle \Big| \leq C\langle \xi, e^{-sB(\tau)}(\mathcal{H}_N+1)(\mathcal{N}_+ +1)e^{sB(\tau)}\xi\rangle \tag{4.112}$$

The second term on the r.h.s. of (4.110) can be bounded as in [14] by

$$\Big| \langle \xi, e^{-sB(\tau)}[B(\tau),\mathcal{K}](\mathcal{N}_+ +1)e^{sB(\tau)}\xi\rangle \Big| \leq C\langle \xi, e^{-sB(\tau)}(\mathcal{H}_N+1)(\mathcal{N}_+ +1)e^{sB(\tau)}\xi\rangle \tag{4.113}$$

Finally, we analyse the third term on the r.h.s. of (4.110). Again, it is convenient to switch to position space. We find

$$\begin{aligned}
[B(\tau),\mathcal{V}_N](\mathcal{N}_+ +1) &= \frac{\kappa}{2}\int_{\Lambda\times\Lambda} dxdy\, N^2 V(N(x-y))\check{\tau}(x-y)(\check{b}_x^*\check{b}_y^* + \check{b}_x\check{b}_y)(\mathcal{N}_+ +1) \\
&\quad + \kappa \int_{\Lambda\times\Lambda} dxdy\; N^2 V(N(x-y)) \big[b_x^* b_y^* a^*(\check{\tau}_y)\check{a}_x + \mathrm{h.c.} \big] (\mathcal{N}_+ +1)
\end{aligned} \tag{4.114}$$

where $\check{\tau}(x) = \sum_{p\in\Lambda_+^*} \tau_p e^{ip\cdot x}$. Using $\|\check{\tau}\|_\infty \leq \|\tau\|_1 \leq C < \infty$ as well as $\|\check{\tau}_y\|_2 = \|\check{\tau}\|_2 = \|\tau\|_2 \leq C < \infty$ independently of $y\in\Lambda$ and of N, it is then simple to check that

$$\Big| \langle \xi, e^{-sB(\tau)}[B(\tau),\mathcal{V}_N](\mathcal{N}_+ +1)e^{sB(\tau)}\xi\rangle \Big| \leq C\langle \xi, e^{-sB(\tau)}(\mathcal{V}_N+\mathcal{N}_+ +1)(\mathcal{N}_+ +1)e^{sB(\tau)}\xi\rangle$$

Combining this bound with (4.112) and (4.113), we obtain

$$\left|\partial_s \langle \xi, e^{-sB(\tau)}(\mathcal{H}_N+1)(\mathcal{N}_++1)e^{sB(\tau)}\xi\rangle\right| \leq C\langle \xi, e^{-sB(\tau)}(\mathcal{H}_N+1)(\mathcal{N}_++1)e^{sB(\tau)}\xi\rangle$$

By Gronwall's inequality and integrating over $s \in [0;1]$ we conclude (4.108).

To prove (4.109), on the other hand, we write

$$e^{-B(\tau)}\mathcal{V}_N e^{B(\tau)} - \mathcal{V}_N = \int_0^1 ds\, e^{-sB(\tau)}[\mathcal{V}_N, B(\tau)]e^{sB(\tau)}$$

With (4.114), it is simple to check that

$$\pm[B(\tau), \mathcal{V}_N] \leq N^{-1/2}(\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)$$

By (4.108) and $\mathcal{N}_+ \leq \mathcal{K}$, the last bound immediately implies

$$\pm\big[e^{-B(\tau)}\mathcal{V}_N e^{B(\tau)} - \mathcal{V}_N\big] \leq CN^{-1/2}(\mathcal{H}_N+1)(\mathcal{N}_++1)\,.$$

$\square$

It follows from Lemma 4.5.2 and Lemma 4.5.3 that the new excitation Hamiltonian $\mathcal{M}_N : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ defined by

$$\mathcal{M}_N = e^{-B(\tau)}\mathcal{J}_N e^{B(\tau)} = e^{-B(\tau)}e^{-A}e^{-B(\eta)}U_N H_N U_N^* e^{B(\eta)}e^{A}e^{B(\tau)}$$

can be decomposed as

$$\mathcal{M}_N = C_{\mathcal{M}_N} + \mathcal{Q}_{\mathcal{M}_N} + \mathcal{V}_N + \mathcal{E}_{\mathcal{M}_N}$$

where

$$C_{\mathcal{M}_N} := C_{\mathcal{J}_N} + \frac{1}{2}\sum_{p\in\Lambda_+^*}\Big[-F_p + \sqrt{F_p^2 - G_p^2}\Big]; \qquad \mathcal{Q}_{\mathcal{M}_N} := \sum_{p\in\Lambda_+^*}\sqrt{F_p^2-G_p^2}\; a_p^* a_p \tag{4.115}$$

with $C_{\mathcal{J}_N}$ as in (4.71) and F_p, G_p as in (4.72) and where the error $\mathcal{E}_{\mathcal{M}_N}$ is such that

$$\pm\mathcal{E}_{\mathcal{M}_N} \leq CN^{-1/4}\big[(\mathcal{H}_N+1)(\mathcal{N}_++1) + (\mathcal{N}_++1)^3\big]$$

To conclude this section, we are going to compute the constant $C_{\mathcal{M}_N}$ and the diagonal coefficients $(F_p^2 - G_p^2)^{1/2}$ appearing in the quadratic opeartor $\mathcal{Q}_{\mathcal{M}_N}$, up to errors that are negligible in the limit $N \to \infty$. To reach this goal, let us introduce some additional notation. For $m \in \mathbb{N}$, we define the Born approximations $\mathfrak{a}_N^{(m)}$ to the finite volume scattering length $\mathfrak{a}_N$ defined in (4.7), by

$$8\pi\mathfrak{a}_N^{(m)} := \kappa\widehat{V}(0) + \kappa\sum_{k=1}^m \frac{(-1)^k\kappa^k}{(2N)^k}\sum_{p_1,..,p_k\in\Lambda_+^*}\frac{\widehat{V}(p_1/N)}{p_1^2}\left(\prod_{i=1}^{k-1}\frac{\widehat{V}((p_i-p_{i+1})/N)}{p_{i+1}^2}\right)\widehat{V}(p_k/N) \tag{4.116}$$

for all $m \geq 1$. Furthermore, we denote by

$$E_{\text{Bog}} := \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[\sqrt{p^4 + 16\pi \mathfrak{a}_0 p^2} - p^2 - 8\pi \mathfrak{a}_0 + \frac{(8\pi \mathfrak{a}_0)^2}{2p^2} \right] \tag{4.117}$$

the usual sum encountered in the computation of the ground state energy in Bogoliubov theory. Recall here that $\mathfrak{a}_0$ is the (infinite volume) scattering length of the interaction κV, as defined in (4.3).

Lemma 4.5.4. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Let $\mathfrak{a}_N^{(m)}$ and E_{Bog} be defined as in (4.116) and (4.117), respectively. Then:*

i) The limit

$$\begin{aligned} \mathfrak{a}_N &= \lim_{m \to \infty} \mathfrak{a}_N^{(m)} \\ &= \kappa \widehat{V}(0) + \kappa \sum_{k=1}^{\infty} \frac{(-1)^k \kappa^k}{(2N)^k} \sum_{p_1, .., p_k \in \Lambda_+^*} \frac{\widehat{V}(p_1/N)}{p_1^2} \left(\prod_{i=1}^{k-1} \frac{\widehat{V}((p_i - p_{i+1})/N)}{p_{i+1}^2} \right) \widehat{V}(p_k/N) \end{aligned} \tag{4.118}$$

exists and it is such that

$$|\mathfrak{a}_N - \mathfrak{a}_0| \leq \frac{C\kappa^2}{N} \tag{4.119}$$

for a constant $C > 0$ independent of κ and of N.

ii) The constant $C_{\mathcal{M}_N}$ in (4.115) is such that

$$C_{\mathcal{M}_N} = 4\pi(N-1)\mathfrak{a}_N + E_{Bog} + \mathcal{O}(N^{-1} \log N)$$

iii) The quadratic operator $\mathcal{Q}_{\mathcal{M}_N}$ in (4.115) is given by

$$\mathcal{Q}_{\mathcal{M}_N} = \sum_{p \in \Lambda_+^*} \sqrt{p^4 + 16\pi \mathfrak{a}_0 p^2}\, a_p^* a_p + \widetilde{\delta}_N$$

where the operator δ_N is bounded by

$$\pm \widetilde{\delta}_N \leq C N^{-1} (\mathcal{K} + 1)$$

Proof. We prove i) first. To show the existence of the limit of $a_N^{(m)}$, as $m \to \infty$, we use the fact that $|\widehat{V}(p/N)| \leq \widehat{V}(0)$ and the estimate

$$\begin{aligned} \Big| \sum_{q \in \Lambda_+^*} \frac{\widehat{V}((p-q)/N)}{q^2} \Big| &\leq \widehat{V}(0) \sum_{q \in \Lambda_+^*, |q| \leq N} \frac{1}{q^2} + \sum_{q \in \Lambda_+^*, |q| \geq N} \frac{\big|\widehat{V}((p-q)/N)\big|}{q^2} \\ &\leq CN + \Big(\sum_{q \in \Lambda_+^*} \widehat{V}^2((p-q)/N) \Big)^{1/2} \Big(\sum_{q \in \Lambda_+^*, |q| \geq N} \frac{1}{q^4} \Big)^{1/2} \\ &\leq CN + CN^{-1/2} \big\| N^3 V(N.) e^{ip\cdot} \big\|_2 \leq CN \end{aligned} \tag{4.120}$$

uniformly in $q \in \Lambda_+^*$. Iterating this bound, we obtain that, for all integer $m < n$,

$$
\begin{aligned}
&\left|8\pi\mathfrak{a}_N^{(n)} - 8\pi\mathfrak{a}_N^{(m)}\right| \\
&\leq \sum_{k=m}^{n} \frac{\kappa^{k+1}}{2^k N^k}\widehat{V}(0) \sum_{p_1,\dots,p_k\in\Lambda_+^*} \left(\prod_{i=1}^{k-1} \frac{|\widehat{V}((p_i-p_{i+1})/N)|}{p_i^2}\right) \frac{|\widehat{V}(p_k/N)|}{p_k^2} \\
&\leq \sum_{k=m}^{n} (C\kappa)^{k+1} \leq (C\kappa)^{m+1}
\end{aligned}
\tag{4.121}
$$

which converges to zero, as $m,n\to\infty$, if $\kappa>0$ is small enough. Hence $\mathfrak{a}_N^{(m)}$ is a Cauchy sequence and $\mathfrak{a}_N = \lim_{m\to\infty}\mathfrak{a}_N^{(m)}$ exists.

To estimate the difference between $\mathfrak{a}_N$ and the infinite volume scattering length $\mathfrak{a}_0$ of the potential κV, we expand $\mathfrak{a}_0$ in a Born series and we compare it then with the Born series for $\mathfrak{a}_N$. From (4.3), we obtain

$$
8\pi\mathfrak{a}_0 = \kappa\widehat{V}(0) - \int \frac{dp}{(2\pi)^3}\,\kappa\widehat{V}(p)\widehat{w}(p) \tag{4.122}
$$

where $\widehat{w}$ denotes the Fourier transform of w. With the zero-energy scattering equation (4.2), we find

$$
\widehat{w}(p) = \frac{\kappa\widehat{V}(p)}{2p^2} - \frac{\kappa}{2p^2}\int \frac{dq}{(2\pi)^3}\,\widehat{V}(p-q)\widehat{w}(q)
$$

Inserting the last identity into (4.122) and iterating, we find

$$
8\pi\mathfrak{a}_0 = \kappa\widehat{V}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k\kappa^{k+1}}{2^k(2\pi)^{3k}} \int dp_1\dots dp_k \frac{\widehat{V}(p_1)}{p_1^2}\left(\prod_{i=1}^{k-1}\frac{\widehat{V}(p_i-p_{i+1})}{p_{i+1}^2}\right)\widehat{V}(p_k) \tag{4.123}
$$

since, for $\kappa>0$ small enough, the absolute convergence of the series can be shown as in (4.121), using $|\widehat{w}(p)| \leq \|Vf\|_1|p|^{-2}$ and $\widehat{V}\in L^2\cap L^\infty(\mathbb{R}^3)$.

To prove (4.119), we compare the summands in (4.118) with the corresponding terms in (4.123). We find

$$
\mathfrak{a}_0 - \mathfrak{a}_N = \sum_{k=1}^{\infty} \frac{(-1)^k\kappa^{k+1}}{2^k(2\pi)^{3k}}\mathrm{I}_k \tag{4.124}
$$

with

$$
\begin{aligned}
I_k = &\int dp_1\dots dp_k \frac{\widehat{V}(p_1)}{p_1^2}\left(\prod_{i=1}^{k-1}\frac{\widehat{V}(p_i-p_{i+1})}{p_{i+1}^2}\right)\widehat{V}(p_k) \\
&- \frac{(2\pi)^{3k}}{N^k}\sum_{p_1,..,p_k\in\Lambda_+^*} \frac{\widehat{V}(p_1/N)}{p_1^2}\left(\prod_{i=1}^{k-1}\frac{\widehat{V}((p_i-p_{i+1})/N)}{p_{i+1}^2}\right)\widehat{V}(p_k/N)
\end{aligned}
$$

Let us consider first the case $k = 1$. For $p \in \mathbb{R}^3$, let $\Omega_r(p)$ denote the cube of side length r centered at p. Then, we have

$$I_1 = \sum_{p \in \frac{2\pi}{N}\mathbb{Z}^3 \backslash \{0\}} \int_{\Omega_{2\pi/N}(p)} \left[\frac{\widehat{V}^2(q)}{q^2} - \frac{\widehat{V}^2(p)}{p^2} \right] dq + \int_{\Omega_{2\pi/N}(0)} \frac{\widehat{V}^2(q)}{q^2} dq =: I_1^{(1)} + I_1^{(2)}$$

From the boundedness of $\widehat{V}$, we easily find that $|I_1^{(2)}| \leq CN^{-1}$. To estimate $I_1^{(1)}$, on the other hand, we write $I_1^{(1)} = I_{1,>}^{(1)} + I_{1,<}^{(1)}$ where

$$I_{1,>}^{(1)} = \sum_{p \in \frac{2\pi}{N}\mathbb{Z}^3 \backslash \{0\} : |p| > N^{1/2}} \int_{\Omega_{2\pi/N}(p)} \left[\frac{\widehat{V}^2(q)}{q^2} - \frac{\widehat{V}^2(p)}{p^2} \right] dq$$

can be estimated by $|I_{1,>}^{(1)}| < CN^{-1}$, using that $\widehat{V} \in L^2(\mathbb{R}^3)$. To control $I_{1,<}^{(1)}$, we Taylor expand $W(q) = \widehat{V}^2(q)/q^2$ around $q = p \in 2\pi N^{-1}\mathbb{Z}^3$, up to second order. We find

$$\begin{aligned} W(q) = W(p) &+ \sum_{i=1}^{3} (\partial_i W)(p)(q_i - p_i) \\ &+ \int_0^1 ds \int_0^s dt \sum_{i,j=1}^{3} \partial_i \partial_j W\big(p + t(q-p)\big)(q_i - p_i)(q_j - p_j) \end{aligned}$$

where, for $0 \neq z \in \mathbb{R}^3$,

$$\begin{aligned} \partial_i \partial_j W(z) = &\frac{2(\partial_j \widehat{V})(z)(\partial_i \widehat{V})(z)}{z^2} + \frac{2(\partial_i \partial_j \widehat{V})(z)\widehat{V}(z)}{z^2} \\ &- \frac{16\pi z_j \widehat{V}(z)\partial_i \widehat{V}(z)}{z^4} - \frac{2\widehat{V}^2(z)}{z^4}\delta_{ij} + \frac{8 z_j z_i \widehat{V}^2(z)}{z^6} \end{aligned}$$

Since $\int_{\Omega_{2\pi/N}(p)} dq \, (q_i - p_i) = 0$, for $i = 1, 2, 3$, we find

$$\mathrm{I}_{1,<}^{(1)} = \kappa^2 \int_0^1 ds \int_0^s dt \sum_{\substack{p \in \frac{2\pi}{N}\mathbb{Z}^3 \backslash \{0\}: \\ |p| < N^{1/2}}} \int_{\Omega_{2\pi/N}(p)} dq \sum_{i,j=1}^{3} \partial_i \partial_j W\big(p + t(q-p)\big)(q_i - p_i)(q_j - p_j)$$

For $q \in \Omega_{2\pi/N}(p)$, $p \in 2\pi N^{-1}\mathbb{Z}^3 \backslash \{0\}$ and $t \in [0;1]$, we have $|q - p| \leq CN^{-1}$ and also $|p + t(q-p)|^{-\alpha} \leq C|q|^{-\alpha}$ for $\alpha \in \{2, 4\}$. Hence,

$$|\mathrm{I}_{1,<}^{(1)}| \leq CN^{-2} \int_{CN^{-1} \leq |q| \leq 1} \frac{dq}{|q|^4} + CN^{-2} \int_{1 \leq |q| \leq N^{1/2}} \frac{dq}{|q|^2} \leq CN^{-1}$$

We conclude that $|I_1| \leq CN^{-1}$. For $k \geq 2$, I_k can be bounded similarly; we find a constant $C > 0$ such that $|I_k| \leq C^k N^{-1}$ for all $k \in \mathbb{N}$. From (4.124), we arrive at

$$|\mathfrak{a}_0 - \mathfrak{a}_N| \leq \frac{1}{N} \sum_{k \geq 1} (C\kappa)^{k+1} \leq \frac{C\kappa^2}{N}$$

if $\kappa > 0$ is small enough.

Let us now prove part ii). To this end, we start from (4.115), with $C_{\mathcal{J}_N}$ as in (4.71) and F_p, G_p as in (4.72). With

$$\sqrt{F_p^2 - G_p^2} = \sqrt{p^4 + 2p^2\kappa\big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N}\big)_p}$$

we obtain

$$\begin{aligned} C_{\mathcal{M}_N} = \frac{(N-1)}{2}\kappa\widehat{V}(0) - \sum_{p\in\Lambda_+^*}\left[\frac{\kappa}{N}\big(\widehat{V}(\cdot/N)*\eta\big)_p\sigma_p\gamma_p + \frac{\kappa^2\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p^2}{4p^2}\right] \\ + \frac{\kappa}{2N}\sum_{p,q\in\Lambda_+^*}\widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p + \frac{1}{N}\sum_{p\in\Lambda^*}\left[p^2{\eta_p}^2 + \frac{\kappa}{2N}\big(\widehat{V}(\cdot/N)*\eta\big)_p\eta_p\right] \\ + E_{\text{Bog},N} \end{aligned}$$

with

$$\begin{aligned} E_{\text{Bog},N} := \frac{1}{2}\sum_{p\in\Lambda_+^*}\Bigg[\sqrt{p^4 + 2p^2\kappa\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p} - p^2 \\ - \kappa\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p + \frac{\kappa^2\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p^2}{2p^2}\Bigg] \end{aligned} \tag{4.125}$$

From (4.52), (4.3), (4.48) and from part i), we find

$$\begin{aligned} -\frac{\kappa}{2}\widehat{V}(0) + \frac{1}{N}\sum_{p\in\Lambda^*}\left[p^2\eta_p + \frac{\kappa}{2N}\big(\widehat{V}(\cdot/N)*\eta\big)_p\right]\eta_p \\ = -\frac{\kappa}{2}\widehat{V}(0) - \frac{\kappa}{2}\sum_{p\in\Lambda^*}\widehat{V}(p/N)\eta_p + \mathcal{O}(N^{-1}) = -4\pi\mathfrak{a}_N + \mathcal{O}(N^{-1}) \end{aligned} \tag{4.126}$$

Next, we compare $E_{\text{Bog},N}$ with its limiting value (4.117). From (4.125), we write $E_{\text{Bog},N} = -(1/2)\sum_{p\in\Lambda_+^*} e_{N,p}$, with

$$e_{N,p} = p^2 + \kappa\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p - \sqrt{p^4 + 2p^2\kappa\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p} - \frac{\kappa^2\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p^2}{2p^2}$$

Taylor expanding the square root, we easily check that $|e_{N,P}| \leq C/|p|^4$, for a constant $C > 0$, independent of N and of p, if $\kappa > 0$ is small enough. Replacing $(\widehat{V}(./N)*\widehat{f}_{\ell,N})_p$ by $(\widehat{V}(./N)*\widehat{f}_{\ell,N})_0$ and then, using Lemma 4.3.1, part ii), by $8\pi\mathfrak{a}_0$, we produce an error that can be estimated by

$$\left|e_{N,p} - \left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right]\right| \leq \frac{C}{N|p|^3}$$

Using this bound for $|p| < N$ and $|e_{N,p}| \le C/|p|^4$ for $|p| > N$, we obtain

$$|E_{\mathrm{Bog},N} - E_{\mathrm{Bog}}| \le CN^{-1}\log N$$

Together with (4.126), this leads to

$$C_{\mathcal{M}_N} = \frac{N}{2}\kappa\widehat{V}(0) - 4\pi\mathfrak{a}_N + E_{\mathrm{Bog}} + D + \mathcal{O}(N^{-1}\log N) \tag{4.127}$$

with

$$\begin{aligned} D = &-\sum_{p\in\Lambda_+^*}\left[\frac{\kappa}{N}\big(\widehat{V}(\cdot/N)*\eta\big)_p\sigma_p\gamma_p + \frac{\kappa^2\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p^2}{4p^2}\right] \\ &+\frac{\kappa}{2N}\sum_{p,q\in\Lambda_+^*}\widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p \end{aligned}$$

We have

$$\frac{\kappa}{2}\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p = \frac{\kappa}{2}\widehat{V}(p/N) + \frac{\kappa}{2N}\sum_{q\in\Lambda_+^*}\widehat{V}((p-q)/N)\eta_q + \frac{\kappa}{2N}\widehat{V}(p/N)\eta_0$$

On the other hand, since $|\sigma_p\gamma_p - \eta_p| \le C/p^6$, we find

$$\begin{aligned} &\sum_{p\in\Lambda_+^*}\frac{\kappa}{N}\big(\widehat{V}(\cdot/N)*\eta\big)_p\sigma_p\gamma_p \\ &\qquad = \sum_{p,q\in\Lambda_+^*}\frac{\kappa}{N}\widehat{V}((p-q)/N)\sigma_q\gamma_q\eta_p + \sum_{q\in\Lambda_+^*}\frac{\kappa}{N}\widehat{V}(q/N)\eta_q\eta_0 + \mathcal{O}(N^{-1}) \end{aligned}$$

and, writing $\sigma_q\gamma_q\sigma_p\gamma_p = (\sigma_q\gamma_q - \eta_q + \eta_q)(\sigma_p\gamma_p - \eta_p + \eta_p)$ and expanding the product,

$$\begin{aligned} &\frac{\kappa}{2N}\sum_{p,q\in\Lambda_+^*}\widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p \\ &\qquad = \frac{\kappa}{N}\sum_{p,q\in\Lambda_+^*}\widehat{V}((p-q)/N)\sigma_q\gamma_q\eta_p - \frac{\kappa}{2N}\sum_{p,q\in\Lambda_+^*}\widehat{V}((p-q)/N)\eta_q\eta_p + \mathcal{O}(N^{-1}) \end{aligned}$$

Since $N^{-2}\sum_{p\in\Lambda_+^*}\widehat{V}^2(p/N)/p^2 \le CN^{-1}$, we obtain

$$\begin{aligned} D = &-\sum_{p\in\Lambda_+^*}\left[\frac{\kappa}{2N}\sum_{q\in\Lambda_+^*}\widehat{V}((p-q)/N)\eta_q\eta_p + \frac{\kappa}{N}\widehat{V}(p/N)\eta_p\eta_0 + \frac{\kappa^2\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p^2}{4p^2}\right] \\ = &-\sum_{p\in\Lambda_+^*}\frac{\kappa^2\widehat{V}^2(p/N)}{4p^2} \\ &-\sum_{p_1,p_2\in\Lambda_+^*}\frac{\kappa\widehat{V}((p_1-p_2)/N)}{2Np_1^2}\left[p_1^2\eta_{p_1} + \kappa\widehat{V}(p_1/N) + \frac{\kappa}{2N}\sum_{p_3\in\Lambda_+^*}\widehat{V}((p_1-p_3)/N)\eta_{p_3}\right]\eta_{p_2} \\ &-\sum_{p\in\Lambda_+^*}\left[p^2\eta_p + \frac{\kappa}{2}\widehat{V}(p/N) + \frac{\kappa}{2N}\sum_{q\in\Lambda_+^*}\widehat{V}((p-q)/N)\eta_q\right]\frac{\kappa\widehat{V}(p/N)}{Np^2}\eta_0 + \mathcal{O}(N^{-1}) \end{aligned}$$

Using the relation (4.52) and discarding negligible contributions, we arrive at

$$
\begin{aligned}
D = & -\sum_{p\in\Lambda_+^*}\frac{\kappa^2\widehat{V}^2(p/N)}{4p^2} \\
& -\frac{1}{4N}\sum_{p_1,p_2\in\Lambda_+^*}\frac{\kappa\widehat{V}((p_1-p_2)/N)}{p_1^2}\Big[\kappa\widehat{V}(p_1/N)+2N^3\lambda_\ell\widehat{\chi}_\ell(p_1)\Big]\eta_{p_2}+\mathcal{O}(N^{-1})
\end{aligned}
$$

Inserting in (4.127), recalling the definition (4.116) of the Born approximations $a_N^{(m)}$ and introducing the sequence

$$
\begin{aligned}
R_N^{(m)} = & \frac{(-1)^m}{(2N)^{m+1}} \\
& \times\sum_{p_1,\dots,p_{m+1}\in\Lambda_+^*}\left(\prod_{i=1}^m\frac{\kappa\widehat{V}((p_i-p_{i+1})/N)}{p_i^2}\right)\Big[\kappa\widehat{V}(p_1/N)+2N^3\lambda_l\widehat{\chi}_l(p_1)\Big]\eta_{p_{m+1}}
\end{aligned}
\tag{4.128}
$$

we obtain

$$
C_{\mathcal{M}_N} = 4\pi N\mathfrak{a}_N^{(1)}+NR_N^{(1)}-4\pi\mathfrak{a}_N+E_{\text{Bog}}+\mathcal{O}(N^{-1}\log N)
$$

We claim that, for all $m\in\mathbb{N}$,

$$
4\pi N\mathfrak{a}_N^{(1)}+NR_N^{(1)} = 4\pi N\mathfrak{a}_N^{(m)}+NR_N^{(m)}+\sum_{j=1}^m\delta_N^{(j)} \tag{4.129}
$$

for a sequence $\delta_N^{(j)}$ such that $|\delta_N^{(j)}|\le(C\kappa)^j/N$ for a constant $C>0$ independent of N,κ and j. We show (4.129) by induction over m. For $m=1$, the claim is obvious. Let us assume that (4.129) holds for a fixed $m\in\mathbb{N}$, $m\ge1$. We show that (4.129) for $m+1$ as well. To this end, we use (4.52) to write

$$
\begin{aligned}
\eta_{p_{m+1}} = & -\frac{\kappa\widehat{V}(p_{m+1}/N)}{2p_{m+1}^2}(1+\eta_0/N)+\frac{2N^3\lambda_\ell\widehat{\chi}(p_{m+1})}{2p_{m+1}^2}+\frac{2N^2\lambda_\ell\eta_{p_{m+1}}}{2p_{m+1}^2} \\
& -\sum_{p_{m+2}\in\Lambda_+^*}\frac{\kappa\widehat{V}((p_{m+1}-p_{m+2})/N)}{2Np_{m+1}^2}\eta_{p_{m+2}}
\end{aligned}
$$

Inserting in (4.128), we find that

$$
NR_N^{(m)} = NR_N^{(m+1)}+4\pi N(\mathfrak{a}_N^{(m+1)}-\mathfrak{a}_N^{(m)})+\delta_N^{(m+1)}
$$

where

$$
\begin{aligned}
& 4\pi N(\mathfrak{a}_N^{(m+1)}-\mathfrak{a}_N^{(m)}) \\
& \quad = \frac{N}{2}\frac{(-1)^{m+1}}{(2N)^{m+1}}\sum_{p_1,\dots,p_{m+1}\in\Lambda_+^*}\frac{\kappa\widehat{V}(p_1/N)}{p_1^2}\left(\prod_{i=1}^m\frac{\kappa\widehat{V}((p_i-p_{i+1})/N)}{p_{i+1}^2}\right)\kappa\widehat{V}(p_{m+1}/N)
\end{aligned}
$$

and where

$$
\begin{aligned}
\delta_N^{(m+1)} := \frac{(-1)^m}{2^{m+2}N^m} \sum_{p_1,\dots,p_{m+1}\in\Lambda_+^*} &\left(\prod_{i=1}^m \frac{\kappa\widehat{V}((p_i-p_{i+1})/N)}{p_i^2}\right)\frac{1}{p_{m+1}^2} \\
\times \Big[&-2N^3\lambda_\ell\kappa\widehat{V}(p_{m+1}/N)\widehat{\chi}_\ell(p_1) + 2N^3\lambda_\ell\kappa\widehat{V}(p_1/N)\widehat{\chi}_\ell(p_{m+1}) \\
&-\big(\kappa\widehat{V}(p_1/N) + 2N^3\lambda_\ell\chi_\ell(p_1)\big)\kappa\widehat{V}(p_{m+1}/N)\eta_0/N \\
&+(2N^3\lambda_\ell)^2\widehat{\chi}_\ell(p_1)\widehat{\chi}_\ell(p_{m+1}) \\
&+\big(\kappa\widehat{V}(p_1/N) + 2N^3\lambda_\ell\chi_\ell(p_1)\big)2N^2\lambda_\ell\eta_{p_{m+1}}\Big]
\end{aligned}
\tag{4.130}
$$

Observe that the contribution proportional to $-2N^3\lambda_\ell\kappa\widehat{V}(p_{m+1}/N)\widehat{\chi}_\ell(p_1)$ cancels exactly with the one proportional to $2N^3\lambda_\ell\kappa\widehat{V}(p_1/N)\widehat{\chi}_\ell(p_{m+1})$. This can be seen with the change of variables $(p_1,p_2,\dots,p_m)\to(p_m,p_{m-1},\dots,p_1)$. Using the estimate (4.120), the bounds in Lemma 4.3.1 and (4.53), we can also control the other terms on the r.h.s. of (4.130). We obtain $|\delta_N^{(m+1)}|\le (C\kappa)^{m+1}/N$, for a constant $C>0$ independent of κ, N and m. This proves (4.129).

Using again the estimate (4.120), we find that $|R_N^{(m)}|\le (C\kappa)^m$ for a constant $C>0$ independent of N,m,κ. Hence, if $\kappa>0$ is sufficiently small, $R_N^{(m)}\to 0$, as $m\to\infty$. Together with part i) we find, letting $m\to\infty$, that

$$
C_{\mathcal{M}_N} = 4\pi(N-1)\mathfrak{a}_N + E_{\text{Bog}} + \mathcal{O}(N^{-1}\log N)
$$

as claimed.

Finally, we prove part iii). Here, we use the two bounds

$$
\left|\sqrt{p^4+2p^2\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p} - \sqrt{p^4+2p^2\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_0}\right| \le CN^{-1}|p|
$$

as well as

$$
\left|\sqrt{p^4+2p^2\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_0} - \sqrt{p^4+16\pi\mathfrak{a}_0p^2}\right| \le CN^{-1}
$$

It follows immediately that

$$
\mathcal{Q}_{\mathcal{M}_N} = \sum_{p\in\Lambda_+^*}\sqrt{p^4+2p^2\big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p}\; a_p^*a_p = \sum_{p\in\Lambda_+^*}\sqrt{p^4+16\pi\mathfrak{a}_0p^2}\; a_p^*a_p + \widetilde{\delta}_N
$$

where the operator $\widetilde{\delta}_N$ is bounded by $\pm\widetilde{\delta}_N\le CN^{-1}(\mathcal{K}+1)$. This concludes the proof of the lemma. □

Combining Proposition 4.3.3 with the results of the last two sections, we obtain the following corollary, which will be used in the next section to show Theorem 4.1.1.

Corollary 4.5.5. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Then the excitation Hamiltonian*

$$\mathcal{M}_N = e^{-B(\tau)}e^{-A}e^{-B(\eta)}UH_NU^*e^{B(\eta)}e^{A}e^{B(\tau)} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

is such that

$$\begin{aligned}\mathcal{M}_N = 4\pi(N-1)\mathfrak{a}_N + \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[-p^2 - 8\pi\mathfrak{a}_0 + \sqrt{p^4 + 16\pi\mathfrak{a}_0p^2} + \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] \\ + \sum_{p\in\Lambda_+^*}\sqrt{p^4+16\pi\mathfrak{a}_0p^2}\; a_p^*a_p + \mathcal{V}_N + \mathcal{E}_{\mathcal{M}_N}\end{aligned} \tag{4.131}$$

and there exists $C > 0$ such that

$$\pm\mathcal{E}_{\mathcal{M}_N} \leq CN^{-1/4}[(\mathcal{H}_N+1)(\mathcal{N}_++1) + (\mathcal{N}_++1)^3]$$

Furthermore, let $\psi_N \in L_s^2(\mathbb{R}^{3N})$ with $\|\psi_N\| = 1$ belong to the spectral subspace of H_N with energies below $E_N + \zeta$, where E_N is the ground state energy of H_N and $\zeta > 0$. In other words, assume that

$$\psi_N = \mathbf{1}_{(-\infty;E_N+\zeta]}(H_N)\psi_N$$

Let $\xi_N = e^{-B(\tau)}e^{-A}e^{-B(\eta)}U\psi_N \in \mathcal{F}_+^{\leq N}$ be the excitation vector associated with ψ_N. Then there exists a constant $C > 0$ such that

$$\langle \xi_N, [(\mathcal{H}_N+1)(\mathcal{N}_++1) + (\mathcal{N}_++1)^3]\xi_N\rangle \leq C(1+\zeta^3) \tag{4.132}$$

Proof. Eq. (4.131) follows from Prop. 4.3.3, Lemma 4.5.2, Lemma 4.5.3 and Lemma 4.5.4. Eq. (4.132) is, on the other hand, a consequence of Corollary 4.4.5. □

4.6 Proof of Theorem 4.1.1

We define

$$E_{\mathcal{M}_N} := 4\pi(N-1)\mathfrak{a}_N + \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[-p^2 - 8\pi\mathfrak{a}_0 + \sqrt{p^4 + 16\pi\mathfrak{a}_0p^2} + \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right]$$

To prove Theorem 4.1.1, we compare the eigenvalues of $\mathcal{M}_N - E_{\mathcal{M}_N}$ below a threshold $\zeta > 0$ with those of the diagonal quadratic operator

$$\mathcal{D} := \sum_{p\in\Lambda_+^*}\varepsilon_p a_p^*a_p \tag{4.133}$$

with the dispersion $\varepsilon_p = (|p|^4 + 16\pi\mathfrak{a}_0p^2)^{1/2}$ for all $p \in \Lambda_+^*$. For $m \in \mathbb{N}$, we denote by λ_m the m-th eigenvalue of $\mathcal{M}_N - E_{\mathcal{M}_N}$ and by ν_m the m-th eigenvalue of $\mathcal{D}$ (in both cases, eigenvalues are counted with multiplicity). To show Theorem 4.1.1, we prove that

$$|\lambda_m - \nu_m| \leq CN^{-1/4}(1+\zeta^3) \tag{4.134}$$

for all $m \in \mathbb{N}\backslash\{0\}$ such that $\lambda_m < \zeta$. Using (4.134), Theorem 4.1.1 can be proven as follows. Taking the expectation of (4.131) in the vacuum, we conclude that $\lambda_1 \leq CN^{-1/4}$. Hence, for N large enough, we have $\lambda_1 \leq \zeta$ and we can apply (4.134) to show that $|\lambda_1 - \nu_1| \leq CN^{-1/4}$. Since $\nu_1 = 0$, we conclude that $|\lambda_1| \leq CN^{-1/4}$ and therefore that

$$|E_N - E_{\mathcal{M}_N}| \leq CN^{-1/4} \tag{4.135}$$

where E_N is the ground state energy of H_N, as defined in (4.1). This proves (4.6). Eq. (4.8), on the other hand, follows from (4.134), from (4.135) and from the observation that the eigenvalues of $\mathcal{D}$ have the form

$$\nu_j = \sum_{p\in\Lambda_+^*} n_p^{(j)} \varepsilon_p$$

for every $j \in \mathbb{N}\backslash\{0\}$. Here the coefficients $n_p^{(j)} \in \mathbb{N}$, for all $j \in \mathbb{N}$ and all $p \in \Lambda_+^*$. Notice that the eigenvector of $\mathcal{D}$ associated with the eigenvalue ν_j is given by

$$\xi_j = C_j \prod_{p\in\Lambda_+^*} (a_p^*)^{n_p^{(j)}} \Omega \tag{4.136}$$

for an appropriate normalization constant $C_j > 0$ (if ν_j is degenerate, the choice of ξ_j is not unique; we will always use eigenvectors of the form (4.136)).

To show (4.134), we will combine a lower and an upper bound for λ_m in terms of ν_m. Since $\mathcal{V}_N \geq 0$, we can ignore the potential energy operator appearing on the r.h.s. of (4.131) when proving the lower bound. For the upper bound, on the other hand, we make use of the following lemma, where we control the expectation of $\mathcal{V}_N$ on low-energy eigenspaces of the quadratic operator $\mathcal{D}$.

Lemma 4.6.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and let $\mathcal{V}_N$ be defined as in (4.59). Let $\zeta > 0$ and $m \in \mathbb{N}$ such that $\nu_m < \zeta$. Let $\xi_1, \dots, \xi_m$ be defined as in (4.136) (ξ_j is an eigenvector of $\mathcal{D}$ associated with the eigenvalue ν_j) and $Y_{\mathcal{D}}^m$ be the subspace spanned by $\xi_1, \dots, \xi_m$. Then there exists $C > 0$ such that*

$$\langle \xi, \mathcal{V}_N \xi \rangle \leq \frac{C(\zeta+1)^{7/2}}{N}$$

for all normalized $\xi \in Y_{\mathcal{D}}^m$.

Proof. The bounds $\varepsilon_p \geq p^2$ and $\nu_1 \leq \dots \leq \nu_m \leq \zeta$ imply that $a_q \xi_j = 0$ for all $q \in \Lambda_+^*$ with $|q| > \zeta^{1/2}$. This also implies that $a_q \xi = 0$ for all $\xi \in Y_{\mathcal{D}}^m$. Hence

$$\begin{aligned} \langle \xi, \mathcal{V}_N \xi \rangle &\leq \frac{1}{N} \sum_{p,q,u\in\Lambda_+^*} |\widehat{V}(u/N)| \|a_{q+u} a_p \xi\| \|a_{p+u} a_q \xi\| \\ &\leq \frac{C}{N} \sum_{p,q,u\in\Lambda_+^* : |p|,|q|,|u| \leq C\zeta^{1/2}} \|a_{q+u} a_p \xi\| \|a_{p+u} a_q \xi\| \leq \frac{C\zeta^{3/2}}{N} \|(\mathcal{N}_+ + 1)\xi\|^2 \end{aligned}$$

Since $\mathcal{N}_+ \leq C\mathcal{D}$, we find

$$\langle \xi, \mathcal{V}_N \xi \rangle \leq \frac{C\zeta^{3/2}}{N} \|(\mathcal{D}+1)\xi\|^2 \leq \frac{C(\zeta+1)^{7/2}}{N}$$

□

In addition to Lemma 4.6.1, we will need the following result which is an extension of Lemma 7.3 in [14] to the Gross-Pitaevskii regime.

Lemma 4.6.2. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric, let $\mathcal{K}, \mathcal{V}_N$ be defined as in (4.59). Then there exists $C > 0$ such that, on $\mathcal{F}_+^{\leq N}$,*

$$\mathcal{V}_N \leq C\mathcal{N}_+\mathcal{K}\,.$$

Proof. We bound

$$\begin{aligned}
\langle \xi, \mathcal{V}_N \xi \rangle &\leq \frac{1}{N} \sum_{p,q\in\Lambda_+^*, u\in\Lambda^*: u\neq -p,-q} |\widehat{V}(u/N)| \|a_{p+u}a_q\xi\| \|a_{q+u}a_p\xi\| \\
&\leq \frac{1}{N} \sum_{p,q\in\Lambda_+^*, u\in\Lambda^*: u\neq -p,-q} \frac{|\widehat{V}(u/N)|}{(q+u)^2}(p+u)^2 \|a_{p+u}a_q\xi\|^2 \\
&\leq \left[\sup_{q\in\Lambda_+^*} \frac{1}{N} \sum_{u\in\Lambda^*: u\neq -q} \frac{|\widehat{V}(u/N)|}{(u+q)^2} \right] \|\mathcal{K}^{1/2}\mathcal{N}^{1/2}\xi\|^2 \leq C\|\mathcal{K}^{1/2}\mathcal{N}^{1/2}\xi\|^2
\end{aligned}$$

□

With the help of Lemma 4.6.1 and Lemma 4.6.2, we are now ready to prove (4.134).

Let us first prove a lower bound for λ_m, under the assumption that $\lambda_m < \zeta$. From the min-max principle, we have

$$\lambda_m = \inf_{\substack{Y\subset\mathcal{F}_+^{\leq N}: \\ \dim Y = m}} \sup_{\substack{\xi\in Y: \\ \|\xi\|=1}} \langle \xi, (\mathcal{M}_N - E_{\mathcal{M}_N})\xi \rangle$$

From the assumption $\lambda_m < \zeta$ we obtain

$$\lambda_m = \inf_{\substack{Y\subset P_\zeta(\mathcal{F}_+^{\leq N}): \\ \dim Y = m}} \sup_{\substack{\xi\in Y: \\ \|\xi\|=1}} \langle \xi, (\mathcal{M}_N - E_{\mathcal{M}_N})\xi \rangle$$

where P_ζ is the spectral projection of $\mathcal{M}_N - E_{\mathcal{M}_N}$ associated with the interval $(-\infty; \zeta]$. Hence, with (4.131), $\mathcal{V}_N \geq 0$ and (4.132) we find

$$\begin{aligned}
\lambda_m &\geq \inf_{\substack{Y\subset P_\zeta(\mathcal{F}_+^{\leq N}): \\ \dim Y = m}} \sup_{\substack{\xi\in Y: \\ \|\xi\|=1}} \langle \xi, \mathcal{D}\xi \rangle - CN^{-1/4}(1+\zeta^3) \\
&\geq \inf_{\substack{Y\subset\mathcal{F}_+^{\leq N}: \\ \dim Y = m}} \sup_{\substack{\xi\in Y: \\ \|\xi\|=1}} \langle \xi, \mathcal{D}\xi \rangle - CN^{-1/4}(1+\zeta^3) = \nu_m - CN^{-1/4}(1+\zeta^3)
\end{aligned}$$

Let us now prove an upper bound for λ_m. From the assumption $\lambda_m < \zeta$ and from the lower bound proven above, it follows that $\nu_m \le \zeta + 1$ (without loss of generality, we can assume $N^{-1/4}\zeta^3 \le 1$, since otherwise the statement of the theorem is trivially satisfied). The min-max principle implies that

$$\lambda_m = \inf_{\substack{Y \subset \mathcal{F}_+^{\le N}: \\ \dim Y = m}} \sup_{\substack{\xi \in Y: \\ \|\xi\|=1}} \langle \xi, (\mathcal{M}_N - E_{\mathcal{M}_N})\xi \rangle \le \sup_{\substack{\xi \in Y_{\mathcal{D}}^m: \\ \|\xi\|=1}} \langle \xi, (\mathcal{M}_N - E_{\mathcal{M}_N})\xi \rangle \tag{4.137}$$

where $Y_{\mathcal{D}}^m$ denotes the subspace spanned by the m vectors $\xi_1, \dots, \xi_m$ defined in (4.136). From Lemma 4.6.2 and the inequalities $\mathcal{N} \le C\mathcal{K} \le C\mathcal{D} \le C\nu_m \le C(\zeta+1)$ on $Y_{\mathcal{D}}^m$, we find that

$$\langle \xi, [(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3]\xi \rangle \le C \langle \xi, (\mathcal{N}_+ + 1)^2 (\mathcal{K} + 1)\xi \rangle \le C(1 + \zeta^3)$$

for all normalized $\xi \in Y_{\mathcal{D}}^m$. Inserting the last inequality and the bound from Lemma 4.6.1 in (4.131), we obtain that

$$\langle \xi, (\mathcal{M}_N - E_{\mathcal{M}_N})\xi \rangle \le \langle \xi, \mathcal{D}\xi \rangle + CN^{-1/4}(1 + \zeta^3)$$

for all $\xi \in Y_{\mathcal{D}}^m$. From (4.137), we conclude that

$$\lambda_m \le \sup_{\xi \in Y_{\mathcal{D}}^m : \|\xi\|=1} \langle \xi, \mathcal{D}\xi \rangle + CN^{-1/4}(1+\zeta^3) \le \nu_m + CN^{-1/4}(1+\zeta^3)$$

Combining lower and upper bound, we showed that $|\lambda_m - \nu_m| \le CN^{-1/4}(1+\zeta^3)$, for all $m \in \mathbb{N}$ such that $\lambda_m < \zeta$. This completes the proof of Theorem 4.1.1.

To conclude this section, we come back to the remark after Theorem 4.1.1, concerning the eigenvectors of the Hamilton operator H_N introduced in (4.1). Theorem 4.1.1 shows that the eigenvalues of H_N can be approximated in terms of the eigenvalues of the diagonal quadratic operator $\mathcal{D}$ defined in (4.133). Following standard arguments one can also approximate the eigenvectors of H_N through the (appropriately transformed) eigenvectors of $\mathcal{D}$. More precisely, let $\theta_1 \le \theta_2 \le \dots$ denote the ordered eigenvalues of H_N (i.e. $\theta_j = \lambda_j + E_{\mathcal{M}_N}$, with the notation introduced after (4.133)) and let $0 = \nu_1 \le \nu_2 \le \dots$ denote the eigenvalues of the diagonal quadratic operator $\mathcal{D}$ defined in (4.133). Fix $j \in \mathbb{N}\backslash\{0\}$ with $\nu_j < \nu_{j+1}$. From (4.134) we obtain that also $\theta_j < \theta_{j+1}$, if N is large enough. We denote by P_j the spectral projection onto the eigenspace of H_N associated with the eigenvalues $\theta_1 \le \dots \le \theta_j$ and by Q_j the orthogonal projection onto the eigenspace of $\mathcal{D}$ associated with the eigenvalues $0 = \nu_1 \le \dots \le \nu_j$. Then, we find

$$\left\| e^{-B(\tau)} e^{-A} e^{-B(\eta)} U_N P_j U_N^* e^{B(\eta)} e^{A} e^{B(\tau)} - Q_j \right\|_{\text{HS}}^2 \le \frac{C(j+1)(1+\nu_j^3)}{\nu_{j+1} - \nu_j} N^{-1/4} \tag{4.138}$$

In particular, if ψ_N denotes a ground state vector of the Hamiltonian H_N, there exists a phase $\omega \in [0; 2\pi)$ such that

$$\left\| \psi_N - e^{i\omega} U_N^* e^{B(\eta)} e^{A} e^{B(\tau)} \Omega \right\|^2 \le \frac{C}{\theta_1 - \theta_0} N^{-1/4} \tag{4.139}$$

The proof of (4.138) and (4.139) can be obtained, using the results of Theorem 4.1.1, analogously as in [46, Section7]. We omit the details.

4.7 Analysis of $\mathcal{G}_N$

In this section, we prove Proposition 4.3.2, devoted to the properties of the excitation Hamiltonian $\mathcal{G}_N$ defined in (4.58). In particular, we will show part b) of Prop. 4.3.2, since part a) was proven already in [13, Prop. 3.2]. In fact, the bound (4.60) is a bit more precise than the estimate appearing in [13, Prop. 3.2] but it can be easily obtained, combining the results of Prop. 4.2, Prop. 4.3, Prop. 4.4, Prop. 4.5 and Prop.4.7 in [13]. As for the bound (4.61), it was not explicitly shown in [13]; however, it follows from the analysis in [13] by noticing that the commutator $[i\mathcal{N}_+, \Delta_N]$ is given by the sum of the same monomials in creation and annihilation operators contributing to Δ_N, multiplied with a constant λ (given by the difference between the number of creation and the number of annihilation operators in the monomial). To be more precise, it follows from [13] that Δ_N can be written as a sum $\Delta_N = \sum_{k=0}^{\infty} \Delta_N^{(k)}$ where the errors $\Delta_N^{(k)}, k \in \mathbb{N}$, are sums of monomials of creation and annihilation operators that satisfy $\pm\Delta_N^{(k)} \leq (C\kappa)^k(\mathcal{H}_N + 1)$ for some constant $C > 0$, independent of N. Moreover, the commutator of a given monomial in $\Delta_N^{(k)}$ with $\mathcal{N}_+$ is given by the same monomial, multiplied by some constant $\lambda^{(k)}$ which is bounded by $|\lambda^{(k)}| \leq (2k+1) \leq C^k$ (if, w.l.o.g., the constant C is sufficiently large). Hence, terms in $[i\mathcal{N}_+, \Delta_N]$, and analogously in higher commutators of Δ_N with $i\mathcal{N}_+$, can be estimated exactly like terms in Δ_N (up to an unimportant additional constant), leading to (4.61). From now on, we will therefore focus on part b) of Proposition 4.3.2.

Using (4.42), we write

$$\mathcal{G}_N = \mathcal{G}_N^{(0)} + \mathcal{G}_N^{(2)} + \mathcal{G}_N^{(3)} + \mathcal{G}_N^{(4)}, \tag{4.140}$$

with $\mathcal{G}_N^{(j)} = e^{-B(\eta)}\mathcal{L}_N^{(j)}e^{B(\eta)}$, for $j = 0, 2, 3, 4$. In the rest of this section, we will compute the operators on the r.h.s. of (4.140), up to errors that are negligible in the limit of large N (on low-energy states). To quickly discard some of the error terms, it will be useful to have a rough estimate on the action of the Bogoliubov transformation $e^{-B(\eta)}$; this is the goal of the next lemma.

Lemma 4.7.1. *Let $B(\eta)$ be defined as in (4.20), with η as in (4.50). Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Let $\mathcal{K}, \mathcal{V}_N$ be defined as in (4.59). Then for every $j \in \mathbb{N}$ there exists a constant $C > 0$ such that*

$$\begin{aligned} e^{-B(\eta)}\mathcal{K}(\mathcal{N}_+ + 1)^j e^{B(\eta)} &\leq C\mathcal{K}(\mathcal{N}_+ + 1)^j + CN(\mathcal{N}_+ + 1)^{j+1} \\ e^{-B(\eta)}\mathcal{V}_N(\mathcal{N}_+ + 1)^j e^{B(\eta)} &\leq C\mathcal{V}_N(\mathcal{N}_+ + 1)^j + CN(\mathcal{N}_+ + 1)^j \end{aligned}$$

Proof. To prove the bound for the kinetic energy operator, we apply Gronwall's inequality. For $\xi \in \mathcal{F}_+^{\leq N}$ and $s \in \mathbb{R}$ we define $\Phi_s = e^{sB(\eta)}\xi$ and we consider

$$\begin{aligned} &\partial_s\langle\Phi_s, \mathcal{K}(\mathcal{N}_+ + 1)^j\Phi_s\rangle \\ &= \langle\Phi_s, [\mathcal{K}(\mathcal{N}_+ + 1)^j, B(\eta)]\Phi_s\rangle \\ &= \langle\Phi_s, [\mathcal{K}, B(\eta)](\mathcal{N}_+ + 1)^j\Phi_s\rangle + \langle\Phi_s, \mathcal{K}[(\mathcal{N}_+ + 1)^j, B(\eta)]\Phi_s\rangle \end{aligned} \tag{4.141}$$

With

$$[\mathcal{K}, B(\eta)] = \sum_{p\in\Lambda_+^*} p^2\eta_p \left(b_p b_{-p} + b_p^* b_{-p}^*\right)$$

the first term on the r.h.s. of (4.141) can be bounded by

$$\begin{aligned}\Big|\langle\Phi_s,&[\mathcal{K},B(\eta)](\mathcal{N}_+ +1)^j\Phi_s\rangle\Big|\\ &\leq C\sum_{p\in\Lambda_+^*} p^2|\eta_p|\|b_p(\mathcal{N}_+ +1)^{j/2}\Phi_s\|\|(\mathcal{N}_+ +1)^{(j+1)/2}\Phi_s\|\\ &\leq C\langle\Phi_s,\mathcal{K}(\mathcal{N}_+ +1)^j\Phi_s\rangle + CN\langle\xi,(\mathcal{N}_+ +1)^{j+1}\xi\rangle\end{aligned} \tag{4.142}$$

Here, we used Cauchy-Schwarz, the estimate (4.54) and Lemma 4.2.1 to replace, in the second term on the r.h.s., Φ_s by ξ. As for the second term on the r.h.s. of (4.141), we have

$$\begin{aligned}\langle\Phi_s,&\mathcal{K}[(\mathcal{N}_+ +1)^j, B(\eta)]\Phi_s\rangle\\ &=\sum_{k=0}^{j-1}\sum_{p\in\Lambda_+^*}\eta_p\langle\Phi_s,\mathcal{K}(\mathcal{N}_+ +1)^{j-k-1}(b_p^*b_{-p}^* + b_p b_{-p})(\mathcal{N}_+ +1)^k\Phi_s\rangle\end{aligned}$$

Writing $\mathcal{K} = \sum_{q\in\Lambda_+^*} q^2 a_q^* a_q$ and normal ordering field operators, we arrive at

$$\begin{aligned}\big|\langle\Phi_s,&\mathcal{K}[(\mathcal{N}_+ +1)^j, B(\eta)]\Phi_s\rangle\big|\\ &\leq C\sum_{p,q\in\Lambda_+^*} q^2|\eta_p|\|a_q a_p(\mathcal{N}_+ +1)^{(j-1)/2}\Phi_s\|\|a_q(\mathcal{N}_+ +1)^{j/2}\Phi_s\|\\ &\quad + C\sum_{p\in\Lambda_+^*} p^2|\eta_p|\|a_p(\mathcal{N}_+ +1)^{(j-1)/2}\Phi_s\|\|(\mathcal{N}_+ +1)^{j/2}\Phi_s\|\\ &\leq C\langle\Phi_s,\mathcal{K}(\mathcal{N}_+ +1)^j\Phi_s\rangle + CN\langle\xi,(\mathcal{N}_+ +1)^j\xi\rangle.\end{aligned}$$

Inserting the last bound and (4.142) into the r.h.s. of (4.141) and applying Gronwall, we obtain the bound for the kinetic energy operator.

To show the estimate for the potential energy operator, we proceed similarly. Using again the notation $\Phi_s = e^{sB(\eta)}\xi$, we compute

$$\begin{aligned}\partial_s\langle\langle\Phi_s,&\mathcal{V}_N(\mathcal{N}_+ +1)^j\Phi_s\rangle\\ &= \langle\Phi_s,\left[\mathcal{V}_N,B(\eta)\right](\mathcal{N}_+ +1)^j\Phi_s\rangle + \langle\Phi_s,\mathcal{V}_N\left[(\mathcal{N}_+ +1)^j,B(\eta)\right]\Phi_s\rangle\end{aligned} \tag{4.143}$$

Using the identity

$$\begin{aligned}\left[\mathcal{V}_N,B(\eta)\right] = &\frac{\kappa}{2N}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q}\widehat{V}(r/N)\eta_{q+r}b_q^*b_{-q}^*\\ &+\frac{\kappa}{N}\sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq p,-q}\widehat{V}(r/N)\eta_{q+r}b_{p+r}^*b_q^*a_{-q-r}^*a_p + \text{h.c.}\end{aligned}$$

and switching to position space, we can bound the expectation of the first term on the r.h.s. of (4.143) by

$$\begin{aligned}
&\big|\langle \Phi_s, \big[\mathcal{V}_N, B(\eta)\big](\mathcal{N}_+ +1)^j \Phi_s\rangle\big| \\
&\quad\leq \left|\frac{\kappa}{2}\int_{\Lambda^2} dxdy\, N^2 V(N(x-y))\check{\eta}(x-y)\big\langle \Phi_s, \check{b}_x^*\check{b}_y^*(\mathcal{N}_+ +1)^j\Phi_s\big\rangle\right| \\
&\qquad + \left|\kappa\int_{\Lambda^2} dxdy\, N^2 V(N(x-y))\big\langle \Phi_s, \check{b}_x^*\check{b}_y^* a^*(\check{\eta}_y)\check{a}_x(\mathcal{N}_+ +1)^j\Phi_s\big\rangle\right| \\
&\quad\leq C\int_{\Lambda^2} dxdy\, N^3 V(N(x-y))\big\|(\mathcal{N}_+ +1)^{j/2}\check{b}_x\check{b}_y\Phi_s\big\|\big\|(\mathcal{N}_+ +1)^{j/2}\Phi_s\big\| \\
&\qquad + C\int_{\Lambda^2} dxdy\, N^2 V(N(x-y))\|\check{\eta}_y\|_2\,\big\|(\mathcal{N}_+ +1)^{j/2}\check{b}_x\check{b}_y\Phi_s\big\|\big\|\check{a}_x(\mathcal{N}_+ +1)^{(j+1)/2}\Phi_s\big\| \\
&\quad\leq C\langle \Phi_s, \mathcal{V}_N(\mathcal{N}_+ +1)^j\Phi_s\rangle + CN\langle \xi, (\mathcal{N}_+ +1)^j\xi\rangle
\end{aligned}$$

where we used Cauchy-Schwarz, the bound $\|\check{\eta}_y\|_2 \leq C$, the fact that $\mathcal{N}_+ \leq N$ on $\mathcal{F}_+^{\leq N}$ and, in the last step, Lemma 4.2.1 to replace Φ_s with ξ in the second term. As for the second term on the r.h.s. of (4.143), it can be controlled similarly, using the identity

$$\mathcal{V}_N\big[(\mathcal{N}_+ +1)^j, B(\eta)\big] = \sum_{k=1}^{j}\sum_{p\in\Lambda_+^*}\eta_p(\mathcal{N}_+ +1)^{j-k-1}\mathcal{V}_N(b_pb_{-p}+b_p^*b_{-p}^*)(\mathcal{N}_+ +1)^k$$

and expressing $\mathcal{V}_N$ in position space. We conclude that

$$\big|\partial_s\langle\langle \Phi_s, \mathcal{V}_N(\mathcal{N}_+ +1)^j\Phi_s\rangle\big| \leq C\langle \Phi_s, \mathcal{V}_N(\mathcal{N}_+ +1)^j\Phi_s\rangle + CN\langle \xi, (\mathcal{N}_+ +1)^j\xi\rangle$$

Gronwall's lemma gives the desired bound. □

4.7.1 Analysis of $\mathcal{G}_N^{(0)} = e^{-B(\eta)}\mathcal{L}_N^{(0)}e^{B(\eta)}$

From (4.43), recall that

$$\mathcal{L}_N^{(0)} = \frac{N-1}{2N}\kappa\widehat{V}(0)(N-\mathcal{N}_+) + \frac{\kappa\widehat{V}(0)}{2N}\mathcal{N}_+(N-\mathcal{N}_+)$$

With Lemma 4.2.1, we immediately obtain that

$$\mathcal{G}_N^{(0)} = \frac{(N-1)}{2}\kappa\widehat{V}(0) + \mathcal{E}_N^{(0)}$$

where the error operator $\mathcal{E}_N^{(0)}$ is such that, on $\mathcal{F}_+^{\leq N}$,

$$\pm\mathcal{E}_N^{(0)} \leq \frac{C}{N}(\mathcal{N}_+ +1)^2 \tag{4.144}$$

4.7.2 Analysis of $\mathcal{G}_N^{(2)} = e^{-B(\eta)}\mathcal{L}_N^{(2)}e^{B(\eta)}$

We define the error operator $\mathcal{E}_N^{(2)}$ by the identity

$$\mathcal{G}_N^{(2)} = \mathcal{G}_N^{(2,K)} + \mathcal{G}_N^{(2,V)} + \mathcal{E}_N^{(2)} \tag{4.145}$$

where we set

$$\begin{aligned}\mathcal{G}_N^{(2,K)} = \mathcal{K} &+ \sum_{p\in\Lambda_+^*} \Big[p^2\sigma_p^2\left(1+\frac{1}{N}-\frac{\mathcal{N}_+}{N}\right) + p^2\sigma_p\gamma_p\big(b_pb_{-p}+b_p^*b_{-p}^*\big) + 2p^2\sigma_p^2 b_p^*b_p\Big]\\ &+ \sum_{p\in\Lambda_+^*}\frac{1}{N}p^2\sigma_p^2\sum_{q\in\Lambda_+^*}\left[\left(\gamma_q^2+\sigma_q^2\right)b_q^*b_q+\sigma_q^2\right]\\ &+ \sum_{p\in\Lambda_+^*}\frac{1}{N}p^2\sigma_p^2\sum_{q\in\Lambda_+^*}\left(\gamma_q\sigma_q b_{-q}b_q+\text{h.c.}\right)\\ &+ \sum_{p\in\Lambda_+^*}\left[p^2\eta_p b_{-p}d_p+\text{h.c.}\right]\end{aligned} \tag{4.146}$$

and $\mathcal{G}_N^{(2,V)}$ is defined as in

$$\begin{aligned}\mathcal{G}_N^{(2,V)} = &\sum_{p\in\Lambda_+^*}\left[\kappa\widehat{V}(p/N)\sigma_p^2 + \kappa\widehat{V}(p/N)\sigma_p\gamma_p\left(1-\frac{\mathcal{N}_+}{N}\right)\right]\\ &+ \sum_{p\in\Lambda_+^*}\kappa\widehat{V}(p/N)(\gamma_p+\sigma_p)^2 b_p^*b_p\\ &+ \frac{1}{2}\sum_{p\in\Lambda_+^*}\kappa\widehat{V}(p/N)(\gamma_p+\sigma_p)^2(b_pb_{-p}+b_p^*b_{-p}^*)\\ &+ \sum_{p\in\Lambda_+^*}\left[\frac{\kappa}{2}\widehat{V}(p/N)\left(\gamma_pb_p+\sigma_pb_{-p}^*\right)d_p + \frac{\kappa}{2}\widehat{V}(p/N)d_p\left(\gamma_pb_p+\sigma_pb_{-p}^*\right)\right]+\text{h.c.}\end{aligned} \tag{4.147}$$

The goal of this subsection consists in proving the following lemma, where we bound the error term $\mathcal{E}_N^{(2)}$.

Lemma 4.7.2. *Let $\mathcal{E}_N^{(2)}$ be as defined in (4.145). Then, under the same assumptions as in Proposition 4.3.2, we find $C>0$ such that*

$$\pm\,\mathcal{E}_N^{(2)} \le CN^{-1/2}(\mathcal{K}+\mathcal{N}_+^2+1)(\mathcal{N}_++1) \tag{4.148}$$

Proof. From (4.43), we have $\mathcal{L}_N^{(2)} = \mathcal{K} + \mathcal{L}_N^{(2,V)}$, with

$$\mathcal{L}_N^{(2,V)} = \kappa\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\left[b_p^*b_p - \frac{1}{N}a_p^*a_p\right] + \frac{\kappa}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\left[b_p^*b_{-p}^*+b_pb_{-p}\right] \tag{4.149}$$

We consider first the contribution of the kinetic energy operator $\mathcal{K}$. We write

$$\mathcal{K} = \frac{N-1}{N} \sum_{p\in\Lambda_+^*} p^2 b_p^* b_p + \sum_{p\in\Lambda_+^*} p^2 b_p^* b_p \frac{\mathcal{N}_+}{N} + \mathcal{K}\,\frac{(\mathcal{N}_+-1)^2}{N^2}$$

Writing $\mathcal{N}_+ = \sum_{p\in\Lambda_+^*} \big(b_p^* b_p + N^{-1}\, a_p^* \mathcal{N}_+ a_p\big)$ in the second term, we find

$$e^{-B(\eta)}\mathcal{K}e^{B(\eta)} = \sum_{p\in\Lambda_+^*} p^2 e^{-B(\eta)} b_p^* b_p e^{B(\eta)} + \frac{1}{N} \sum_{p,q\in\Lambda_+^*} p^2 e^{-B(\eta)} b_p^* b_q^* b_q b_p e^{B(\eta)} + \widetilde{\mathcal{E}}_1 \tag{4.150}$$

where, with Lemma 4.7.1,

$$\pm\,\widetilde{\mathcal{E}}_1 \le CN^{-2} e^{-B(\eta)} \mathcal{K}(\mathcal{N}_+ + 1)^2 e^{B(\eta)} \le CN^{-1}\mathcal{K}(\mathcal{N}_+ + 1) + CN^{-1}(\mathcal{N}_+ + 1)^3 \tag{4.151}$$

Next, we study the first term on the r.h.s. of (4.150). We claim that

$$\begin{aligned} \sum_{p\in\Lambda_+^*} p^2 e^{-B(\eta)} b_p^* b_p e^{B(\eta)} = &\,\mathcal{K} + \sum_{p\in\Lambda_+^*} p^2 \sigma_p^2 \left(1 - \frac{\mathcal{N}_+}{N}\right) \\ &+ \sum_{p\in\Lambda_+^*} \Big[p^2 \sigma_p \gamma_p \big(b_p b_{-p} + b_p^* b_{-p}^*\big) + 2p^2 \sigma_p^2 b_p^* b_p\Big] \\ &+ \sum_{p\in\Lambda_+^*} \big[p^2 \eta_p b_{-p} d_p + \text{h.c.}\big] + \widetilde{\mathcal{E}}_2 \end{aligned} \tag{4.152}$$

with the error operator $\widetilde{\mathcal{E}}_2$ such that

$$\pm\,\widetilde{\mathcal{E}}_2 \le CN^{-1/2}(\mathcal{K}+1)(\mathcal{N}_+ + 1) \tag{4.153}$$

To prove (4.153), we use (4.33) to decompose

$$\sum_{p\in\Lambda_+^*} p^2 e^{-B(\eta)} b_p^* b_p e^{B(\eta)} = \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3,$$

with

$$\begin{aligned} \mathrm{E}_1 &:= \sum_{p\in\Lambda_+^*} p^2 (\gamma_p b_p^* + \sigma_p b_{-p})(\gamma_p b_p + \sigma_p b_{-p}^*) \\ \mathrm{E}_2 &:= \sum_{p\in\Lambda_+^*} p^2 \Big[(\gamma_p b_p^* + \sigma_p b_{-p}) d_p + d_p^* (\gamma_p b_p + \sigma_p b_{-p}^*)\Big] \\ \mathrm{E}_3 &:= \sum_{p\in\Lambda_+^*} p^2 d_p^* d_p \end{aligned}$$

The term E_1 can be rewritten as

$$\mathrm{E}_1 = \mathcal{K} + \sum_{p\in\Lambda_+^*} p^2 \sigma_p^2 \left(1 - \frac{\mathcal{N}_+}{N}\right) + \sum_{p\in\Lambda_+^*} \Big[p^2 \sigma_p \gamma_p \big(b_p b_{-p} + b_p^* b_{-p}^*\big) + 2p^2 \sigma_p^2 b_p^* b_p\Big] + \widetilde{\mathcal{E}}_3,$$

where

$$\widetilde{\mathcal{E}}_3 = \frac{1}{N} \sum_{p \in \Lambda_+^*} p^2 \Big[a_p^* \mathcal{N}_+ a_p + \sigma_p^2 a_p^* a_p \Big]$$

is such that, for any $\xi \in \mathcal{F}_+^{\leq N}$,

$$|\langle \xi, \widetilde{\mathcal{E}}_3 \xi \rangle| \leq \frac{1}{N} \sum_{p \in \Lambda_+^*} \Big[p^2 \|a_p \mathcal{N}_+^{1/2} \xi\|^2 + p^2 \sigma_p^2 \|a_p \xi\|^2 \Big] \leq C N^{-1} \langle \xi, \mathcal{K}(\mathcal{N}_+ + 1)\xi \rangle \tag{4.154}$$

The term E_2 can be split as

$$\mathrm{E}_2 = \sum_{p \in \Lambda_+^*} \big[p^2 \eta_p b_{-p} d_p + \text{h.c.} \big] + \widetilde{\mathcal{E}}_4$$

where

$$\begin{aligned}
|\langle \xi, \widetilde{\mathcal{E}}_4 \xi \rangle| &\leq \sum_{p \in \Lambda_+^*} p^2 |\sigma_p - \eta_p| |\langle \xi, b_{-p} d_p \xi \rangle| + \sum_{p \in \Lambda_+^*} p^2 |\gamma_p| |\langle \xi, b_p^* d_p \xi \rangle| \\
&\leq \frac{1}{N} \sum_{p \in \Lambda_+^*} p^2 |\eta_p|^3 \|(\mathcal{N}_+ + 1)\xi\|^2 \\
&\quad + \frac{1}{N} \sum_{p \in \Lambda_+^*} p^2 \|b_p (\mathcal{N}_+ + 1)^{1/2} \xi\| \Big[|\eta_p| \|(\mathcal{N}_+ + 1)\xi\| + \|b_p (\mathcal{N}_+ + 1)^{1/2} \xi\| \Big] \\
&\leq C N^{-1/2} \|\mathcal{K}^{1/2} (\mathcal{N}_+ + 1)^{1/2} \xi\|^2
\end{aligned}$$

As for the term E_3, we estimate

$$\begin{aligned}
|\langle \xi, \mathrm{E}_3 \xi \rangle| &\leq \sum_{p \in \Lambda_+^*} p^2 \|d_p \xi\|^2 \\
&\leq \frac{C}{N^2} \sum_{p \in \Lambda_+^*} p^2 \Big[|\eta_p|^2 \|(\mathcal{N}_+ + 1)^{3/2} \xi\|^2 + \|b_p (\mathcal{N}_+ + 1)\xi\|^2 \Big] \\
&\leq C N^{-1} \|(\mathcal{N}_+ + 1)^{3/2} \xi\|^2 + C N^{-1} \|(\mathcal{K} + 1)^{1/2} (\mathcal{N}_+ + 1)^{1/2} \xi\|^2
\end{aligned}$$

for any $\xi \in \mathcal{F}_+^{\leq N}$. This concludes the proof of (4.152), with the estimate (4.153).

Next, we consider the second term on the r.h.s. of (4.150). We claim that

$$\begin{aligned}
&\frac{1}{N} \sum_{p,q \in \Lambda_+^*} p^2 e^{-B(\eta)} b_p^* b_q^* b_q b_p e^{B(\eta)} \\
&\quad = \frac{1}{N} \sum_{p \in \Lambda_+^*} p^2 \sigma_p^2 + \frac{1}{N} \sum_{p,q \in \Lambda_+^*} p^2 \sigma_p^2 \sigma_q^2 + \frac{1}{N} \sum_{p,q \in \Lambda_+^*} p^2 \sigma_p^2 (\gamma_q^2 + \sigma_q^2) b_q^* b_q \\
&\qquad + \frac{1}{N} \sum_{p,q \in \Lambda_+^*} p^2 \sigma_p^2 \gamma_q \sigma_q \big(b_q^* b_{-q}^* + \text{h.c.} \big) + \widetilde{\mathcal{E}}_5
\end{aligned} \tag{4.155}$$

with an error term $\widetilde{\mathcal{E}}_5$ such that

$$\pm\,\widetilde{\mathcal{E}}_5 \leq CN^{-1/2}(\mathcal{K}+\mathcal{N}_+^2+1)(\mathcal{N}_++1)\,. \tag{4.156}$$

To prove (4.156), we consider first the operator

$$\begin{aligned}
\mathrm{D} &= \sum_{q\in\Lambda_+^*} e^{-B(\eta)} b_q^* b_q e^{B(\eta)} \\
&= \sum_{q\in\Lambda_+^*} (\gamma_q b_q^* + \sigma_q b_{-q} + d_q^*)(\gamma_q b_q + \sigma_q b_{-q}^* + d_q) \\
&= \sum_{q\in\Lambda_+^*} \big[(\gamma_q^2+\sigma_q^2) b_q^* b_q + \sigma_q\gamma_q (b_q^* b_{-q}^* + b_q b_{-q}) + \sigma_q^2\big] + \widetilde{\mathcal{E}}_6
\end{aligned}$$

where the error $\widetilde{\mathcal{E}}_6$ is such that

$$\pm\,\widetilde{\mathcal{E}}_6 \leq CN^{-1}(\mathcal{N}_++1)^2 \tag{4.157}$$

as can be easily checked using the commutation relations (1.23) and the bound (4.34). We go back to (4.155), and we observe that

$$\begin{aligned}
&\frac{1}{N}\sum_{p,q\in\Lambda_+^*} p^2 e^{-B(\eta)} b_p^* b_q^* b_q b_p e^{B(\eta)} \\
&\qquad = \frac{1}{N}\sum_{p\in\Lambda_+^*} p^2 e^{-B(\eta)} b_p^* e^{B(\eta)} \mathrm{D} e^{-B(\eta)} b_p e^{B(\eta)} \\
&\qquad = \frac{1}{N}\sum_{p\in\Lambda_+^*} p^2 (\gamma_p b_p^* + \sigma_p b_{-p} + d_p^*)\,\mathrm{D}\,(\gamma_p b_p + \sigma_p b_{-p}^* + d_p) \\
&\qquad = \frac{1}{N}\sum_{p\in\Lambda_+^*} p^2 \sigma_p^2 b_p \mathrm{D} b_p^* + \widetilde{\mathcal{E}}_7
\end{aligned} \tag{4.158}$$

where

$$\begin{aligned}
\widetilde{\mathcal{E}}_7 = &\frac{1}{N}\sum_{p\in\Lambda_+^*} p^2\sigma_p(\gamma_p b_p^* + d_p^*)\mathrm{D} b_{-p}^* + \frac{1}{N}\sum_{p\in\Lambda_+^*} p^2 \sigma_p b_{-p}\mathrm{D}(\gamma_p b_p + d_p) \\
&+ \frac{1}{N}\sum_{p\in\Lambda_+^*} p^2 (\gamma_p b_p^* + d_p^*)\mathrm{D}(\gamma_p b_p + d_p)
\end{aligned}$$

can be bounded using (4.34) and the fact that, by Lemma 4.2.1, $D \leq C(\mathcal{N}_+ + 1)$, by

$$
\begin{aligned}
|\langle \xi, \widetilde{\mathcal{E}}_7 \, \xi \rangle| &\leq \frac{2}{N} \sum_{p\in\Lambda_+^*} p^2 |\sigma_p| \left[\|\mathrm{D}^{1/2} b_p \xi\| + \|\mathrm{D}^{1/2} d_p \xi\| \right] \|\mathrm{D}^{1/2} b_p^* \xi\| \\
&\quad + \frac{1}{N} \sum_{p\in\Lambda_+^*} p^2 \left[\|\mathrm{D}^{1/2} b_p \xi\| + \|\mathrm{D}^{1/2} d_p \xi\| \right] \left[\|\mathrm{D}^{1/2} b_p \xi\| + \|\mathrm{D}^{1/2} d_p \xi\| \right] \\
&\leq \frac{C}{N} \sum_{p\in\Lambda_+^*} p^2 |\eta_p| \left[\|b_p (\mathcal{N}_+ + 1)^{1/2} \xi\| + N^{-1/2} |\eta_p| \|(\mathcal{N}_+ + 1)^{3/2} \xi\| \right] \|(\mathcal{N}_+ + 1) \xi\| \\
&\quad + \frac{C}{N} \sum_{p\in\Lambda_+ *} p^2 \left[\|b_p (\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + N^{-1} |\eta_p|^2 \|(\mathcal{N}_+ + 1)^{3/2} \xi\|^2 \right] \\
&\leq C N^{-1/2} \|(\mathcal{K} + \mathcal{N}_+^2 + 1)^{1/2} (\mathcal{N}_+ + 1)^{1/2} \xi\|^2
\end{aligned}
$$

As for the other term on the r.h.s. of (4.158), we have, by (4.157),

$$
\begin{aligned}
\frac{1}{N} \sum_{p\in\Lambda_+^*} p^2 \sigma_p^2 b_p \mathrm{D} b_p^* &= \frac{1}{N} \sum_{p,q\in\Lambda_+^*} p^2 \sigma_p^2 (\gamma_q^2 + \sigma_q^2) b_p b_q^* b_q b_p^* + \frac{1}{N} \sum_{p,q\in\Lambda_+^*} p^2 \sigma_p^2 \sigma_q^2 b_p b_p^* \\
&\quad + \frac{1}{N} \sum_{p,q\in\Lambda_+^*} p^2 \sigma_p^2 \gamma_q \sigma_q b_p (b_q^* b_{-q}^* + \text{h.c.}) b_p^* + \widetilde{\mathcal{E}}_8
\end{aligned}
\tag{4.159}
$$

where, using (4.157), it is easy to check that $\pm\widetilde{\mathcal{E}}_8 \leq C N^{-1} (\mathcal{N}_+ + 1)^2$. Rearranging the other terms on the r.h.s. of (4.159) in normal order and using the commutator relations (1.23), we obtain (4.155) with an error term satisfying (4.156).

Finally, we focus on the contribution of (4.149). We claim that

$$
\begin{aligned}
& e^{-B(\eta)} \mathcal{L}_N^{(2,V)} e^{B(\eta)} \\
&= \sum_{p\in\Lambda_+^*} \left[\kappa \widehat{V}(p/N) \sigma_p^2 + \kappa \widehat{V}(p/N) \sigma_p \gamma_p \left(1 - \frac{\mathcal{N}_+}{N} \right) \right] \\
&\quad + \sum_{p\in\Lambda_+^*} \kappa \widehat{V}(p/N) (\gamma_p + \sigma_p)^2 b_p^* b_p + \frac{1}{2} \sum_{p\in\Lambda_+^*} \kappa \widehat{V}(p/N) (\gamma_p + \sigma_p)^2 (b_p b_{-p} + b_p^* b_{-p}^*) \\
&\quad + \sum_{p\in\Lambda_+^*} \left[\frac{\kappa}{2} \widehat{V}(p/N) \left(\gamma_p b_{-p} + \sigma_p b_p^* \right) d_p + \frac{\kappa}{2} \widehat{V}(p/N) d_p \left(\gamma_p b_{-p} + \sigma_p b_p^* \right) \right] + \text{h.c.} \\
&\quad + \widetilde{\mathcal{E}}_9
\end{aligned}
\tag{4.160}
$$

where

$$
\pm \widetilde{\mathcal{E}}_9 \leq C N^{-1/2} (\mathcal{K} + \mathcal{N}_+^2 + 1)(\mathcal{N}_+ + 1) \tag{4.161}
$$

To prove (4.160), (4.161), we start from (4.149) and decompose

$$
\begin{aligned}
e^{-B(\eta)}\mathcal{L}_N^{(2,V)}e^{B(\eta)} &= \kappa \sum_{p\in\Lambda_+^*} \widehat{V}(p/N)e^{-B(\eta)}b_p^*b_pe^{B(\eta)} - \frac{\kappa}{N}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N)e^{B(\eta)}a_p^*a_pe^{-B(\eta)} \\
&\quad + \frac{\kappa}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N)e^{-B(\eta)}\big[b_pb_{-p}+b_p^*b_{-p}^*\big]e^{B(\eta)} \\
&=: \mathrm{F}_1+\mathrm{F}_2+\mathrm{F}_3
\end{aligned} \tag{4.162}
$$

The operators F_1 and F_2 can be handled exactly as in the proof [14, Prop. 7.6] (notice that the bounds are independent of $\beta \in (0;1)$ and they can be readily extended to the case Gross-Pitaevskii case $\beta = 1$). We obtain that

$$
\mathrm{F}_1 = \kappa\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)[\gamma_pb_p^*+\sigma_pb_{-p}][\gamma_pb_p+\sigma_pb_{-p}^*]+\widetilde{\mathcal{E}}_{10}
$$

where

$$
\pm\widetilde{\mathcal{E}}_{10}\le CN^{-1}(\mathcal{N}_++1)^2
$$

and that

$$
\pm\mathrm{F}_2\le CN^{-1}(\mathcal{N}_++1)
$$

Let us consider the last term on the r.h.s. of (4.162). With (4.33), we obtain

$$
\begin{aligned}
\mathrm{F}_3 = {}& \frac{\kappa}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\left[\gamma_pb_p+\sigma_pb_{-p}^*\right]\left[\gamma_pb_{-p}+\sigma_pb_p^*\right] \\
&+\frac{\kappa}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\ \left[(\gamma_pb_p+\sigma_pb_{-p}^*)\,d_{-p}+d_p\,(\gamma_pb_{-p}+\sigma_pb_p^*)\right] \\
&+\widetilde{\mathcal{E}}_{11}+\mathrm{h.c.}
\end{aligned}
$$

where the error term $\widetilde{\mathcal{E}}_{11} = \frac{\kappa}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)d_pd_{-p}$ can be bounded, using (4.34), by

$$
\begin{aligned}
|\langle\xi,\widetilde{\mathcal{E}}_{11}\xi\rangle| &\le C\sum_{p\in\Lambda_+^*}|\widehat{V}(p/N)|\|d_p^*\xi\|\|d_{-p}\xi\| \\
&\le \frac{C}{N^2}\sum_{p\in\Lambda_+^*}|\widehat{V}(p/N)|\|(\mathcal{N}_++1)^{3/2}\xi\|\Big[|\eta_p|\|(\mathcal{N}_++1)^{3/2}\xi\|+\|b_p(\mathcal{N}_++1)\xi\|\Big] \\
&\le CN^{-1/2}\|(\mathcal{N}_++1)^{3/2}\xi\|^2
\end{aligned}
$$

since $\|\widehat{V}(./N)\|_2\le CN^{3/2}$. This concludes the proof of (4.160) and (4.161). Comparing (4.145), (4.146) and (4.147) with (4.150). (4.152), (4.155) and (4.160), we conclude that the bounds (4.151), (4.153), (4.154), (4.156) and (4.161) imply the desired estimate (4.148). $\square$

4.7.3 Analysis of $\mathcal{G}_N^{(3)} = e^{-B(\eta)}\mathcal{L}_N^{(3)} e^{B(\eta)}$

From (4.43), we have

$$\mathcal{G}_N^{(3)} = \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N) e^{-B(\eta)} b^*_{p+q} a^*_{-p} a_q e^{B(\eta)} + \text{h.c.}$$

We define the error operator $\mathcal{E}_N^{(3)}$ through the identity

$$\mathcal{G}_N^{(3)} = \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N) \left[b^*_{p+1} b^*_{-p}(\gamma_q b_q + \sigma_q b^*_{-q}) + \text{h.c.}\right] + \mathcal{E}_N^{(3)} \tag{4.163}$$

The goal of this subsection is to prove the next lemma, where we estimate $\mathcal{E}_N^{(3)}$.

Lemma 4.7.3. *Let $\mathcal{E}_N^{(3)}$ be as defined in (4.163). Then, under the same assumptions as in Proposition 4.3.2, we find $C > 0$ such that*

$$\pm\, \mathcal{E}_N^{(3)} \leq CN^{-1/2}(\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1) \tag{4.164}$$

Proof. With

$$a^*_{-p} a_q = b^*_{-p} b_q + N^{-1} a^*_{-p}\mathcal{N}_+ a_q$$

we obtain

$$\mathcal{G}_N^{(3)} = \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N) e^{-B(\eta)} b^*_{p+q} b^*_{-p} b_q e^{B(\eta)} + \widetilde{\mathcal{E}}_1 + \text{h.c.} \tag{4.165}$$

where

$$\widetilde{\mathcal{E}}_1 = \frac{\kappa}{N^{3/2}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N) e^{-B(\eta)} b^*_{p+q} a^*_{-p} a_q e^{B(\eta)}$$

can be bounded, switching to position space, by

$$\begin{aligned}
&\big|\langle \xi, \widetilde{\mathcal{E}}_1 \xi\rangle\big| \\
&\quad\leq\ \kappa \int_{\Lambda^2} dxdy\ N^{3/2} V(N(x-y)) \big\| \check{a}_x \check{a}_y (\mathcal{N}_+ + 1)^{1/2} e^{B(\eta)}\xi \big\| \big\| \check{a}_x (\mathcal{N}_+ + 1)^{1/2} e^{B(\eta)} \xi \big\| \\
&\quad\leq\ CN^{-3/2} \langle \xi, e^{-B(\eta)} \mathcal{V}_N (\mathcal{N}_+ + 1) e^{B(\eta)} \xi\rangle + CN^{-1/2} \langle \xi, e^{-B(\eta)} (\mathcal{N}_+ + 1)^2 e^{B(\eta)} \xi\rangle
\end{aligned}$$

With Lemma 4.2.1 and Lemma 4.7.1 we conclude that

$$\big|\langle \xi, \widetilde{\mathcal{E}}_1 \xi\rangle\big| \leq CN^{-1/2} \langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)\xi\rangle \tag{4.166}$$

To control the first term on the r.h.s. of (4.165), we use (4.33) to decompose

$$\frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N) e^{-B(\eta)} b^*_{p+q} b^*_{-p} b_q e^{B(\eta)} = \mathrm{M}_0 + \mathrm{M}_1 + \mathrm{M}_2 + \mathrm{M}_3 \tag{4.167}$$

where

$$
\begin{aligned}
\mathrm{M}_0 := \frac{\kappa}{\sqrt{N}} \sum_{p,q}^{*} \widehat{V}(p/N) \big[&\gamma_{p+q}\gamma_p b^*_{p+q} b^*_{-p} + \gamma_{p+q}\sigma_p b^*_{p+q} b_p + \sigma_{p+q}\sigma_p b_{-p-q} b_p \\
&+ \sigma_{p+q}\gamma_p b^*_{-p} b_{-p-q} - N^{-1}\sigma_{p+q}\gamma_p a^*_{-p} a_{-p-q} \big] \big[\gamma_q b_q + \sigma_q b^*_{-q} \big]
\end{aligned} \tag{4.168}
$$

and

$$
\begin{aligned}
\mathrm{M}_1 :=& \frac{\kappa}{\sqrt{N}} \sum_{p,q}^{*} \widehat{V}(p/N) \big[\gamma_{p+q} b^*_{p+q} d^*_{-p} + \sigma_{p+q} b_{-p-q} d^*_{-p} + \gamma_p d^*_{p+q} b^*_{-p} + \sigma_p d^*_{p+q} b_p d^*_{p+q} \big] \\
&\quad \times \big[\gamma_q b_q + \sigma_q b^*_{-q} \big]; \\
&+ \frac{\kappa}{\sqrt{N}} \sum_{p,q}^{*} \widehat{V}(p/N) \big[\gamma_{p+q}\gamma_p b^*_{p+q} b^*_{-p} + \gamma_{p+q}\sigma_p b^*_{p+q} b_p + \sigma_{p+q}\sigma_p b_{-p-q} b_p \\
&\quad + \sigma_{p+q}\gamma_p b^*_{-p} b_{-p-q} - N^{-1}\sigma_{p+q}\gamma_p a^*_{-p} a_{-p-q} \big] d_q; \\
\mathrm{M}_2 :=& \frac{\kappa}{\sqrt{N}} \sum_{p,q}^{*} \widehat{V}(p/N) \big[\gamma_{p+q} b^*_{p+q} d^*_{-p} + \sigma_{p+q} b_{-p-q} d^*_{-p} + \gamma_p d^*_{p+q} b^*_{-p} + \sigma_p d^*_{p+q} b_p \big] d_q \\
&+ \frac{\kappa}{\sqrt{N}} \sum_{p,q}^{*} \widehat{V}(p/N) d^*_{p+q} d^*_{-p} \big[\gamma_q b_q + \sigma_q b^*_{-q} \big]; \\
\mathrm{M}_3 :=& \frac{\kappa}{\sqrt{N}} \sum_{p,q}^{*} \widehat{V}(p/N) d^*_{p+q} d^*_{-p} d_q
\end{aligned}
$$

Here, we introduced the shorthand notation $\sum^*_{p,q} \equiv \sum_{p,q\in\Lambda^*_+, p+q\neq 0}$ and we used the identity $b_{-p-q}b^*_{-p} = b^*_{-p}b_{-p-q} - N^{-1}a^*_{-p}a_{-p-q}$, for all $q \in \Lambda^*_+$. Notice that the index i in M_i counts the number of d-operators it contains.

Let us start by analysing M_3. Switching to position space we find, using (4.35) and the bound $\|\check{\eta}\|_\infty \leq CN$ (as follows from (4.49) since, by the definition (4.50), we have $\check{\eta}(x) = -Nw_\ell(Nx)$),

$$
\begin{aligned}
|\langle \xi, \mathrm{M}_3 \xi \rangle| \leq& \int dxdy\, N^{5/2} V(N(x-y)) \|(\mathcal{N}_+ + 1)^{-1} \check{d}_x \check{d}_y \xi\| \|(\mathcal{N}_+ + 1) \check{d}_x \xi\| \\
\leq& \frac{C}{N^3} \int dxdy\, N^{5/2} V(N(x-y)) \Big[\|(\mathcal{N}_+ + 1)^{5/2}\xi\| + \|\check{a}_x (\mathcal{N}_+ + 1)^2 \xi\| \Big] \\
&\times \Big[N \|(\mathcal{N}_+ + 1)\xi\| + \|\check{a}_x (\mathcal{N}_+ + 1)^{3/2}\xi\| \\
&\quad + \|\check{a}_y (\mathcal{N}_+ + 1)^{3/2}\xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1)\xi\| \Big] \\
\leq& CN^{-1/2} \langle \xi, (\mathcal{V}_N + \mathcal{N}_+^2 + 1)\xi \rangle
\end{aligned} \tag{4.169}
$$

As for M_2, it reads in position space

$$\begin{aligned}\mathrm{M}_2 &= \kappa \int dxdy\, N^{5/2} V(N(x-y)) \big[b^*(\check{\gamma}_x) \check{d}^*_y + b(\check{\sigma}_x) \check{d}^*_y + \check{d}^*_x b^*(\check{\gamma}_y) + \check{d}^*_x b(\check{\sigma}_y) \big] \check{d}_x \\ &\quad + \kappa \int dxdy\, N^{5/2} V(N(x-y)) \check{d}^*_x \check{d}^*_y \big[b(\check{\gamma}_x) + b^*(\check{\sigma}_x) \big] \\ &=: \mathrm{M}_{21} + \mathrm{M}_{22}\end{aligned}$$

To control M_{22}, we use the bound (4.35) to estimate

$$\begin{aligned}\big| \langle \xi, \mathrm{M}_{22} \xi \rangle \big| &\leq \kappa \int dxdy\, N^{5/2} V(N(x-y)) \big\| (\mathcal{N}_+ + 1)^{-1} \check{d}_y \check{d}_x \xi \big\| \\ &\qquad \times \big\| (\mathcal{N}_+ + 1) \big[b(\check{\gamma}_x) + b^*(\check{\sigma}_x) \big] \xi \big\| \\ &\leq C N^{-2} \int dxdy N^{5/2} V(N(x-y)) \Big[\|(\mathcal{N}_+ + 1)^{3/2} \xi\| + \|\check{a}_x (\mathcal{N}_+ + 1) \xi\| \Big] \\ &\quad \times \Big[N \|(\mathcal{N}_+ + 1) \xi\| + \|\check{a}_x (\mathcal{N}_+ + 1)^{3/2} \xi\| \\ &\qquad + \|\check{a}_y (\mathcal{N}_+ + 1)^{3/2} \xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1) \xi\| \Big] \\ &\leq C N^{-1} \langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1) \xi \rangle\end{aligned}$$

With the first and the second bounds in (4.35), we can also control M_{21}. We find

$$\begin{aligned}|\langle \xi, \mathrm{M}_{21} \xi \rangle| &\leq C \int dxdy\, N^{5/2} V(N(x-y)) \|\check{d}_x \xi\| \\ &\qquad \times \big[\|\check{d}_y b(\check{\gamma}_x) \xi\| + \|\check{d}_y b^*(\check{\sigma}_x) \xi\| + \|b(\check{\gamma}_y) \check{d}_x \xi\| + \|b^*(\check{\sigma}_y) \check{d}_x \xi\| \big] \\ &\leq C \int dxdy\, N^{1/2} V(N(x-y)) \Big[\|(\mathcal{N}_+ + 1)^{3/2} \xi\| + \|\check{a}_x (\mathcal{N}_+ + 1) \xi\| \Big] \\ &\qquad \times \Big[N \|(\mathcal{N}_+ + 1) \xi\| + \|\check{a}_x (\mathcal{N}_+ + 1)^{3/2} \xi\| + \|\check{a}_y (\mathcal{N}_+ + 1)^{3/2} \xi\| \\ &\qquad\qquad + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1) \xi\| \Big] \\ &\leq C N^{-1} \langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1) \xi \rangle\end{aligned} \tag{4.170}$$

where we used that $\|\check{\eta}\|_\infty \leq CN$. Hence, we proved that

$$|\langle \xi, \mathrm{M}_2 \xi \rangle| \leq C N^{-1} \langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1) \xi \rangle \tag{4.171}$$

Next, let us consider the operator M_1. In position space, we find

$$\begin{aligned}\mathrm{M}_1 &= \kappa \int dxdy\, N^{5/2} V(N(x-y)) \big[b^*(\check{\gamma}_x) \check{d}^*_y + b(\check{\sigma}_x) \check{d}^*_y + \check{d}^*_x b^*(\check{\gamma}_y) + \check{d}^*_x b(\check{\sigma}_y) \big] \\ &\qquad \times \big[b(\check{\gamma}_x) + b^*(\check{\sigma}_x) \big] \\ &\quad + \kappa \int dxdy\, N^{5/2} V(N(x-y)) \big[b^*(\check{\gamma}_x) b^*(\check{\gamma}_y) + b^*(\check{\gamma}_x) b(\check{\sigma}_y) + b(\check{\sigma}_x) b(\check{\sigma}_y) \\ &\qquad + b^*(\check{\gamma}_x) b(\check{\sigma}_y) - N^{-1} a^*(\check{\gamma}_x) a(\check{\sigma}_y) \big] \check{d}_x \\ &=: \mathrm{M}_{11} + \mathrm{M}_{12}\end{aligned}$$

To estimate M_{11}, we proceed as in (4.170). With (4.35), using again $\|\check{\eta}\|_\infty \le CN$, we find

$$\begin{aligned}\big|\langle \xi, \mathrm{M}_{11}\xi\rangle\big| \le & \kappa \int dxdy\, N^{5/2} V(N(x-y))\, [\|b(\check{\gamma}_x)\xi\| + \|b^*(\check{\sigma}_x)\xi\|] \\ & \times \big[\|\check{d}_y b(\check{\gamma}_x)\xi\| + \|\check{d}_y b^*(\check{\sigma}_x)\xi\| + \|b(\check{\gamma}_y)\check{d}_x\xi\| + \|b^*(\check{\sigma}_y)\check{d}_x\xi\|\big] \\ \le & C \int dxdy N^{3/2} V(N(x-y)) \Big[\|(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\check{a}_x\xi\|\Big] \\ & \times \Big[N\|(\mathcal{N}_+ +1)\xi\| + \|\check{a}_x(\mathcal{N}_+ +1)^{3/2}\xi\| + \|\check{a}_y(\mathcal{N}_+ +1)^{3/2}\xi\| \\ & \qquad + \|\check{a}_x\check{a}_y(\mathcal{N}_+ +1)\xi\|\Big] \\ \le & CN^{-1/2}\langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ +1)\xi\rangle\end{aligned}$$

As for the term M_{12}, we use the bound

$$\begin{aligned}&\Big\|(\mathcal{N}_+ +1)^{1/2}\big[b^*(\check{\gamma}_x)b^*(\check{\gamma}_y) + b^*(\check{\gamma}_x)b(\check{\sigma}_y) + b(\check{\sigma}_x)b(\check{\sigma}_y) \\ &\qquad + b^*(\check{\gamma}_x)b(\check{\sigma}_y) - N^{-1}a^*(\check{\gamma}_x)a(\check{\sigma}_y)\big]\xi\Big\| \\ &\le C\Big[\big\|(\mathcal{N}_+ +1)^{3/2}\xi\big\| + \big\|\check{a}_x(\mathcal{N}_+ +1)\xi\big\| + \big\|\check{a}_y(\mathcal{N}_+ +1)\xi\big\| + \big\|\check{a}_x\check{a}_y(\mathcal{N}_+ +1)^{1/2}\xi\big\|\Big]\end{aligned}$$

to conclude that

$$\begin{aligned}&|\langle \xi, \mathrm{M}_{12}\xi\rangle| \\ &\le C\int dxdy\, N^{3/2}V(N(x-y))\,\Big[\|(\mathcal{N}_+ +1)\xi\| + \|\check{a}_x(\mathcal{N}_+ +1)^{1/2}\xi\|\Big] \\ &\quad \times \Big[\big\|(\mathcal{N}_+ +1)^{3/2}\xi\big\| + \big\|\check{a}_x(\mathcal{N}_+ +1)\xi\big\| + \big\|\check{a}_y(\mathcal{N}_+ +1)\xi\big\| + \big\|\check{a}_x\check{a}_y(\mathcal{N}_+ +1)^{1/2}\xi\big\|\Big] \\ &\le CN^{-1/2}\langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ +1)\xi\rangle\end{aligned}$$

Thus,

$$|\langle \xi, \mathrm{M}_1\xi\rangle| \le CN^{-1/2}\langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ +1)\xi\rangle \tag{4.172}$$

Finally, we consider (4.168). We split $\mathrm{M}_0 = \mathrm{M}_{01} + \mathrm{M}_{02}$, with

$$\begin{aligned}\mathrm{M}_{01} &:= \frac{\kappa}{\sqrt{N}}\sum_{p,q}^{*}\widehat{V}(p/N)\gamma_{p+q}\gamma_p b^*_{p+q}b^*_{-p}\big[\gamma_q b_q + \sigma_q b^*_{-q}\big]; \\ \mathrm{M}_{02} &:= \frac{\kappa}{\sqrt{N}}\sum_{p,q}^{*}\widehat{V}(p/N)\big[\gamma_{p+q}\sigma_p b^*_{p+q}b_p + \sigma_{p+q}\sigma_p b_{-p-q}b_p + \sigma_{p+q}\gamma_p b^*_{-p}b_{-p-q} \\ &\qquad - N^{-1}\sigma_{p+q}\gamma_p a^*_{-p}a_{-p-q}\big]\big[\gamma_q b_q + \sigma_q b^*_{-q}\big]\end{aligned}$$

Switching to position space, we find

$$\begin{aligned}\big|\langle \xi, \mathrm{M}_{02}\xi\rangle\big| \le & C\int dxdy\, N^{5/2}V(N(x-y))\Big[\|\check{a}_x\xi\| + \|(\mathcal{N}_+ +1)^{1/2}\xi\|\Big] \\ & \times \Big[\|(\mathcal{N}_+ +1)\xi\| + \|\check{a}_x(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\check{a}_y(\mathcal{N}_+ +1)^{1/2}\xi\|\Big] \\ \le & CN^{-1/2}\langle \xi, (\mathcal{N}_+ +1)^2\xi\rangle\end{aligned}$$

As for M_{01}, we write $\gamma_p = 1+(\gamma_p-1)$ and $\gamma_{p+q} = 1+(\gamma_{p+q}-1)$. Using that $|\gamma_p-1| \leq C/p^4$ and σ_q are square summable, it is easy to check that

$$\mathrm{M}_{01} = \frac{\kappa}{\sqrt{N}} \sum_{p,q}^{*} \widehat{V}(p/N) b^*_{p+q} b^*_{-p} \left[\gamma_q b_q + \sigma_q b^*_{-q}\right] + \widetilde{\mathcal{E}}_2$$

where $\widetilde{\mathcal{E}}_2$ is such that

$$|\langle \xi, \widetilde{\mathcal{E}}_2 \xi \rangle| \leq C N^{-1/2} \langle \xi, (\mathcal{N}_+ + 1)^2 \xi \rangle$$

Combining the last bound, with the bounds (4.166), (4.169), (4.171), (4.172) and the decompositions (4.165) and (4.167), we obtain (4.164). □

4.7.4 Analysis of $\mathcal{G}_N^{(4)} = e^{-B(\eta)} \mathcal{L}_N^{(4)} e^{B(\eta)}$

From (4.43) we have

$$\mathcal{G}_N^{(4)} = \frac{\kappa}{2N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p,-q} \widehat{V}(r/N) e^{-B(\eta)} a^*_{p+r} a^*_q a_p a_{q+r} e^{B(\eta)} \tag{4.173}$$

We define the error operator $\mathcal{E}_N^{(4)}$ through the identity

$$\begin{aligned}
\mathcal{G}_N^{(4)} = \mathcal{V}_N &+ \frac{\kappa}{2N} \sum_{p,q \in \Lambda_+^*} \widehat{V}((p-q)/N) \sigma_q \gamma_q \sigma_p \gamma_p \big(1 + 1/N - 2\,\mathcal{N}_+/N\big) \\
&+ \frac{\kappa}{2N} \sum_{p \in \Lambda_+^*, q \in \Lambda^*} \widehat{V}((p-q)/N) \eta_q \\
&\qquad \times \Big[\gamma_p^2 b^*_p b^*_{-p} + 2\gamma_p \sigma_p b^*_p b_p + \sigma_p^2 b_p b_{-p} + d_p\big(\gamma_p b_{-p} + \sigma_p b^*_p\big) \\
&\qquad\qquad + \big(\gamma_p b_p + \sigma_p b^*_{-p}\big) d_{-p} + \text{h.c.}\Big] \\
&+ \frac{\kappa}{N^2} \sum_{p,q,u \in \Lambda_+^*} \widehat{V}((p-q)/N) \eta_p \eta_q \\
&\qquad \times \Big[\gamma_u^2 b^*_u b_u + \sigma_u^2 b^*_u b_u + \gamma_u \sigma_u b^*_u b^*_{-u} + \gamma_u \sigma_u b_u b_{-u} + \sigma_u^2\Big] \\
&+ \mathcal{E}_N^{(4)}
\end{aligned} \tag{4.174}$$

The goal of this subsection is to bound the error term $\mathcal{E}_N^{(4)}$.

Lemma 4.7.4. *Let $\mathcal{E}_N^{(4)}$ be as defined in (4.174). Then, under the same assumptions as in Proposition 4.3.2, we find $C > 0$ such that*

$$\pm\, \mathcal{E}_N^{(4)} \leq C N^{-1/2} (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1) \tag{4.175}$$

Proof. First of all, we replace, on the r.h.s. of (4.173), all a-operators by b-operators. To this end, we notice that

$$a^*_{p+r}a^*_q a_p a_{q+r} = b^*_{p+r}b^*_q b_p b_{q+r}\left(1-\frac{3}{N}+\frac{2\mathcal{N}_+}{N}\right) + a^*_{p+r}a^*_q a_p a_{q+r}\,\Theta_{\mathcal{N}_+}$$

where

$$\begin{aligned}\Theta_{\mathcal{N}_+} :=& \left[\frac{\big(N-\mathcal{N}_++2\big)}{N}\frac{\big(\mathcal{N}_+-1\big)}{N}+\frac{\big(\mathcal{N}_+-2\big)}{N}\right]^2 \\ &\qquad + \left[-\frac{\mathcal{N}_+^2}{N^2}+\frac{3\mathcal{N}_+}{N^2}-\frac{2}{N^2}\right]\left[\frac{\big(N-\mathcal{N}_++2\big)}{N}\frac{\big(N-\mathcal{N}_++1\big)}{N}\right]\end{aligned}$$

is such that $\pm\Theta_{\mathcal{N}_+}\leq C(\mathcal{N}_++1)^2/N^2$ on $\mathcal{F}_+^{\leq N}$. With Lemma 4.7.1 we conclude that

$$\begin{aligned}\mathcal{G}_N^{(4)} =& \frac{\kappa(N+1)}{2N^2}\sum_{\substack{p,q\in\Lambda^*_+, r\in\Lambda^*:\\ r\neq -p,-q}} \widehat{V}(r/N)e^{-B(\eta)}b^*_{p+r}b^*_q b_p b_{q+r}e^{B(\eta)} \\ &+ \frac{\kappa}{N^2}\sum_{\substack{p,q,u\in\Lambda^*_+, r\in\Lambda^*:\\ r\neq -p,-q}} \widehat{V}(r/N)e^{-B(\eta)}b^*_{p+r}b^*_q b^*_u b_u b_p b_{q+r}e^{B(\eta)} + \widetilde{\mathcal{E}}_1\end{aligned} \tag{4.176}$$

with the error $\widetilde{\mathcal{E}}_1$ satisfying

$$\pm\widetilde{\mathcal{E}}_1 \leq CN^{-1}(\mathcal{V}_N+\mathcal{N}_++1)(\mathcal{N}_++1)$$

We split the rest of the proof in two steps, where we analyze separately the two terms on the r.h.s. of (4.176).

Step 1. The first term on the r.h.s. of (4.176) can be written as

$$\begin{aligned}&\frac{\kappa(N+1)}{2N^2}\sum_{\substack{p,q\in\Lambda^*_+, r\in\Lambda^*:\\ r\neq -p,-q}} \widehat{V}(r/N)e^{-B(\eta)}b^*_{p+r}b^*_q b_p b_{q+r}e^{B(\eta)} \\ &\quad = \mathcal{V}_N + \frac{\kappa}{2N}\sum_{p,q\in\Lambda^*_+}\widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p\left(1+1/N-2\mathcal{N}_+/N\right) \\ &\qquad + \frac{\kappa}{2N}\sum_{p,q\in\Lambda^*_+}\widehat{V}((p-q)/N)\sigma_q\gamma_q\Big[\gamma_p^2 b^*_p b^*_{-p}+2\gamma_p\sigma_p b^*_p b_p+\sigma_p^2 b_p b_{-p} \\ &\qquad\qquad + d_p\big(\gamma_p b_{-p}+\sigma_p b^*_p\big)+\big(\gamma_p b_p+\sigma_p b^*_{-p}\big)d_{-p}+\text{h.c.}\Big]+\widetilde{\mathcal{E}}_2\end{aligned} \tag{4.177}$$

where the error $\widetilde{\mathcal{E}}_2$ is such that

$$\pm\widetilde{\mathcal{E}}_2 \leq CN^{-1/2}(\mathcal{V}_N+\mathcal{N}_++1)(\mathcal{N}_++1)$$

To show (4.177), we write

$$\frac{\kappa(N+1)}{2N^2}\sum_{\substack{p,q\in\Lambda_+^*,r\in\Lambda^*:\\ r\neq -p,-q}}\widehat{V}(r/N)e^{-B(\eta)}b^*_{p+r}b^*_q b_p b_{q+r}e^{B(\eta)} = \mathrm{V}_0+\mathrm{V}_1+\mathrm{V}_2+\mathrm{V}_3+\mathrm{V}_4$$

with

$$\begin{aligned}\mathrm{V}_0 := \frac{\kappa(N+1)}{2N^2}\sum^*_{p,q,r}\widehat{V}(r/N)\Big[&\gamma_{p+r}\gamma_q b^*_{p+r}b^*_q+\gamma_{p+r}\sigma_q b^*_{p+r}b_{-q}+\sigma_{p+r}\sigma_q b_{-p-r}b_{-q}\\ &+\sigma_{p+r}\gamma_q\big(b^*_q b_{-p-r}-N^{-1}a^*_q a_{-p-r}\big)\Big]\Big[\sigma_p\sigma_{q+r}b^*_{-p}b^*_{-q-r}\\ &+\sigma_p\gamma_{q+r}b^*_{-p}b_{q+r}+\gamma_p\gamma_{q+r}b_p b_{q+r}+\gamma_p\sigma_{q+r}\big(b^*_{-q-r}b_p-N^{-1}a^*_{-q-r}a_p\big)\Big]\\ +\frac{\kappa(N+1)}{2N^2}\sum_{p,q\in\Lambda^*_+}&\widehat{V}((p-q)/N)\sigma_q\gamma_q\Big[\big(\gamma_p^2 b^*_p b^*_{-p}+2\gamma_p\sigma_p b^*_p b_p-N^{-1}\gamma_p\sigma_p a^*_p a_p\\ &\qquad+\sigma_p^2 b_p b_{-p}\big)\big(1-\mathcal{N}_+/N\big)+\text{h.c.}\Big]\\ +\frac{\kappa(N+1)}{2N^2}\sum_{p,q\in\Lambda^*_+}&\widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p\left(1-\mathcal{N}_+/N\right)^2,\end{aligned} \tag{4.178}$$

$$\begin{aligned}\mathrm{V}_1 := \frac{\kappa(N+1)}{2N^2}\sum^*_{p,q,r}\widehat{V}(r/N)\Big[&\gamma_{p+r}\gamma_q b^*_{p+r}b^*_q+\gamma_{p+r}\sigma_q b^*_{p+r}b_{-q}\\ &+\sigma_{p+r}\sigma_q b_{-p-r}b_{-q}+\sigma_{p+r}\gamma_q\big(b^*_q b_{-p-r}-N^{-1}a^*_q a_{-p-r}\big)\Big]\\ &\times\Big[\big(\gamma_p b_p+\sigma_p b^*_{-p}\big)d_{q+r}+d_p\big(\gamma_{q+r}b_{q+r}+\sigma_{q+r}b^*_{-q-r}\big)\Big]\\ +\frac{\kappa(N+1)}{2N^2}\sum_{p,q\in\Lambda^*_+}&\widehat{V}((p-q)/N)\sigma_q\gamma_q\big(1-\mathcal{N}_+/N\big)\\ &\times\Big[d_p\big(\gamma_p b_{-p}+\sigma_p b^*_p\big)+\big(\gamma_p b_p+\sigma_p b^*_{-p}\big)d_{-p}\Big]+\text{h.c.}\ ,\end{aligned} \tag{4.179}$$

and

$$
\begin{aligned}
\mathrm{V}_2 :=& \frac{\kappa(N+1)}{2N^2} \sum_{p,q,r}^{*} \widehat{V}(r/N)\Big[\big(\gamma_{p+r}b^*_{p+r} + \sigma_{p+r}b_{-p-r}\big)d^*_q + d^*_{p+r}\big(\gamma_q b^*_q + \sigma_q b_{-q}\big)\Big] \\
&\qquad\qquad\qquad\qquad \times\Big[\big(\gamma_p b_p + \sigma_p b^*_{-p}\big)d_{q+r} + d_p\big(\gamma_{q+r}b_{q+r} + \sigma_{q+r}b^*_{-q-r}\big)\Big] \\
&+ \frac{\kappa(N+1)}{2N^2} \sum_{p,q\in\Lambda^*_+} \widehat{V}((p-q)/N)\sigma_q\gamma_q\Big[d^*_{-p}d^*_p\big(1-\mathcal{N}_+/N\big) + \big(1-\mathcal{N}_+/N\big)d_p d_{-p}\Big]\,, \\
\mathrm{V}_3 :=& \frac{\kappa(N+1)}{2N^2} \sum_{p,q,r}^{*} \widehat{V}(r/N)\Big[\big(\gamma_{p+r}b^*_{p+r} + \sigma_{p+r}b_{-p-r}\big)d^*_q + d^*_{p+r}\big(\gamma_q b^*_q + \sigma_q b_{-q}\big)\Big]d_p d_{q+r} \\
&+ \text{h.c.}\,; \\
\mathrm{V}_4 :=& \frac{\kappa(N+1)}{2N^2} \sum_{p,q,r}^{*} \widehat{V}(r/N)d^*_{p+r}d^*_q d_p d_{q+r}
\end{aligned}
\tag{4.180}
$$

Here, we used the notation $\sum^*_{p,q,r} := \sum_{p,q\in\Lambda^*_+, r\in\Lambda^*: r\neq -p,-q}$ for simplicity. Notice that the index of V_j refers to the number of d-operators it contains.

Let us consider V_4. Switching to position space and using (4.35), we find

$$
\begin{aligned}
|\langle \xi, \mathrm{V}_4\xi\rangle| &\le C\int dxdy\, N^2 V(N(x-y))\|\check{d}_x\check{d}_y\xi\|\,\|\check{d}_x\check{d}_y\xi\| \\
&\le C\int dxdy\, V(N(x-y))\Big[\|(\mathcal{N}_++1)^2\xi\| + N\|(\mathcal{N}_++1)\xi\| \\
&\qquad\qquad + \|\check{a}_x(\mathcal{N}_++1)^{3/2}\xi\| + \|\check{a}_y(\mathcal{N}_++1)^{3/2}\xi\| + \|\check{a}_x\check{a}_y(\mathcal{N}_++1)\xi\|\Big]^2 \\
&\le CN^{-1}\langle\xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_++1)\xi\rangle
\end{aligned}
$$

Next, we switch to the contribution V_3, defined in (4.180). Switching again to position space, using (4.35) and the bound $\mathcal{N}_+\le N$, we obtain

$$
\begin{aligned}
|\langle \xi, \mathrm{V}_3\xi\rangle| &\le C\int dxdy\, N^2 V(N(x-y))\|\check{d}_x\check{d}_y\xi\| \\
&\qquad\qquad \times\Big[\|\check{d}_y b(\check{\gamma}_x)\xi\| + \|\check{d}_y b^*(\check{\sigma}_x)\xi\| + \|b(\check{\gamma}_y)\check{d}_x\xi\| + \|b^*(\check{\sigma}_y)\check{d}_x\xi\|\Big] \\
&\le C\int dxdy\, V(N(x-y))\Big[\|(\mathcal{N}_++1)^2\xi\| + \|\check{a}_x(\mathcal{N}_++1)^{3/2}\xi\| \\
&\qquad\qquad + \|\check{a}_y(\mathcal{N}_++1)^{3/2}\xi\| + \|\check{a}_x\check{a}_y(\mathcal{N}_++1)\xi\| + N\|(\mathcal{N}_++1)\xi\|\Big]^2 \\
&\le CN^{-1}\langle\xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_++1)\xi\rangle
\end{aligned}
$$

Proceeding similarly, V_2 can be bounded, switching to position space, by

$$\begin{aligned}
|\langle \xi, \mathrm{V}_2 \xi\rangle| \leq C \int dxdy\, N^2 V(N(x-y)) \\
\times \Big[\|\check{d}_y b(\check{\gamma}_x)\xi\| + \|\check{d}_y b^*(\check{\sigma}_x)\xi\| + \|b(\check{\gamma}_y)\check{d}_x\xi\| + \|b^*(\check{\sigma}_y)\check{d}_x\xi\|\Big]^2 \\
+ C \int dxdy\, N^2 V(N(x-y)) |(\check{\sigma} * \check{\gamma})(x-y)| \|(\mathcal{N}_+ + 1)^{-1} \check{d}_x \check{d}_y \xi\| \|(\mathcal{N}_+ + 1)\xi\| \\
\leq C \int dxdy V(N(x-y)) \Big[\|(\mathcal{N}_+ + 1)^2\xi\| + \|\check{a}_x (\mathcal{N}_+ + 1)^{3/2}\xi\| \\
+ \|\check{a}_y (\mathcal{N}_+ + 1)^{3/2}\xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1)\xi\| + N\|(\mathcal{N}_+ + 1)\xi\|\Big]^2 \\
+ C \int dxdy N V(N(x-y)) \|(\mathcal{N}_+ + 1)\xi\| \Big[\|(\mathcal{N}_+ + 1)^2\xi\| + N\|(\mathcal{N}_+ + 1)\xi\| \\
+ \|\check{a}_x (\mathcal{N}_+ + 1)^{3/2}\xi\| + \|\check{a}_y (\mathcal{N}_+ + 1)^{3/2}\xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1)\xi\|\Big] \\
\leq C N^{-1} \langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)\xi\rangle
\end{aligned}$$

Here we used the bound $\|\check{\sigma} * \check{\gamma}\|_\infty \leq CN$ from (4.57).

Let us now study the term V_1. We write

$$\begin{aligned}
\mathrm{V}_1 = \frac{\kappa}{2N} \sum_{p,q \in \Lambda_+^*} \widehat{V}((p-q)/N) \sigma_q \gamma_q \Big[d_p\big(\gamma_p b_{-p} + \sigma_p b_p^*\big) + \big(\gamma_p b_p + \sigma_p b_{-p}^*\big) d_{-p}\Big] + \text{h.c.} \\
+ \mathrm{V}_{12} + \mathrm{V}_{13}
\end{aligned} \tag{4.181}$$

where V_{13} denotes the first sum on the r.h.s. of (4.179) and V_{12} is the difference between the second term on the r.h.s. of (4.179) and the term on the r.h.s. of (4.181). Switching to position space and using (4.35), we find easily

$$\begin{aligned}
|\langle \xi, \mathrm{V}_{12}\xi\rangle| \leq C \int dxdy\, N V(N(x-y)) |(\check{\sigma} * \check{\gamma})(x-y)| \|(\mathcal{N}_+ + 1)\xi\| \\
\times \| \left[\|\check{d}_x b(\check{\gamma}_y)\xi\| + \|\check{d}_x b^*(\sigma_y)\xi\| + \|b(\check{\gamma}_x)\check{d}_y\xi\| + \|b^*(\check{\sigma}_x)\check{d}_y\xi\|\right] \\
\leq C N^{-1} \langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)\xi\rangle
\end{aligned}$$

and

$$\begin{aligned}
|\langle \xi, \mathrm{V}_{13}\xi\rangle| \leq C \int dxdy N^2 V(N(x-y)) \\
\times \Big[\|(\mathcal{N}_+ + 1)\xi\| + \|a_x (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|a_y (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|a_x a_y \xi\|\Big] \\
\times \Big[\|b(\check{\gamma}_x)\check{d}_y\xi\| + \|b^*(\check{\sigma}_x)\check{d}_y\xi\| + \|\check{d}_x b(\check{\gamma}_y)\xi\| + \|\check{d}_y b^*(\check{\sigma}_y)\xi\|\Big] \\
\leq C N^{-1} \langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)\xi\rangle
\end{aligned}$$

Finally, we analyse V_0, as defined in (4.178). We write $\mathrm{V}_0 = \mathrm{V}_{01} + \mathrm{V}_{02} + \mathrm{V}_{03}$, where

$$\mathrm{V}_{01} := \frac{\kappa(N+1)}{2N^2} \sum_{p,q\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p \left(1-\mathcal{N}_+/N\right)^2;$$

$$\begin{aligned}\mathrm{V}_{02} := \frac{\kappa(N+1)}{2N^2} \sum_{p,q\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q \Big[\big(\gamma_p^2 b_p^* b_{-p}^* + 2\gamma_p\sigma_p b_p^* b_p - N^{-1}\gamma_p\sigma_p a_p^* a_p \\ + \sigma_p^2 b_p b_{-p}\big)\big(1-\mathcal{N}_+/N\big) + \text{h.c.}\Big];\end{aligned}$$

$$\begin{aligned}\mathrm{V}_{03} := \frac{\kappa(N+1)}{2N^2} \sum_{p,q,r}^{*} \widehat{V}(r/N) \Big[&\gamma_{p+r}\gamma_q b_{p+r}^* b_q^* + \gamma_{p+r}\sigma_q b_{p+r}^* b_{-q} + \sigma_{p+r}\sigma_q b_{-p-r} b_{-q} \\ &+ \sigma_{p+r}\gamma_q \big(b_q^* b_{-p-r} - N^{-1} a_q^* a_{-p-r}\big)\Big]\Big[\sigma_p\sigma_{q+r} b_{-p}^* b_{-q-r}^* + \sigma_p\gamma_{q+r} b_{-p}^* b_{q+r} \\ &+ \gamma_p\gamma_{q+r} b_p b_{q+r} + \gamma_p\sigma_{q+r}\big(b_{-q-r}^* b_p - N^{-1} a_{-q-r}^* a_p\big)\Big]\end{aligned}$$

Proceeding similarly as above, switching to position space and using (in the estimate for $\widetilde{\mathcal{E}}_4$) the bound $\|\check{\sigma} * \check{\gamma}\|_\infty \leq CN$, we find that

$$\mathrm{V}_{01} = \frac{\kappa}{2N} \sum_{p,q\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p \left(1 + 1/N - 2\mathcal{N}_+/N\right) + \widetilde{\mathcal{E}}_3$$

$$\mathrm{V}_{02} = \frac{\kappa}{2N} \sum_{p,q\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q \Big[\gamma_p^2 b_p^* b_{-p}^* + 2\gamma_p\sigma_p b_p^* b_p + \sigma_p^2 b_p b_{-p} + \text{h.c.}\Big] + \widetilde{\mathcal{E}}_4$$

$$\mathrm{V}_{03} = \mathcal{V}_N + \widetilde{\mathcal{E}}_5$$

Combining with (4.181) and with all other bounds for the error terms, we arrive at (4.177).

Step 2. We claim that

$$\begin{aligned}&\frac{\kappa}{N^2} \sum_{\substack{p,q,u\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq -p,-q}} \widehat{V}(r/N) e^{-B(\eta)} b_{p+r}^* b_q^* b_u^* b_u b_p b_{q+r} e^{B(\eta)} \\ &= \frac{\kappa}{N^2} \sum_{p,q,u\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p \Big[\gamma_u^2 b_u^* b_u + \sigma_u^2 b_u^* b_u + \gamma_u\sigma_u b_u^* b_{-u}^* + \gamma_u\sigma_u b_u b_{-u}\Big] \\ &\quad + \frac{\kappa}{N^2} \sum_{p,q,u\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p\sigma_u^2 + \widetilde{\mathcal{E}}_6\end{aligned} \tag{4.182}$$

where the error $\widetilde{\mathcal{E}}_6$ is such that, on $\mathcal{F}_+^{\leq N}$,

$$\pm \mathcal{E}_{3,N}^{(4)} \leq CN^{-1/2}(\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)$$

To show (4.182), we split

$$\frac{\kappa}{N^2} \sum_{\substack{p,q,u\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq -p,-q}} \widehat{V}(r/N) e^{-B(\eta)} b_{p+r}^* b_q^* b_u^* b_u b_p b_{q+r} e^{B(\eta)} = \mathrm{W}_0 + \mathrm{W}_1 + \mathrm{W}_2$$

where

$$\mathrm{W}_0 := \frac{\kappa}{N^2} \sum_{p,q,u\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p \left[1-\mathcal{N}_+/N\right]\left[e^{-B(\eta)}b_u^*b_u e^{B(\eta)}\right]\left[1-\mathcal{N}_+/N\right];$$

and

$$\begin{aligned}\mathrm{W}_1 := \frac{\kappa}{N^2} \sum_{p,q,u\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q \Big[&\gamma_p^2 b_p^* b_{-p}^* + 2\gamma_p\sigma_p b_p^* b_p - N^{-1}\gamma_p\sigma_p a_p^* a_p + \sigma_p^2 b_p b_{-p} \\ &+ \gamma_p b_{-p}^* d_p^* + \sigma_p b_p d_p^* + \gamma_p d_{-p}^* b_p^* + \sigma_p d_p^* b_p + d_{p+r}^* d_q^*\Big] \\ &\times \left[e^{-B(\eta)}b_u^* b_u e^{B(\eta)}\right]\left[1-\mathcal{N}_+/N\right] + \text{h.c.};\end{aligned}$$

$$\begin{aligned}\mathrm{W}_2 := \frac{\kappa}{N^2} \sum_{p,q,r,u}^{*} \widehat{V}(r/N)\Big[&\gamma_{p+r}\gamma_q b_{p+r}^* b_q^* + \gamma_{p+r}\sigma_q b_{p+r}^* b_{-q} + \sigma_{p+r}\sigma_q b_{-p-r} b_{-q} \\ &+ \sigma_{p+r}\gamma_q\big(b_q^* b_{-p-r} - N^{-1}a_q^* a_{-p-r}\big) + \big(\gamma_{p+r}b_{p+r}^* + \sigma_{p+r}b_{-p-r}\big)d_q^* \\ &+ d_{p+r}^*\big(\gamma_q b_q^* + \sigma_q b_{-q}\big) + d_{p+r}^* d_q^*\Big] \times \Big[e^{-B(\eta)}b_u^* b_u e^{B(\eta)}\Big] \times \Big[\sigma_p\sigma_{q+r}b_{-p}^* b_{-q-r}^* \\ &+ \sigma_p\gamma_{q+r}b_{-p}^* b_{q+r} + \gamma_p\gamma_{q+r}b_p b_{q+r} + \gamma_p\sigma_{q+r}\big(b_{-q-r}^* b_p - N^{-1}a_{-q-r}^* a_p\big) \\ &+ \big(\gamma_p b_p + \sigma_p b_{-p}^*\big)d_{q+r} + d_p\big(\gamma_{q+r}b_{q+r} + \sigma_{q+r}b_{-q-r}^*\big) + d_p d_{q+r}\Big]\end{aligned}$$

Here, we introduced the notation $\sum^*_{p,q,r,u} := \sum_{p,q,u\in\Lambda_+^*, r\in\Lambda^*: r\neq -p,-q}$ for simplicity. Using Lemma 4.2.1 to get rid of the factor $\sum_{u\in\Lambda_+^*} e^{-B(\eta)}b_u^* b_u e^{B(\eta)} = e^{-B(\eta)}\mathcal{N}_+(1-\mathcal{N}_+/N)e^{B(\eta)}$ and then proceeding similarly as in Step 1, we obtain that

$$\begin{aligned}|\langle\xi, \mathrm{W}_1\xi\rangle| &\le CN^{-1/2}\langle\xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)\xi\rangle \\ |\langle\xi, \mathrm{W}_2\xi\rangle| &\le CN^{-1}\langle\xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)\xi\rangle\end{aligned}$$

As for W_0, we write

$$\begin{aligned}\mathrm{W}_0 = \frac{\kappa}{N^2} &\sum_{p,q,u\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p \\ &\qquad\times \Big[\gamma_u^2 b_u^* b_u + \sigma_u^2 b_u^* b_u + \gamma_u\sigma_u b_u^* b_{-u}^* + \gamma_u\sigma_u b_u b_{-u} + \sigma_u^2\Big] \\ &+ \widetilde{\mathcal{E}}_7\end{aligned}$$

Using (4.33) to decompose

$$e^{-B(\eta)}b_u^* b_u e^{B(\eta)} = \big(\gamma_u b_u^* + \sigma_u b_{-u} + d_u^*\big)\big(\gamma_u b_u + \sigma_u b_{-u}^* + d_u\big)$$

and then the bounds (4.34), it is easy to estimate the remainder operator $\widetilde{\mathcal{E}}_7$, on $\mathcal{F}_+^{\le N}$, by

$$\pm\widetilde{\mathcal{E}}_7 \le CN^{-1}(\mathcal{N}_+ + 1)$$

Hence, we obtain (4.182).

Step 3. Conclusion of the proof. Combining (4.177) and (4.182) with (4.176), we conclude that

$$\begin{aligned}
\mathcal{G}_N^{(4)} &= \mathcal{V}_N + \frac{\kappa}{2N} \sum_{p,q\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p \big(1 + 1/N - 2\,\mathcal{N}_+/N\big) \\
&\quad + \frac{\kappa}{2N} \sum_{p,q\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q \Big[\gamma_p^2 b_p^* b_{-p}^* + 2\gamma_p\sigma_p b_p^* b_p + \sigma_p^2 b_p b_{-p} + d_p\big(\gamma_p b_{-p} + \sigma_p b_p^*\big) \\
&\qquad\qquad + \big(\gamma_p b_p + \sigma_p b_{-p}^*\big) d_{-p} + \text{h.c.}\Big] \\
&\quad + \frac{\kappa}{N^2} \sum_{p,q,u\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p \\
&\qquad\qquad \times \Big[\gamma_u^2 b_u^* b_u + \sigma_u^2 b_u^* b_u + \gamma_u\sigma_u b_u^* b_{-u}^* + \gamma_u\sigma_u b_u b_{-u} + \sigma_u^2\Big] \\
&\quad + \widetilde{\mathcal{E}}_8
\end{aligned} \tag{4.183}$$

with an error $\widetilde{\mathcal{E}}_8$ such that, on $\mathcal{F}_+^{\leq N}$,

$$\pm\widetilde{\mathcal{E}}_8 \leq CN^{-1/2}(\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)$$

To conclude the proof of (4.175), we just observe that in the term appearing on the second line on the r.h.s. of (4.183), we can replace the product $\sigma_q\gamma_q$ simply by η_q. Since $|\sigma_q\gamma_q - \eta_q| \leq C|q|^{-6}$ (or, in position space, $\|(\check{\sigma} * \check{\gamma}) - \check{\eta}\|_\infty \leq C$), it is easy to show that the difference can be incorporated in the error term. Similarly, in the term appearing on the third and fourth lines on the r.h.s. of (4.183), we can replace $\sigma_p\gamma_p\sigma_q\gamma_q$ by $\eta_p\eta_q$; also in this case, the contribution of the difference is small and can be included in the remainder. This concludes the proof of the lemma. □

4.7.5 Proof of Proposition 4.3.2

Collecting the results of (4.144), Lemma 4.7.2, Lemma 4.7.3 and Lemma 4.7.4, we obtain that

$$\mathcal{G}_N = \widetilde{C}_{\mathcal{G}_N} + \widetilde{\mathcal{Q}}_{\mathcal{G}_N} + \mathcal{D}_N + \mathcal{H}_N + \mathcal{C}_N + \widetilde{\mathcal{E}}_{\mathcal{G}_N} \tag{4.184}$$

where $\mathcal{C}_N$ is the cubic term defined in (4.66), $\widetilde{\mathcal{E}}_{\mathcal{G}_N}$ an error term controlled by

$$\pm\widetilde{\mathcal{E}}_{\mathcal{G}_N} \leq CN^{-1/2}(\mathcal{H}_N + \mathcal{N}_+^2)(\mathcal{N}_+ + 1),$$

and where $\widetilde{C}_{\mathcal{G}_N}$, $\widetilde{\mathcal{Q}}_{\mathcal{G}_N}$ and $\mathcal{D}_N$ are given by

$$\begin{aligned}
\widetilde{C}_{\mathcal{G}_N} &= \frac{(N-1)}{2}\kappa\widehat{V}(0) + \sum_{p\in\Lambda_+^*} \Big[p^2\sigma_p^2\left(1 + \frac{1}{N}\right) + \kappa\widehat{V}(p/N)\big(\sigma_p\gamma_p + \sigma_p^2\big)\Big] \\
&\quad + \frac{\kappa}{2N} \sum_{q\in\Lambda_+^*} \widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p\big(1 + 1/N\big) \\
&\quad + \frac{1}{N} \sum_{u\in\Lambda_+^*} \sigma_u^2 \sum_{p\in\Lambda_+^*} \big[p^2\sigma_p^2 + \frac{\kappa}{N} \sum_{q\in\Lambda_+^*} \widehat{V}((p-q)/N)\eta_p\eta_q\big]
\end{aligned} \tag{4.185}$$

$$
\begin{aligned}
\widetilde{\mathcal{Q}}_{\mathcal{G}_N} = &\sum_{p\in\Lambda_+^*} b_p^* b_p \big[2\sigma_p^2 p^2 + \kappa \widehat{V}(p/N)\,(\gamma_p+\sigma_p)^2 + \frac{2\kappa}{N}\gamma_p\sigma_p \sum_{q\in\Lambda^*}\widehat{V}((p-q)/N)\eta_q\big] \\
&+ \sum_{p\in\Lambda_+^*}\big(b_p^* b_{-p}^* + b_p b_{-p}\big) \\
&\qquad \times \big[p^2\sigma_p\gamma_p + \frac{\kappa}{2}\widehat{V}(p/N)(\gamma_p+\sigma_p)^2 + \frac{\kappa}{2N}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N)\eta_q(\gamma_p^2+\sigma_p^2)\big] \\
&- \frac{\mathcal{N}_+}{N}\sum_{p\in\Lambda_+^*}\big[p^2\sigma_p^2 + \kappa\widehat{V}(p/N)\gamma_p\sigma_p + \frac{\kappa}{N}\sum_{q\in\Lambda_+^*}\widehat{V}((p-q)/N)\gamma_p\sigma_p\gamma_q\sigma_q\big] \\
&+ \frac{1}{N}\sum_{u\in\Lambda_+^*}\big[(\gamma_u^2+\sigma_u^2)b_u^* b_u + \gamma_u\sigma_u(b_u^* b_{-u}^* + b_u b_{-u})\big] \\
&\qquad\qquad \times \sum_{p\in\Lambda_+^*}\big[p^2\sigma_p^2 + \frac{\kappa}{N}\sum_{q\in\Lambda_+^*}\widehat{V}((p-q)/N)\eta_p\eta_q\big]
\end{aligned}
\tag{4.186}
$$

and

$$
\begin{aligned}
\mathcal{D}_N = &\sum_{p\in\Lambda_+^*}\Big[p^2\eta_p b_p d_{-p} + \frac{\kappa}{2}\widehat{V}(p/N)b_p d_{-p} + \frac{\kappa}{2N}\big(\widehat{V}(\cdot/N)*\eta\big)_p b_p d_{-p} + \text{h.c.}\Big] \\
&+ \frac{\kappa}{2}\sum_{p\in\Lambda_+^*}(\widehat{V}(./N)*\widehat{f}_{\ell,N})(p)\Big[(\gamma_p-1)b_p d_{-p} + \sigma_p b_{-p}^* d_{-p} + \text{h.c.}\Big] \\
&+ \frac{\kappa}{2}\sum_{p\in\Lambda_+^*}(\widehat{V}(./N)*\widehat{f}_{\ell,N})(p)\Big[\gamma_p d_p b_{-p} + \sigma_p d_p b_p^* + \text{h.c.}\Big]
\end{aligned}
\tag{4.187}
$$

with $\widehat{f}_{\ell,N}$ defined as in (4.46). Next, we analyse the operator $\mathcal{D}_N$, which still contains d-operators, to extract the important contributions. To this end, we write $\mathcal{D}_N = \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3$, where $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3$ denote the operators on the first, second and, respectively, third line on the r.h.s. of (4.187).

Using the relation (4.52) and the bound (4.34), we find

$$
\begin{aligned}
|\langle\xi, \mathrm{D}_1\xi\rangle| &\le \sum_{p\in\Lambda_+^*}|(\widehat{\chi}_\ell * \widehat{f}_{\ell,N})(p)|\|(\mathcal{N}_++1)\xi\|\|(\mathcal{N}_++1)^{-1/2}d_{-p}\xi\| \\
&\le \frac{C}{N}\|(\mathcal{N}_++1)\xi\|\sum_{p\in\Lambda_+^*}\Big[\frac{1}{p^4}\|(\mathcal{N}_++1)\xi\| + \frac{1}{p^2}\|b_p(\mathcal{N}_++1)^{1/2}\xi\|\Big] \\
&\le \frac{C}{N}\|(\mathcal{N}_++1)\xi\|^2
\end{aligned}
$$

Similarly, using (4.34), we find

$$
\pm\mathrm{D}_2 \le CN^{-1}(\mathcal{N}_++1)^2
$$

Thus, we switch to D_3. We split $\mathrm{D}_3 = \mathrm{D}_{31} + \mathrm{D}_{32} + \mathrm{D}_{33}$, with

$$\begin{aligned}
\mathrm{D}_{31} &:= \frac{\kappa}{2} \sum_{p\in\Lambda_+^*} (\widehat{V}(./N) * \widehat{f}_{\ell,N})(p)\gamma_p d_p b_{-p} + \text{h.c.} \\
\mathrm{D}_{32} &:= \frac{\kappa}{2} \sum_{p\in\Lambda_+^*} (\widehat{V}(./N) * \widehat{f}_{\ell,N})(p)(\sigma_p - \eta_p) d_p b_p^* + \text{h.c.} \\
\mathrm{D}_{33} &:= \frac{\kappa}{2} \sum_{p\in\Lambda_+^*} (\widehat{V}(./N) * \widehat{f}_{\ell,N})(p)\eta_p d_p b_p^* + \text{h.c.}
\end{aligned}$$

Switching to position space and using (4.35), we observe that

$$\begin{aligned}
|\langle \xi, \mathrm{D}_{31}\xi\rangle| &\leq C \int dxdy N^3 V(N(x-y)) f_\ell(N(x-y)) \|(\mathcal{N}_+ + 1)\xi\| \|(\mathcal{N}_+ + 1)^{-1} \check{d}_x b(\check{\gamma}_y)\xi\| \\
&\leq C \int dxdy N^2 V(N(x-y)) f_\ell(N(x-y)) \|(\mathcal{N}_+ + 1)\xi\| \\
&\qquad \times \Big[\|(\mathcal{N}_+ + 1)\xi\| + \|\check{a}_x (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_y (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_x \check{a}_y \xi\| \Big] \\
&\leq C N^{-1/2} \langle \xi, (\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)\xi\rangle
\end{aligned}$$

As for D_{32}, we can use the decay of $|\sigma_p - \eta_p| \leq C|p|^{-6}$ to prove that

$$\pm \mathrm{D}_{32} \leq C N^{-1} (\mathcal{N}_+ + 1)^2$$

We are left with D_{33}; here we cannot apply (4.34) because of the lack of decay in p. This term contains contributions that are relevant in the limit of large N. To isolate these contributions, it is useful to rewrite the remainder operator d_p as

$$\begin{aligned}
d_p &= e^{-B(\eta)} b_p e^{B(\eta)} - \gamma_p b_p - \sigma_p b_{-p}^* \\
&= (1-\gamma_p) b_p - \sigma_p b_{-p}^* + \eta_p \int_0^1 ds\, e^{-sB(\eta)} \frac{N - \mathcal{N}_+}{N} b_{-p}^* e^{sB(\eta)} \\
&\qquad\qquad - \frac{1}{N} \int_0^1 ds \sum_{q\in\Lambda_+^*} \eta_q e^{-sB(\eta)} b_q^* a_{-q}^* a_p e^{sB(\eta)} \\
&= \eta_p \int_0^1 ds\, d_{-p}^{(s)*} - \frac{\eta_p}{N} \int_0^1 ds\; e^{-sB(\eta)} \mathcal{N}_+ b_{-p}^* e^{sB(\eta)} \\
&\qquad\qquad - \frac{1}{N} \int_0^1 ds \sum_{q\in\Lambda_+^*} \eta_q e^{-sB(\eta)} b_q^* a_{-q}^* a_p e^{sB(\eta)}
\end{aligned} \tag{4.188}$$

where, in the last step, we wrote $e^{-sB(\eta)} b_{-p}^* e^{sB(\eta)} = \gamma_p^{(s)} b_{-p}^* + \sigma_p^{(s)} b_p + d_{-p}^{(s)*}$ (the label s indicates that the coefficients $\gamma_p^{(s)}$, $\sigma_p^{(s)}$ and the operator $d_{-p}^{(s)*}$ are defined with η replaced by $s\eta$, for an $s \in [0;1]$) and we integrated $\gamma_p^{(s)}$ and $\sigma_p^{(s)}$ over $s \in [0;1]$. Inserting (4.188) into D_{33} and using the additional factor η_p appearing in the first two terms on the r.h.s. of (4.188), we conclude that

$$\mathrm{D}_{33} = -\frac{\kappa}{2N}\int_0^1 ds \sum_{p,q\in\Lambda_+^*} (\widehat{V}(./N) * \widehat{f}_{\ell,N})(p)\eta_p\eta_q \Big[e^{-sB(\eta)} b_q^* a_{-q}^* a_p e^{sB(\eta)} b_p^* + \text{h.c.}\Big] + \widetilde{\mathcal{E}}_1$$

with an error operator $\widetilde{\mathcal{E}}_1$ such that

$$\pm\, \widetilde{\mathcal{E}}_1 \leq CN^{-1}(\mathcal{N}_+ + 1)^2 \tag{4.189}$$

We expand

$$-e^{-sB(\eta)} a_{-q}^* a_p e^{sB(\eta)} = -a_{-q}^* a_p - \int_0^s dt\; e^{-tB(\eta)} \big(\eta_p b_{-q}^* b_{-p}^* + \eta_q b_q b_p\big) e^{tB(\eta)}$$

Again, the contribution containing the additional factor η_p is small. Hence, we have

$$\begin{aligned}\mathrm{D}_{33} = &-\frac{\kappa}{2N}\int_0^1 ds \sum_{p,q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N}\big)(p) \Big[\eta_p\eta_q e^{-sB(\eta)} b_q^* e^{sB(\eta)} \\ &\times \Big(a_{-q}^* a_p b_p^* + \int_0^s dt\; \eta_q e^{-tB(\eta)} b_q b_p e^{tB(\eta)} b_p^*\Big) + \text{h.c.}\Big] + \widetilde{\mathcal{E}}_2\end{aligned}$$

where, similarly to (4.189), $\pm\widetilde{\mathcal{E}}_2 \leq CN^{-1}(\mathcal{N}_+ + 1)^2$. In the contribution proportional to $a_{-q}^* a_p b_p^*$ we commute b_p^* to the left. In the other term, we expand $e^{-tB(\eta)} b_p e^{tB(\eta)} = \gamma_p^{(t)} b_p + \sigma_p^{(t)} b_{-p}^* + d_p^{(t)}$ using the notation introduced after (4.188) and we commute the contribution $\gamma_p^{(t)} b_p$ to the right of b_p^*. We obtain $\mathrm{D}_{33} = \mathrm{D}_{331} + \mathrm{D}_{332} + \widetilde{\mathcal{E}}_2$, with

$$\begin{aligned}\mathrm{D}_{331} := &-\frac{\kappa}{2N}\int_0^1 ds \sum_{p,q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N}\big)_p \eta_p\eta_q e^{-sB(\eta)} b_q^* e^{sB(\eta)} b_{-q}^* \\ &-\frac{\kappa}{2N}\int_0^1 ds \int_0^s dt \sum_{p,q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N}\big)_p \eta_p\eta_q^2 e^{-sB(\eta)} b_q^* e^{sB(\eta)} e^{-tB(\eta)} b_q e^{tB(\eta)} \\ &+ \text{h.c.}, \\ \mathrm{D}_{332} := &-\frac{\kappa}{2N}\int_0^1 ds \sum_{p,q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N}\big)_p \eta_p\eta_q e^{-sB(\eta)} b_q^* e^{sB(\eta)} b_{-q}^* a_p^* a_p \\ &-\frac{\kappa}{2N}\int_0^1 ds \int_0^s dt \sum_{p,q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N}\big)_p \eta_p\eta_q^2 e^{-sB(\eta)} b_q^* e^{sB(\eta)} e^{-tB(\eta)} b_q e^{tB(\eta)} \\ &\times \Big[b_p^* b_p - N^{-1}\mathcal{N}_+ - N^{-1} a_p^* a_p + \big(\gamma_p^{(t)} - 1\big) b_p b_p^* + \sigma_p^{(t)} b_{-p}^* b_p^* + d_p^{(t)} b_p^*\Big] + \text{h.c.}\end{aligned}$$

Since $(\gamma_p^{(t)} - 1) \leq C\eta_p$ and $\sigma_p^{(t)} \leq C\eta_p$, we can bound

$$\big|\langle \xi, \mathrm{D}_{332}\xi\rangle\big| \leq\; CN^{-1}\langle \xi, (\mathcal{N}_+ + 1)^2 \xi\rangle$$

We are left with the operator D_{331}, which is quadratic in the b-fields. Expanding $e^{-sB(\eta)}b_q^* e^{sB(\eta)} = \gamma_q^{(s)} b_q^* + \sigma_q^{(s)} b_{-q} + d_q^{(s),*}$ and $e^{-tB(\eta)} b_q e^{tB(\eta)} = \gamma_q^{(t)} b_q + \sigma_q^{(t)} b_{-q}^* + d_q^{(t)}$ and using the bounds (4.34), we obtain

$$\begin{aligned}
\mathrm{D}_{331} = & -\frac{\kappa}{2N}\int_0^1 ds \sum_{p,q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p \eta_p\eta_q \Big[\gamma_q^{(s)} b_q^* b_{-q}^* + \sigma_q^{(s)} b_q^* b_q + \sigma_q^{(s)} + \text{h.c.}\Big] \\
& -\frac{\kappa}{2N}\int_0^1 ds \int_0^s dt \sum_{p,q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)_p \eta_p\eta_q^2 \Big[\gamma_q^{(s)}\gamma_q^{(t)} b_q^* b_q + \gamma_q^{(s)}\sigma_q^{(t)} b_q^* b_{-q}^* \\
& \qquad + \sigma_q^{(s)}\gamma_q^{(t)} b_q b_q + \sigma_q^{(s)}\sigma_q^{(t)} b_q^* b_q + \sigma_q^{(s)}\sigma_q^{(t)} + \text{h.c.}\Big] + \widetilde{\mathcal{E}}_2
\end{aligned} \tag{4.190}$$

where the operator $\widetilde{\mathcal{E}}_2$ is such that $\pm\widetilde{\mathcal{E}}_2 \leq CN^{-1}(\mathcal{N}_+ + 1)^2$. Integrating (4.190) over t and s, we conclude that

$$\begin{aligned}
\mathcal{D}_N = & -\frac{\kappa}{2N}\sum_{p\in\Lambda^*, q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)(p)\,\eta_p\Big[\gamma_q\sigma_q\big(b_q^* b_{-q}^* + b_q b_{-q}\big) + (\sigma_q^2+\gamma_q^2) b_q^* b_q + \sigma_q^2\Big] \\
& + \frac{\kappa}{2N}\sum_{p\in\Lambda^*, q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)(p)\,\eta_p\, b_q^* b_q + \widetilde{\mathcal{E}}_3
\end{aligned} \tag{4.191}$$

where the error $\widetilde{\mathcal{E}}_3$ satisfies

$$\pm\widetilde{\mathcal{E}}_3 \leq CN^{-1/2}(\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1)$$

Notice that, in (4.191), we are summing also over the mode $p=0$ (this contribution is small, it can be inserted in the operator $\widetilde{\mathcal{E}}_3$).

Inserting (4.191) into (4.184), we obtain the decomposition (4.67) of the Hamiltonian $\mathcal{G}_N$. In fact, combining (4.185) with the constant contribution in (4.191), we find

$$\begin{aligned}
\widetilde{C}_{\mathcal{G}_N} & - \frac{\kappa}{2N}\sum_{p\in\Lambda^*, q\in\Lambda_+^*} \big(\widehat{V}(\cdot/N)*\widehat{f}_{\ell,N}\big)(p)\,\eta_p\sigma_q^2 \\
= & \frac{(N-1)}{2}\kappa\widehat{V}(0) + \sum_{p\in\Lambda_+^*}\Big[p^2\sigma_p^2 + \kappa\widehat{V}(p/N)\left(\sigma_p\gamma_p + \sigma_p^2\right)\Big] \\
& + \frac{\kappa}{2N}\sum_{q\in\Lambda_+^*}\widehat{V}((p-q)/N)\sigma_q\gamma_q\sigma_p\gamma_p + \frac{1}{N}\sum_{p\in\Lambda_+^*}\Big[p^2\eta_p^2 + \frac{\kappa}{2N}(\widehat{V}(./N)*\eta)(p)\eta_p\Big] \\
& + \frac{1}{N}\sum_{u\in\Lambda_+^*}\sigma_u^2\sum_{p\in\Lambda_+^*}\left[p^2\eta_p^2 - \frac{\kappa}{2}\widehat{V}(p/N)\eta_p + \frac{\kappa}{2N}\sum_{q\in\Lambda_+^*}(\widehat{V}(./N)*\eta)(p)\,\eta_p\right] + \mathcal{O}(N^{-1})
\end{aligned} \tag{4.192}$$

(the error $\mathcal{O}(N^{-1})$ arises from the substitution $\sigma_p \to \eta_p$ in the terms appearing at the end of the third and on the fourth line). Using the relation (4.51), we have

$$p^2\eta_p^2 - \frac{\kappa}{2}\widehat{V}(p/N)\eta_p + \frac{\kappa}{2N}\sum_{q\in\Lambda_+^*}(\widehat{V}*\eta)(p)\,\eta_p = -\kappa\widehat{V}(p/N)\eta_p + N^3\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_{\ell,N})(p)\eta_p \quad (4.193)$$

Since $N^3\lambda_\ell = \mathcal{O}(1)$ and $\|(\widehat{\chi}_\ell * \widehat{f}_{\ell,N})\eta\|_1 \leq \|\widehat{\chi}_\ell * \widehat{f}_{\ell,N}\|_2\|\eta\|_2 = \|\chi_\ell f_\ell\|_2\|\eta\|_2 \leq \|\chi_\ell\|_2\|\eta\|_2 \leq C$, uniformly in N, we conclude that the r.h.s. (4.192) coincides with (4.62), up to errors of order N^{-1}. Similarly, combining the quadratic term (4.186) with the quadratic terms on the r.h.s. of (4.191) (and using again the relation (4.193), we obtain (4.65), up to terms that can be incorporated in the error. We omit these last details.

4.8 Analysis of the excitation Hamiltonian $\mathcal{J}_N$

In this section we analyse the excitation Hamiltonian

$$\mathcal{J}_N = e^{-A}e^{-B(\eta)}U_N H_N U_N^* e^{B(\eta)}e^A = e^{-A}\mathcal{G}_N e^A$$

to show Proposition 4.3.3. The starting point is part b) of Proposition 4.3.2, stating that

$$\mathcal{G}_N = C_{\mathcal{G}_N} + \mathcal{Q}_{\mathcal{G}_N} + \mathcal{C}_N + \mathcal{H}_N + \mathcal{E}_{\mathcal{G}_N}\,,$$

where $C_{\mathcal{G}_N}$, $\mathcal{Q}_{\mathcal{G}_N}$ and $\mathcal{C}_N$ are defined in (4.62), (4.65) and (4.66) and the error term $\mathcal{E}_{\mathcal{G}_N}$ is such that

$$\pm\mathcal{E}_{\mathcal{G}_N} \leq \frac{C}{\sqrt{N}}\left[(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1) + (\mathcal{N}_+ + 1)^3\right].$$

From Proposition 4.4.2 and Proposition 4.4.4 we conclude that

$$\pm e^{-A}\mathcal{E}_{\mathcal{G}_N}e^A \leq \frac{C}{\sqrt{N}}\left[(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1) + (\mathcal{N}_+ + 1)^3\right]$$

In the following sections we study the action of e^A on $\mathcal{Q}_{\mathcal{G}_N}$, $\mathcal{C}_N$ and $\mathcal{H}_N$ separately. At the end, in Section 4.8.4, we combine these results to prove Theorem 4.3.3.

4.8.1 Analysis of $e^{-A}\mathcal{Q}_{\mathcal{G}_N}e^A$.

The action of e^A on the quadratic operator $\mathcal{Q}_{\mathcal{G}_N}$ defined in Prop. 4.3.2 is determined by the next proposition.

Proposition 4.8.1. *Let A and $\mathcal{Q}_{\mathcal{G}_N}$ be defined as in* (4.69) *and, respectively,* (4.65)*. Then, under the assumptions of Proposition 4.3.3, we have*

$$e^{-A}\mathcal{Q}_{\mathcal{G}_N}\,e^A = \mathcal{Q}_{\mathcal{G}_N} + \mathcal{E}_N^{(\mathcal{Q})}\,,$$

where the error $\mathcal{E}_N^{(\mathcal{Q})}$ is such that, for a constant $C > 0$,

$$\pm\,\mathcal{E}_N^{(\mathcal{Q})} \leq \frac{C}{\sqrt{N}}\,(\mathcal{N}_+ + 1)^2$$

To prove the proposition we use the following lemma.

Lemma 4.8.2. *Let A be defined as in (4.69) and let Φ_p and $\mathcal{G}_p$ such that $|\Phi_p| \leq C$ and $|\mathcal{G}_p| \leq C|p|^{-2}$. Then, under the assumptions of Proposition 4.3.3,*

$$\pm \sum_{p\in\Lambda_+^*} \Phi_p \, [b_p^* b_p, A] \leq \frac{C}{\sqrt{N}} \, (\mathcal{N}_+ + 1)^2 \tag{4.194}$$

$$\pm \sum_{p\in\Lambda_+^*} \mathcal{G}_p \, [(b_p b_{-p} + b_p^* b_{-p}^*), A] \leq \frac{C}{\sqrt{N}} \, (\mathcal{N}_+ + 1)^2 \tag{4.195}$$

Proof of Proposition 4.8.1. We write

$$e^{-A} \mathcal{Q}_{\mathcal{G}_N} \, e^{A} = \mathcal{Q}_{\mathcal{G}_N} + \int_0^1 ds \, e^{-sA} \, [\mathcal{Q}_N, A] e^{sA} \, .$$

We recall the expression for $\mathcal{Q}_{\mathcal{G}_N}$ in (4.65), where the coefficients of the diagonal and off-diagonal terms are given by Φ_p in (4.63) and, respectively, by Γ_p in (4.64). With (4.52), one can show that $|\Phi_p| \leq C$ and $|\Gamma_p| \leq C|p|^{-2}$. Hence, Proposition 4.8.1 follows from Lemma 4.8.2 and Proposition 4.4.2. □

Proof of Lemma 4.8.2. We start from the proof of (4.194). We use the formula

$$\begin{aligned}
&[b_p^*, A] \\
&= \frac{1}{\sqrt{N}} \sum_{\substack{r\in P_H \\ v\in P_L}} \eta_r \Big[\, (\gamma_v b_v^* + \sigma_v b_{-v}) b_{-r} \Big(1 - \frac{\mathcal{N}_+}{N}\Big) \, \mathrm{d}_{p,r+v} \\
&\qquad\qquad + (\gamma_v b_v^* + \sigma_v b_{-v}) b_{r+v} \Big(1 - \frac{\mathcal{N}_+ - 1}{N}\Big) \, \mathrm{d}_{p,-r} \\
&\qquad\qquad + \sigma_v \Big(1 - \frac{\mathcal{N}_+}{N}\Big) b_{r+v} b_{-r} \, \mathrm{d}_{p,-v} - \gamma_v b_{r+v}^* b_{-r}^* \Big(1 - \frac{\mathcal{N}_+}{N}\Big) \, \mathrm{d}_{p,v} \, \Big] \\
&\quad - \frac{1}{N\sqrt{N}} \sum_{\substack{r\in P_H \\ v\in P_L}} \eta_r \big[\, (\gamma_v b_v^* + \sigma_v b_{-v})(b_{-r} a_p^* a_{r+v} + a_p^* a_{-r} b_{r+v}) \\
&\qquad\qquad\qquad + \sigma_v a_p^* a_{-v} b_{-r} b_{r+v} - \gamma_v b_{r+v}^* b_{-r}^* a_p^* a_v \, \big]
\end{aligned} \tag{4.196}$$

and the fact that $[b_p, A] = [b_p^*, A]^*$ to compute $[b_p^* b_p, A]$. We get:

$$\sum_{p\in\Lambda_+^*} \Phi_p \, [b_p^* b_p, A] = \sum_{j=1}^{8} \Delta_j + \text{h.c.}$$

where

$$\Delta_1 = \frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Phi_{r+v}\eta_r \left(1 - \frac{\mathcal{N}_+ - 1}{N}\right) b^*_{r+v} b^*_{-r} (\gamma_v b_v + \sigma_v b^*_{-v})$$

$$\Delta_2 = \frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Phi_r\eta_r \left(1 - \frac{\mathcal{N}_+ - 2}{N}\right) b^*_{-r} b^*_{r+v} (\gamma_v b_v + \sigma_v b^*_{-v}),$$

$$\Delta_3 = \frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Phi_v\eta_r \left(1 - \frac{\mathcal{N}_+ - 3}{N}\right) b^*_{r+v} b^*_{-r} \sigma_v b^*_{-v}$$

$$\Delta_4 = -\frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Phi_v\eta_r \left(1 - \frac{\mathcal{N}_+ - 2}{N}\right) b^*_{-r} b^*_{r+v} \gamma_v b_v$$

$$\Delta_5 = -\frac{1}{N^{3/2}} \sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} \Phi_p\eta_r\, b^*_p a^*_{r+v} a_p b^*_{-r} (\gamma_v b_v + \sigma_v b^*_{-v})$$

$$\Delta_6 = -\frac{1}{N^{3/2}} \sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} \Phi_p\eta_r\, b^*_p b^*_{r+v} a^*_{-r} a_p (\gamma_v b_v + \sigma_v b^*_{-v})$$

$$\Delta_7 = -\frac{1}{N^{3/2}} \sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} \Phi_p\eta_r\sigma_v\, b^*_p b^*_{r+v} b^*_{-r} a^*_{-v} a_p$$

$$\Delta_8 = \frac{1}{N^{3/2}} \sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} \Phi_p\eta_r\gamma_v\, b^*_p a^*_v a_p b_{-r} b_{r+v}$$

Using $|\Phi_p| \leq C$, $|\eta_r| \leq C|r|^{-2}$, $|\sigma_v| \leq C|v|^{-2}$, we estimate, for any $\xi \in \mathcal{F}^{\leq N}_+$,

$$\begin{aligned} |\langle \xi, \Delta_1 \xi\rangle| &\leq \frac{C}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} |\eta_r| \|b_{-r} b_{r+v}\xi\| \big(|\sigma_v| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|b_v\xi\|\big) \\ &\leq \frac{C}{\sqrt{N}} \|(\mathcal{N}_+ + 1)\xi\|^2 \end{aligned}$$

The terms Δ_j with $j = 2,3,4$ are bounded in a similar way. To bound Δ_5 we first move b^*_{-r} to the left, obtaining

$$\begin{aligned} \Delta_5 = &-\frac{1}{N^{3/2}} \sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} \Phi_p\eta_r\, b^*_p b^*_{-r} a^*_{r+v} a_p (\gamma_v b_v + \sigma_v b^*_{-v}) \\ &- \frac{1}{N^{3/2}} \sum_{r\in P_H,\, v\in P_L} \Phi_r\eta_r b^*_{-r} b^*_{r+v} (\gamma_v b_v + \sigma_v b^*_{-v}) = \Delta_5^{(1)} + \Delta_5^{(2)} \end{aligned}$$

The cubic term $\Delta_5^{(2)}$ can be estimated similarly as Δ_1, while

$$\begin{aligned}
|\langle \xi, \Delta_5^{(1)} \xi\rangle| &\leq \frac{C}{N^{3/2}} \sum_{p\in\Lambda_+^*} \sum_{r\in P_H,\, v\in P_L} |\eta_r| \|a_{r+v} b_{-r} b_p \xi\| \|a_p(\gamma_v b_v + \sigma_v b^*_{-v})\xi\| \\
&\leq \frac{C}{N} \Big(\frac{1}{N} \sum_{p\in\Lambda_+^*} \sum_{r\in P_H,\, v\in P_L} \|a_{r+v} b_{-r} b_p\xi\|^2 \Big)^{1/2} \\
&\qquad\qquad \times \Big(\sum_{p\in\Lambda_+^*} \sum_{r\in P_H,\, v\in P_L} |\eta_r|^2 \left[\|a_p b_v \xi\|^2 + |\sigma_v|^2 \|a_p b^*_{-v}\xi\|^2 \right] \Big)^{1/2} \\
&\leq \frac{C}{N} \|(\mathcal{N}_+ + 1)\xi\|^2 .
\end{aligned}$$

(In the last step, to bound the contribution proportional to $\|a_p b^*_{-v}\xi\|$, we first estimated the sum over r, p with fixed v by $|\sigma_v|^2 \|\mathcal{N}_+^{1/2} b^*_{-v}\xi\|^2 \leq |\sigma_v|^2 \|(\mathcal{N}_+ + 1)\xi\|$ and then we summed over v). The terms Δ_j with $j = 6, 7, 8$ can be treated as $\Delta_5^{(1)}$. Hence for all $j = 1, \dots, 8$ we have

$$\pm \left(\Delta_j + \text{h.c.} \right) \leq \frac{C}{\sqrt{N}} (\mathcal{N}_+ + 1)^2 .$$

This concludes the proof of (4.194).

To show (4.195) we use (4.196) and its conjugate to compute

$$\sum_{p\in\Lambda_+^*} \Gamma_p \, [(b_p b_{-p} + b^*_p b^*_{-p}), A] = \sum_{j=1}^{9} \Upsilon_j + \text{h.c.}$$

where

$$\begin{aligned}
\Upsilon_1 &= \frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Gamma_{r+v} \eta_r \left(1 - \frac{\mathcal{N}_+ + 1}{N} \right) (b^*_{-r} b_{-r-v} - \frac{1}{N} a^*_{-r} a_{-r-v})(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Upsilon_2 &= \frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Gamma_{r+v} \eta_r \left(1 - \frac{\mathcal{N}_+}{N} \right) b^*_{-r} (\gamma_v b_v + \sigma_v b^*_{-v}) b_{-r-v} \\
\Upsilon_3 &= \frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Gamma_r \eta_r \left(1 - \frac{\mathcal{N}_+}{N} \right) (b^*_{r+v} b_r - \frac{1}{N} a^*_{r+v} a_r)(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Upsilon_4 &= \frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Gamma_r \eta_r \left(1 - \frac{\mathcal{N}_+ - 1}{N} \right) b^*_{r+v} (\gamma_v b_v + \sigma_v b^*_{-v}) b_r \\
\Upsilon_5 &= \frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Gamma_v \eta_r \sigma_v \left(1 - \frac{\mathcal{N}_+ - 1}{N} \right) (b^*_{r+v} b_v - \frac{1}{N} a^*_{r+v} a_v) b^*_{-r} \\
\Upsilon_6 &= \frac{1}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \Gamma_v \eta_r \sigma_v \left(1 - \frac{\mathcal{N}_+ - 2}{N} \right) b^*_{r+v} b^*_{-r} b_v
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon_7 = &-\frac{1}{\sqrt{N}}\sum_{r\in P_H,\, v\in P_L}\Gamma_v\eta_r\gamma_v\left[\left(1-\frac{\mathcal{N}_+}{N}\right)+\left(1-\frac{\mathcal{N}_++1}{N}\right)\right]b_{r+v}b_{-r}b_{-v}\\
\Upsilon_8 = &-\frac{1}{N\sqrt{N}}\sum_{p\in\Lambda_+^*}\sum_{r\in P_H,\, v\in P_L}\Gamma_p\eta_r\big[\, b_p(a^*_{r+v}a_{-p}b^*_{-r}+b^*_{r+v}a^*_{-r}a_{-p})(\gamma_v b_v+\sigma_v b^*_{-v})\\
&\qquad\qquad\qquad\qquad\qquad\qquad +\sigma_v b_p b^*_{r+v}b^*_{-r}a^*_{-v}a_{-p}-\gamma_v b_p a^*_v a_{-p}b_{-r}b_{r+v}\,\big]\\
\Upsilon_9 = &-\frac{1}{N\sqrt{N}}\sum_{p\in\Lambda_+^*}\sum_{r\in P_H,\, v\in P_L}\Gamma_p\eta_r\big[\,(a^*_{r+v}a_p b^*_{-r}+b^*_{r+v}a^*_{-r}a_p)(\gamma_v b_v+\sigma_v b^*_{-v})b_{-p}\\
&\qquad\qquad\qquad\qquad\qquad\qquad +\sigma_v b^*_{r+v}b^*_{-r}a^*_{-v}a_p b_{-p}-\gamma_v a^*_v a_p b_{-r}b_{r+v}b_{-p}\,\big]
\end{aligned}$$

We show now that for all $j=1,\dots,9$ we have, on $\mathcal{F}_+^{\leq N}$,

$$\pm\,\big(\Upsilon_j+\text{h.c.}\big)\leq\frac{C}{\sqrt{N}}\,(\mathcal{N}_++1)^2 \tag{4.197}$$

We observe that

$$\begin{aligned}
|\langle\xi,\Upsilon_1\xi\rangle|&\leq\frac{1}{\sqrt{N}}\sum_{r\in P_H,\, v\in P_L}|\Gamma_{r+v}||\eta_r|\|a_{-r}\xi\|\|a_{-r-v}(\gamma_v b_v+\sigma_v b_v^*)\xi\|\\
&\leq\frac{C}{\sqrt{N}}\Big(\sum_{r\in P_H,\, v\in P_L}|\Gamma_{r+v}|^2\|a_{-r}\xi\|^2\Big)^{1/2}\\
&\qquad\qquad\times\Big(\sum_{r\in P_H,\, v\in P_L}|\eta_r|^2\left[\|b_v(\mathcal{N}_++1)^{1/2}\xi\|^2+|\sigma_v|^2\|(\mathcal{N}_++1)\xi\|^2\right]\Big)^{1/2}\\
&\leq\frac{C}{\sqrt{N}}\,\|(\mathcal{N}_++1)\xi\|^2,
\end{aligned}$$

and similar bounds hold for Υ_j with j=2,3,4. Next, we bound

$$\begin{aligned}
|\langle\xi,\Upsilon_5\xi\rangle|&\leq\frac{1}{\sqrt{N}}\sum_{r\in P_H,\, v\in P_L}|\Gamma_v||\eta_r||\sigma_v|\|a_{r+v}\xi\|\|a_v b^*_{-r}\xi\|\\
&\leq\frac{C}{\sqrt{N}}\Big(\sum_{r\in P_H,\, v\in P_L}|\Gamma_v|^2\|a_{r+v}\xi\|^2\Big)^{1/2}\Big(\sum_{r\in P_H,\, v\in P_L}|\eta_r|^2|\sigma_v|^2\|(\mathcal{N}_++1)\xi\|^2\Big)^{1/2}\\
&\leq\frac{C}{\sqrt{N}}\,\|(\mathcal{N}_++1)\xi\|^2\,.
\end{aligned}$$

and similarly for Υ_6. To bound Υ_7 we use

$$\begin{aligned}
|\langle\xi,\Upsilon_7\xi\rangle|&\leq\frac{1}{\sqrt{N}}\sum_{r\in P_H,\, v\in P_L}|\Gamma_v||\eta_r|\|b_{r+v}\,\mathcal{N}_+^{-1/2}b_{-r}b_{-v}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|\\
&\leq\frac{C}{\sqrt{N}}\Big(\sum_{r\in P_H,\, v\in P_L}\|b_{-r}b_{-v}\xi\|^2\Big)^{1/2}\Big(\sum_{r\in P_H,\, v\in P_L}|\mathcal{G}_v|^2|\eta_r|^2\|(\mathcal{N}_++1)\xi\|^2\Big)^{1/2}\\
&\leq\frac{C}{\sqrt{N}}\,\|(\mathcal{N}_++1)\xi\|^2\,.
\end{aligned}$$

We consider now Υ_8. In the first term we move b^*_{-r} to the left:

$$\begin{aligned}
\Upsilon_8^{(1)} &:= -\frac{1}{N^{3/2}} \sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} \Gamma_p \eta_r \, b_p a^*_{r+v} a_{-p} b^*_{-r} (\gamma_v b_v + \sigma_v b^*_{-v}) \\
&= -\frac{1}{N^{3/2}} \sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} \Gamma_p \eta_r \, b_p b^*_{-r} a^*_{r+v} a_{-p} (\gamma_v b_v + \sigma_v b^*_{-v}) \\
&\quad - \frac{1}{N^{3/2}} \sum_{r\in P_H,\, v\in P_L} \Gamma_r \eta_r \, b_r b^*_{r+v} (\gamma_v b_v + \sigma_v b^*_{-v}) = \Upsilon_8^{(1a)} + \Upsilon_8^{(1b)}
\end{aligned}$$

To bound the quintic term we use that $|P_L| \leq CN^{3/2}$ and $\sum_{r\in P_H} |\eta_r|^2 \leq CN^{-1/2}$:

$$\begin{aligned}
|\langle \xi, \Upsilon_8^{(1a)} \xi\rangle| &\leq \frac{1}{N^{3/2}} \sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} |\Gamma_p||\eta_r| \, \|b^*_p a_{r+v}\xi\| \|a_{-p} b^*_{-r}(\gamma_v b_v + \sigma_v b^*_{-v})\xi\| \\
&\leq \frac{1}{N} \Big(\sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} |\mathcal{G}_p|^2 \|(\mathcal{N}_+ + 1)^{1/2} a_{r+v}\xi\|^2 \Big)^{1/2} \\
&\quad \times \Big(\frac{1}{N} \sum_{p\in\Lambda^*_+} \sum_{r\in P_H,\, v\in P_L} |\eta_r|^2 \left[\|a_{-p} b^*_{-r} b_v \xi\|^2 + |\sigma_v|^2 \|a_{-p} b^*_{-r} b^*_{-v}\xi\|^2 \right] \Big)^{1/2} \\
&\leq \frac{C}{\sqrt{N}} \|(\mathcal{N}_+ + 1)\xi\|^2
\end{aligned}$$

where, in the last step, to bound the contribution proportional to $\|a_{-p} b^*_{-r} b_v \xi\|^2$, we first estimated the sum over p by $\|\mathcal{N}_+^{1/2} b^*_{-r} b_v \xi\| \leq \|b_v(\mathcal{N}_+ + 1)\xi\|$ and then we summed over r and v (and similarly for the term proportional to $\|a_{-p} b^*_{-r} b^*_{-v}\xi\|$). As for the cubic term, we have

$$\begin{aligned}
|\langle \xi, \Upsilon_8^{(1b)} \xi\rangle| &\leq \frac{1}{N^{3/2}} \Big(\sum_{r\in P_H,\, v\in P_L} |\Gamma_r|^2 \|b^*_r b_{r+v}\xi\|^2 \Big)^{1/2} \\
&\qquad \times \Big(\sum_{r\in P_H,\, v\in P_L} |\eta_r|^2 \left[\|b_v\xi\|^2 + |\sigma_v|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \right] \Big)^{1/2} \\
&\leq \frac{1}{N^{3/2}} \|(\mathcal{N}_+ + 1)\xi\|^2
\end{aligned}$$

The remaining terms in Υ_8 and Υ_9 can be treated with the same arguments shown above, thus concluding the proof of (4.197). □

4.8.2 Analysis of $e^{-A}\mathcal{C}_N e^A$.

In this section, we analyze the action of the cubic exponential on the cubic term $\mathcal{C}_N$, defined in (4.66).

Proposition 4.8.3. *Let A be defined as in (4.69) and let $\mathcal{C}_N$ be defined as in (4.66). Then, under the assumptions of Proposition 4.3.3, we have*

$$\begin{aligned} e^{-A}\mathcal{C}_N\, e^{A} = \mathcal{C}_N + \frac{2}{N} \sum_{r\in P_H, v\in P_L} &\kappa\big(\widehat{V}(r/N) + \widehat{V}((r+v)/N)\big)\eta_r \\ &\times \Big[\sigma_v^2 + (\gamma_v^2+\sigma_v^2)\, b_v^* b_v + \gamma_v\sigma_v\,\big(b_v b_{-v} + b_v^* b_{-v}^*\big)\Big] \\ + \,\mathcal{E}_N^{(\mathcal{C})}, \end{aligned} \tag{4.198}$$

where the error $\mathcal{E}_N^{(\mathcal{C})}$ satisfies

$$\pm\mathcal{E}_N^{(\mathcal{C})} \leq \frac{C}{\sqrt{N}}\left[(\mathcal{N}_+ + 1)(\mathcal{H}_N+1) + (\mathcal{N}_+ + 1)^3\right]. \tag{4.199}$$

To prove the proposition we use the following lemma.

Lemma 4.8.4. *Let A be defined as in (4.69) and $\mathcal{C}_N$ be defined as in (4.62). Then, under the assumptions of Proposition 4.3.3,*

$$[\mathcal{C}_N, A] = \sum_{j=0}^{14} \Xi_j + \text{h.c.} \tag{4.200}$$

where

$$\Xi_0 = \frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}} \big(\widehat{V}(r/N) + \widehat{V}((r+v)/N)\big)\eta_r\left[\sigma_v^2 + (\gamma_v^2+\sigma_v^2)\, b_v^* b_v + \gamma_v\sigma_v\,\big(b_v b_{-v} + b_v^* b_{-v}^*\big)\right]$$

$$\begin{aligned}\Xi_1 = &\frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}} \widehat{V}((r+v)/N)\eta_r\,(\gamma_v b_v^* + \sigma_v b_{-v})\left[\left(1-\frac{\mathcal{N}_+}{N}\right)^2 - 1\right](\gamma_v b_v + \sigma_v b_{-v}^*) \\ &+ \frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}} \widehat{V}(r/N)\eta_r\,(\gamma_v b_v^* + \sigma_v b_{-v})\left[\left(1-\frac{\mathcal{N}_+ + 1}{N}\right)\left(1-\frac{\mathcal{N}_+}{N}\right) - 1\right](\gamma_v b_v + \sigma_v b_{-v}^*)\end{aligned}$$

$$\Xi_2 = \frac{\kappa}{N}\sum_{r\in P_H,\, v\in P_L} \widehat{V}(r/N)\eta_r\,\sigma_v\left(1-\frac{\mathcal{N}_+ + 1}{N}\right)\left(1-\frac{\mathcal{N}_+}{N}\right) b_{r+v}(\gamma_{r+v} b_{-r-v} + \sigma_{r+v} b_{r+v}^*)$$

$$\Xi_3 = \frac{\kappa}{N}\sum_{r\in P_H,\, v\in P_L} \widehat{V}((r+v)/N)\eta_r\,\sigma_v\left(1-\frac{\mathcal{N}_+}{N}\right)^2 b_{-r}(\gamma_r b_r + \sigma_r b_{-r}^*)$$

$$\Xi_4 = -\frac{\kappa}{N^2}\sum_{r\in P_H,\, v\in P_L} \widehat{V}((r+v)/N)\eta_r\,\sigma_v\left(1-\frac{\mathcal{N}_+}{N}\right) b_{-r}(\gamma_r b_r + \sigma_r b_{-r}^*)$$

$$\begin{aligned}\Xi_5 = \frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{p\in\Lambda_+^*\\ p\neq r+v}} &\widehat{V}(p/N)\eta_r(\gamma_v b_v^* + \sigma_v b_{-v})\left(1-\frac{\mathcal{N}_+ + 1}{N}\right)\left(b_{-p}^* b_{-r} - \frac{1}{N}a_{-p}^* a_{-r}\right) \\ &\times(\gamma_{r+v-p} b_{r+v-p} + \sigma_{r+v-p} b_{p-r-v}^*)\end{aligned}$$

and

$$\Xi_6 = \frac{\kappa}{N} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^*\\ p\neq -r}} \widehat{V}(p/N)\eta_r(\gamma_v b_v^* + \sigma_v b_{-v}) \left(1 - \frac{\mathcal{N}_+}{N}\right) (b_{-p}^* b_{r+v} - \frac{1}{N} a_{-p}^* a_{r+v}) \times (\gamma_{r+p} b_{-r-p} + \sigma_{r+p} b_{r+p}^*)$$

$$\Xi_7 = \frac{\kappa}{N} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^*\\ p\neq -v}} \widehat{V}(p/N)\eta_r \left(1 - \frac{\mathcal{N}_+}{N}\right) \sigma_v (b_{-p}^* b_{r+v} - \frac{1}{N} a_{-p}^* a_{r+v}) b_{-r} \times (\gamma_{p+v} b_{-p-v} + \sigma_{p+v} b_{p+v}^*)$$

$$\Xi_8 = -\frac{\kappa}{N} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^*\\ p\neq v}} \widehat{V}(p/N)\eta_r \left(1 - \frac{\mathcal{N}_+ - 2}{N}\right) \gamma_v\, b_{r+v}^* b_{-r}^* b_{-p}^* (\gamma_{p-v} b_{-p+v} + \sigma_{p-v} b_{p-v}^*)$$

$$\Xi_9 = -\frac{\kappa}{N^2} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^*\\ p\neq -v}} \widehat{V}(p/N)\eta_r \left(1 - \frac{\mathcal{N}_+}{N}\right) \sigma_v\, a_{-p}^* a_{-r} b_{r+v} \, (\gamma_{p+v} b_{-p-v} + \sigma_{p+v} b_{p+v}^*)$$

$$\Xi_{10} = \frac{\kappa}{N^2} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{p,q\in\Lambda_+^*\\ p\neq -q}} \widehat{V}(p/N)\eta_r(\gamma_v b_v^* + \sigma_v b_{-v})(a_{p+q}^* a_{r+v} b_{-r} + b_{r+v} a_{p+q}^* a_{-r}) b_{-p}^* \times (\gamma_q b_q + \sigma_q b_{-q}^*)$$

$$\Xi_{11} = \frac{\kappa}{N^2} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{p,q\in\Lambda_+^*\\ p\neq -q}} \widehat{V}(p/N)\eta_r \big(\gamma_v b_{r+v}^* b_{-r}^* a_{p+q}^* a_v + \sigma_v a_{p+q}^* a_{-v} b_{-r} b_{r+v}\big) b_{-p}^* \times (\gamma_q b_q + \sigma_q b_{-q}^*)$$

and

$$\begin{aligned}
\Xi_{12} &= \frac{\kappa}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*\\ p\neq -q}} \widehat{V}(p/N)\, b_{p+q}^*\, [b_{-p}^*, A]\, \big(\gamma_q b_q + \sigma_q b_{-q}^*\big) \\
\Xi_{13} &= \frac{\kappa}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*\\ p\neq -q}} \widehat{V}(p/N)\, \gamma_q\, b_{p+q}^*\, b_{-p}^*\, [b_q, A] \\
\Xi_{14} &= \frac{\kappa}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*\\ p\neq -q}} \widehat{V}(p/N)\, \sigma_q\, b_{p+q}^*\, b_{-p}^*\, [b_{-q}^*, A]
\end{aligned} \tag{4.201}$$

For all $j = 1, \ldots, 14$ (but not for $j = 0$) we have

$$\pm\, \big(\Xi_j + h.c.\big) \leq \frac{C}{\sqrt{N}} \Big[(\mathcal{N}_+ + 1)(\mathcal{K} + 1) + (\mathcal{N}_+ + 1)^3\Big]. \tag{4.202}$$

Moreover,

$$\pm\, [\Xi_0, A] \leq \frac{C}{\sqrt{N}} \big(\mathcal{N}_+ + 1\big)^2. \tag{4.203}$$

Proof of Prop. 4.8.3. We write

$$e^{-A}\mathcal{C}_N\,e^{A} = \mathcal{C}_N + \int_0^1 ds\, e^{-sA}\,[\mathcal{C}_N, A]e^{sA}\,. \tag{4.204}$$

We set $\tilde{\mathcal{E}}_N^{(\mathcal{C})} := [\mathcal{C}_N, A] - 2\Xi_0 = \sum_{j=1}^{14}(\Xi_j + \text{h.c.})$ and rewrite (4.204) as $e^{-A}\mathcal{C}_N\,e^{A} = \mathcal{C}_N + 2\Xi_0 + \mathcal{E}_N^{(\mathcal{C})}$, with

$$\mathcal{E}_N^{(\mathcal{C})} = \int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{-s_2 A}\,\big[2\Xi_0, A\big]e^{s_2 A} + \int_0^1 ds\, e^{-sA}\,\tilde{\mathcal{E}}_N^{(\mathcal{C})} e^{sA}\,.$$

Lemma 4.8.4 together with Proposition 4.4.2 and Proposition 4.4.4 imply (4.199); with the definition of Ξ_0 we obtain (4.198).

□

Proof of Lemma 4.8.4. We have

$$[\mathcal{C}_N, A] = \frac{\kappa}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*\\ p\neq -q}} \widehat{V}(p/N)\,[b^*_{p+q}, A]\,b^*_{-p}\big(\gamma_q b_q + \sigma_q b^*_{-q}\big) + \sum_{j=12}^{14}\Xi_j + \text{h.c.} \tag{4.205}$$

We use the formula (4.196) to compute the first term on the r.h.s. of (4.205). Putting in normal order the quartic terms (but leaving unchanged the parenthesis $(\gamma_v b_v^* + \sigma_v b_{-v})$ and its conjugate) we obtain

$$\begin{aligned}
&\frac{\kappa}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*\\ p\neq -q}} \widehat{V}(p/N)\,[b^*_{p+q}, A]\,b^*_{-p}\big(\gamma_q b_q + \sigma_q b^*_{-q}\big)\\
&= \frac{\kappa}{N} \sum_{\substack{r\in P_H\\ v\in P_L}} \widehat{V}(r/N)\eta_r\,(\gamma_v b_v^* + \sigma_v b_{-v})\left(1 - \frac{\mathcal{N}_+ + 1}{N}\right)\left(1 - \frac{\mathcal{N}_+}{N}\right)(\gamma_v b_v + \sigma_v b^*_{-v})\\
&\quad + \frac{\kappa}{N} \sum_{\substack{r\in P_H\\ v\in P_L}} \widehat{V}((r+v)/N)\eta_r\,(\gamma_v b_v^* + \sigma_v b_{-v})\left(1 - \frac{\mathcal{N}_+}{N}\right)^2(\gamma_v b_v + \sigma_v b^*_{-v}) + \sum_{j=2}^{11}\Xi_j
\end{aligned}$$

The first two terms on the r.h.s. of the last equation can be further decomposed as $\Xi_0 + \Xi_1$. Combining (4.205) with the last equation we obtain the decomposition (4.200).

Next, we prove the bound (4.202). With $\sum_{r\in\Lambda_+^*}\widehat{V}(r/N)\eta_r \le CN$ and $\sum_{v\in P_L}|\sigma_v|^2 \le C$, we obtain that $\pm\Xi_1 \le CN^{-1}(\mathcal{N}_+ + 1)^2$. As for Ξ_2, we find

$$\begin{aligned}
|\langle \xi, \Xi_2\xi\rangle| &\le \frac{C}{N}\Big(\sum_{\substack{r\in P_H\\ v\in P_L}} |\eta_r|^2|\sigma_v|\|\mathcal{N}_+^{1/2}\xi\|^2\Big)^{1/2}\\
&\qquad\times\Big(\sum_{\substack{r\in P_H\\ v\in P_L}} |\sigma_v|\,\Big[\|b_{-r-v}\xi\|^2 + |\sigma_{r+v}|^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\Big]\Big)^{1/2}\\
&\le \frac{C}{\sqrt{N}}\,\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2
\end{aligned}$$

using $\sum_{v\in P_L}|\sigma_v|\le CN^{1/2}$. The terms Ξ_3 and Ξ_4 can be bounded analogously. As for the term $\Xi_5=\Xi_5^{(1)}+\Xi_5^{(2)}$ we use that $|P_L|\le CN^{3/2}$ and $\sum_{r\in P_H}|\eta_r|^2\le CN^{-1/2}$, hence

$$\begin{aligned}
&|\langle\xi,\Xi_5^{(1)}\xi\rangle|\\
&\le\frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{p\in\Lambda_+^*\\ p\neq r+v}}|\widehat{V}(p/N)||\eta_r|\|b_vb_{-p}\xi\|\|b_{-r}(\gamma_{r+v-p}b_{r+v-p}+\sigma_{r+v-p}b^*_{p-r-v})\xi\|\\
&\le\frac{\kappa}{N}\Bigg(\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{p\in\Lambda_+^*\\ p\neq r+v}}|\eta_r|^2\|b_vb_{-p}\xi\|^2\Bigg)^{1/2}\\
&\times\Bigg[\Bigg(\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{p\in\Lambda_+^*\\ p\neq r+v}}\|b_{-r}b_{r+v-p}\xi\|^2\Bigg)^{1/2}+\Bigg(\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{p\in\Lambda_+^*\\ p\neq r+v}}|\sigma_{r+v-p}|^2\|b_{-r}b^*_{p-r-v}\xi\|^2\Bigg)^{1/2}\Bigg]\\
&\le\frac{C}{\sqrt{N}}\|(\mathcal{N}_++1)\xi\|^2
\end{aligned}$$

In the last step, to bound the term proportional to $\|b_{-r}b^*_{p-r-v}\xi\|$, we first shifted $p\to p+r+v$, then we estimated the sum over r by $\|\mathcal{N}_+^{1/2}b^*_p\xi\|\le\|(\mathcal{N}_++1)\xi\|$ and at the end we summed over v and p (using the factor $|\sigma_p|^2$). The bound for $\Xi_5^{(2)}$ and for the terms Ξ_j with $j=6,7,8,9$ can be obtained similarly. As for the term Ξ_{10} we first move the operator b_p^* to the left, obtaining

$$\begin{aligned}
\Xi_{10}=\;&\frac{\kappa}{N^2}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq -q}}\widehat{V}(p/N)\eta_r\,(\gamma_vb^*_v+\sigma_vb_{-v})\,a^*_{p+q}a_{r+v}\big(b^*_{-p}b_{-r}-\frac{1}{N}a^*_{-p}a_{-r}\big)\\
&\qquad\qquad\times(\gamma_qb_q+\sigma_qb^*_{-q})\\
&+\frac{\kappa}{N^2}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{p,q\in\Lambda_+^*\\ p\neq -q}}\widehat{V}(p/N)\eta_r\,(\gamma_vb^*_v+\sigma_vb_{-v})\,b_{r+v}b^*_{-p}a^*_{p+q}a_{-r}(\gamma_qb_q+\sigma_qb^*_{-q})\\
&+\frac{\kappa}{N^2}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{q\in\Lambda_+^*\\ q\neq -r}}\widehat{V}(r/N)\eta_r(\gamma_vb^*_v+\sigma_vb_{-v})\,a^*_{r+q}a_{r+v}(\gamma_qb_q+\sigma_qb^*_{-q})\\
&+\frac{\kappa}{N^2}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{q\in\Lambda_+^*\\ q\neq -r}}\widehat{V}(r/N)\eta_r(\gamma_vb^*_v+\sigma_vb_{-v})\,b_{r+v}b^*_{r+q}(\gamma_qb_q+\sigma_qb^*_{-q})\\
=\;&\sum_{j=1}^4\Xi_{10}^{(j)}
\end{aligned}\tag{4.206}$$

To bound $\Xi_{10}^{(1)}$, we commute the operator a_{r+v} to the right of b^*_{-p}. we find

$$
\begin{aligned}
|\langle \xi, \Xi_{10}^{(1)} \xi\rangle| \leq \; & \frac{C}{N^2} \sum_{\substack{p,q\in\Lambda_+^* \\ p\neq -q}} \sum_{\substack{r\in P_H \\ v\in P_L}} |\widehat{V}(p/N)||\eta_r| \\
& \qquad\qquad \times \|b_{-p}a_{p+q}(\gamma_v b_v + \sigma_v b^*_{-v})\xi\| \|a_{r+v}b_{-r}(\gamma_q b_q + \sigma_q b^*_{-q})\xi\| \\
& + \frac{C}{N^2} \sum_{\substack{p,q\in\Lambda_+^* \\ p\neq -q}} \sum_{\substack{v\in P_L \\ p+v\in P_H}} |\widehat{V}(p/N)||\eta_{p+v}| \\
& \qquad\qquad \times \|b_{p+q}(\gamma_v b_v + \sigma_v b^*_{-v})\xi\| \|b_{p+v}(\gamma_q b_q + \sigma_q b^*_{-q})\xi\|
\end{aligned}
\tag{4.207}
$$

By Cauchy-Schwarz and using the bound $\sum_{p\in\Lambda_+^*} |\widehat{V}(p/N)|^2 \leq CN^3$, we obtain

$$
\begin{aligned}
& |\langle \xi, \Xi_{10}^{(1)} \xi\rangle| \\
& \leq \frac{C}{N^2} \Big(\sum_{\substack{p,q\in\Lambda_+^* \\ p\neq -q}} \sum_{\substack{r\in P_H \\ v\in P_L}} |\eta_r|^2 \left[\|a_{-p}a_{p+q}a_v\xi\|^2 + |\sigma_v|^2 \|a_{-p}a_{p+q}a^*_{-v}\xi\|^2 \right] \Big)^{1/2} \\
& \qquad \times \Big(\sum_{\substack{p,q\in\Lambda_+^* \\ p\neq -q}} \sum_{\substack{r\in P_H \\ v\in P_L}} |\widehat{V}(p/N)|^2 \left[\|a_{r+v}a_{-r}a_q\xi\| + |\sigma_q|^2 \|a_{r+v}a_{-r}a^*_q\xi\|^2 \right] \Big)^{1/2} \\
& \quad + \frac{C}{N^2} \Big(\sum_{\substack{p,q\in\Lambda_+^* \\ p\neq -q}} \sum_{v\in P_L} |\eta_{p+v}|^2 \left[\|a_{p+q}b_v\xi\|^2 + |\sigma_v|^2 \|a_{p+q}b^*_{-v}\xi\|^2 \right] \Big)^{1/2} \\
& \qquad\quad \times \Big(\sum_{\substack{p,q\in\Lambda_+^* \\ p\neq -q}} \sum_{v\in P_L} \left[\|a_{p+v}b_q\xi\|^2 + |\sigma_q|^2 \|a_{p+v}b^*_{-q}\xi\|^2 \right] \Big)^{1/2} \\
& \leq \frac{C}{\sqrt{N}} \langle \xi, (\mathcal{N}_+ + 1)^3 \xi\rangle
\end{aligned}
\tag{4.208}
$$

The bound for $\Xi_{10}^{(2)}$ is similar. As for the quartic operators $\Xi_{10}^{(3)}$ and $\Xi_{10}^{(4)}$, they can be handled like the second term on the r.h.s. of (4.207) (in $\Xi_{10}^{(4)}$ we first commute b_{r+v} and b^*_{r+q}). We obtain that

$$
\pm \Xi_{10} \leq \frac{C}{\sqrt{N}} (\mathcal{N}_+ + 1)^3
$$

The operator Ξ_{11} can be controlled similarly as Ξ_{10}. To estimate Ξ_{12}, Ξ_{13} and Ξ_{14}, we insert (4.196) into (4.201); this produces several terms. The contributions arising from Ξ_{12} are similar to the terms $\Xi_1, \dots, \Xi_{11}$ considered above and their expectation can be estimated analogously. On the other hand, to bound some of the contributions to Ξ_{13} and Ξ_{14} we need to use the kinetic energy operator. To explain this step, let us compute Ξ_{13} explicitly. We find $\Xi_{13} = \sum_{j=1}^{6} \Xi_{13}^{(j)}$ with

$$\begin{aligned}
\Xi_{13}^{(1)} &= \frac{\kappa}{N} \sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^* \\ p\neq -r-v}} \widehat{V}(p/N)\, \eta_r\, \gamma_{r+v} b^*_{p+r+v}\, b^*_{-p} \Big(1 - \frac{\mathcal{N}_+}{N}\Big) b^*_{-r}(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Xi_{13}^{(2)} &= \frac{\kappa}{N} \sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^* \\ p\neq r}} \widehat{V}(p/N)\, \eta_r\, \gamma_r b^*_{p-r}\, b^*_{-p} \Big(1 - \frac{\mathcal{N}_+ - 1}{N}\Big) b^*_{r+v}(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Xi_{13}^{(3)} &= \frac{\kappa}{N} \sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^* \\ p\neq v}} \widehat{V}(p/N)\, \eta_r\, \gamma_v\sigma_v\, b^*_{p-v}\, b^*_{-p}\, b^*_{r+v} b^*_{-r} \Big(1 - \frac{\mathcal{N}_+}{N}\Big) \\
\Xi_{13}^{(4)} &= -\frac{\kappa}{N} \sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^* \\ p\neq -v}} \widehat{V}(p/N)\, \eta_r\, \gamma_v^2\, b^*_{p+v}\, b^*_{-p} \Big(1 - \frac{\mathcal{N}_+}{N}\Big) b_{-r} b_{r+v} \\
\Xi_{13}^{(5)} &= -\frac{\kappa}{N^2} \sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{p,q\in\Lambda_+^* \\ p\neq -q}} \widehat{V}(p/N)\, \eta_r\, \gamma_q b^*_{p+q}\, b^*_{-p}\, (a^*_{r+v} a_q b^*_{-r} + b^*_{r+v} a^*_{-r} a_q)(\gamma_v b_v + \sigma_v b^*_{-v}) \\
\Xi_{13}^{(6)} &= \frac{\kappa}{N^2} \sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{p,q\in\Lambda_+^* \\ p\neq -q}} \widehat{V}(p/N)\, \eta_r\, \gamma_q b^*_{p+q}\, b^*_{-p} (\gamma_v a^*_v a_q b_{-r} b_{r+v} - \sigma_v b^*_{r+v} b^*_{-r} a^*_{-v} a_q)\,.
\end{aligned}$$

To bound $\Xi_{13}^{(1)}$ we use Cauchy-Schwarz. We find (with appropriate shifts of the summation variables)

$$\begin{aligned}
&|\langle \xi, \Xi_{13}^{(1)} \xi\rangle| \\
&\leq \frac{C}{N} \Big(\sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^* \\ p\neq r+v}} |p|^2\, \|b_{-r} b_{-p} b_{p+r+v}\, (\mathcal{N}_+ + 1)^{-1/2} \xi\|^2 \Big)^{1/2} \\
&\qquad \times \Big(\sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{p\in\Lambda_+^* \\ p\neq r+v}} \frac{|\widehat{V}(p/N)|^2}{|p|^2}\, |\eta_r|^2 \Big[\|b_v\, (\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + |\sigma_v|^2 \|(\mathcal{N}_+ + 1)\xi\|^2 \Big] \Big)^{1/2} \\
&\leq \frac{C}{\sqrt{N}} \Big[\|(\mathcal{N}_+ + 1)^{1/2} (\mathcal{K} + 1)^{1/2} \xi\|^2 + \|(\mathcal{N}_+ + 1)\xi\|^2 \Big]\,.
\end{aligned}$$

The bounds for $\Xi_{13}^{(j)}$ with $j = 2, 3, 4$ can be obtained similarly. As for the terms $\Xi_{13}^{(5)}$ and $\Xi_{13}^{(6)}$, they can be estimated proceeding as we did for Ξ_{10}. We conclude that

$$\pm\, \Xi_{13} \leq \frac{C}{\sqrt{N}} (\mathcal{N}_+ + 1)(\mathcal{K} + 1) + (\mathcal{N}_+ + 1)^3 \tag{4.209}$$

Also the term Ξ_{14} can be controlled analogously. To avoid repetitions, we skip the details.

To conclude the proof of the lemma it remains to show (4.203), which follows from Lemma 4.8.2 since, for any $v \in P_L$, we have

$$\Big| \frac{\kappa}{N} \sum_{r \in P_H} \big(\widehat{V}(r/N) + \widehat{V}((r+v)/N)\big) \eta_r \, (\gamma_v^2 + \sigma_v^2) \Big| \leq C,$$
$$\Big| \frac{\kappa}{N} \sum_{r \in P_H} \big(\widehat{V}(r/N) + \widehat{V}((r+v)/N)\big) \eta_r \, \gamma_v \sigma_v \Big| \leq \frac{C}{v^2}$$

□

4.8.3 Analysis of $e^{-A}\mathcal{H}_N e^A$.

In this section, we analyse the action of the cubic exponential on $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$.

Proposition 4.8.5. *Let A be defined as in* (4.69) *and $\mathcal{H}_N$ as defined after (4.59). Then, under the assumptions of Proposition 4.3.3, we have*

$$\begin{aligned} e^{-A}\mathcal{H}_N \, e^A = \mathcal{H}_N &- \frac{1}{\sqrt{N}} \sum_{\substack{r \in P_H \\ v \in P_L}} \kappa \widehat{V}(r/N) \Big[b^*_{r+v} b^*_{-r} \big(\gamma_v b_v + \sigma_v b^*_{-v} \big) + \text{h.c.} \Big] \\ &- \frac{1}{N} \sum_{\substack{r \in P_H \\ v \in P_L}} \kappa \big(\widehat{V}(r/N) + \widehat{V}((r+v)/N) \big) \eta_r \\ &\qquad \times \Big[\sigma_v^2 + (\gamma_v^2 + \sigma_v^2) \, b_v^* b_v + \gamma_v \sigma_v \, \big(b_v b_{-v} + b_v^* b_{-v}^* \big) \Big] \\ &+ \mathcal{E}_N^{(\mathcal{H})} , \end{aligned}$$

where the error $\mathcal{E}_N^{(\mathcal{H})}$ satisfies

$$\pm \mathcal{E}_N^{(\mathcal{H})} \leq C N^{-1/4} \Big[(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1) + (\mathcal{N}_+ + 1)^3 \Big] .$$

To show the proposition we use the following lemma.

Lemma 4.8.6. *Let A be defined as in* (4.69) *and let*

$$\Theta_0 = - \frac{1}{\sqrt{N}} \sum_{r \in P_H, \, v \in P_L} \kappa \widehat{V}(r/N) \, b^*_{r+v} b^*_{-r} \big(\gamma_v b_v + \sigma_v b^*_{-v} \big)$$

as defined in Lemma 4.4.3. Then, under the assumptions of Proposition 4.3.3,

$$[\Theta_0 + \Theta_0^*, A] = \sum_{j=0}^{12} \Pi_j + \text{h.c.}$$

with

$$\Pi_0 = -\frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}} \big(\,\widehat{V}(r/N)+\widehat{V}((r+v)/N)\,\big)\eta_r\big[\,\sigma_v^2+(\gamma_v^2+\sigma_v^2)b_v^*b_v+\gamma_v\sigma_v(b_vb_{-v}+b_v^*b_{-v}^*)\,\big]$$

$$\begin{aligned}
\Pi_1 = &-\frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}} \widehat{V}((r+v)/N)\eta_r(\gamma_vb_v^*+\sigma_vb_{-v})\left[\left(1-\frac{\mathcal{N}_+}{N}\right)^2-1\right](\gamma_vb_v+\sigma_vb_{-v}^*)\\
&-\frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}} \widehat{V}(r/N)\eta_r(\gamma_vb_v^*+\sigma_vb_{-v})\left[\left(1-\frac{\mathcal{N}_++1}{N}\right)\left(1-\frac{\mathcal{N}_+}{N}\right)-1\right]\\
&\qquad\times(\gamma_vb_v+\sigma_vb_{-v}^*)
\end{aligned}$$

$$\Pi_2 = -\frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}} \widehat{V}(r/N)\eta_r\sigma_v\left(1-\frac{\mathcal{N}_++1}{N}\right)\left(1-\frac{\mathcal{N}_+}{N}\right)b_{r+v}(\gamma_{r+v}b_{-r-v}+\sigma_{r+v}b_{r+v}^*)$$

$$\begin{aligned}
\Pi_3 = &-\frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{w\in P_L\\ -w+r+v\in P_H}} \widehat{V}((r+v-w)/N)\eta_r(\gamma_vb_v^*+\sigma_vb_{-v})\left(1-\frac{\mathcal{N}_++1}{N}\right)\\
&\qquad\times\big(b_{w-r-v}^*b_{-r}-\frac{1}{N}a_{w-r-v}^*a_{-r}\big)(\gamma_wb_w+\sigma_wb_{-w}^*)
\end{aligned}$$

$$\begin{aligned}
\Pi_4 = &-\frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{w\in P_L\\ w+r\in P_H}} \widehat{V}((r+w)/N)\eta_r(\gamma_vb_v^*+\sigma_vb_{-v})\left(1-\frac{\mathcal{N}_+}{N}\right)\\
&\qquad\times\big(b_{r+w}^*b_{r+v}-\frac{1}{N}a_{r+w}^*a_{r+v}\big)(\gamma_wb_w+\sigma_wb_{-w}^*)
\end{aligned}$$

$$\begin{aligned}
\Pi_5 = &-\frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{w\in P_L\\ w+v\in P_H}} \widehat{V}((v+w)/N)\eta_r\sigma_v\left(1-\frac{\mathcal{N}_+}{N}\right)\\
&\qquad\times\big(b_{w+v}^*b_{r+v}-\frac{1}{N}a_{w+v}^*a_{r+v}\big)b_{-r}(\gamma_wb_w+\sigma_wb_{-w}^*)
\end{aligned}$$

$$\Pi_6 = \frac{\kappa}{N^2}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{w\in P_L\\ w+v\in P_H}} \widehat{V}((v+w)/N)\eta_r\sigma_v\left(1-\frac{\mathcal{N}_+}{N}\right)a_{v+w}^*a_{-r}b_{r+v}(\gamma_wb_w+\sigma_wb_{-w}^*)$$

$$\Pi_7 = \frac{\kappa}{N}\sum_{\substack{r\in P_H\\ v\in P_L}}\sum_{\substack{w\in P_L\\ v-w\in P_H}} \widehat{V}((v-w)/N)\eta_r\gamma_vb_{r+v}^*b_{-r}^*\left(1-\frac{\mathcal{N}_+}{N}\right)b_{w-v}^*(\gamma_wb_w+\sigma_wb_{-w}^*)$$

$$\Pi_8 = \frac{\kappa}{N^2}\sum_{\substack{r,s\in P_H\\ v,w\in P_L}} \widehat{V}(s/N)\eta_r(\gamma_vb_v^*+\sigma_vb_{-v})(b_{-r}a_{s+w}^*a_{r+v}+a_{s+w}^*a_{-r}b_{r+v})b_{-s}^*(\gamma_wb_w+\sigma_wb_{-w}^*)$$

$$\Pi_9 = \frac{\kappa}{N^2}\sum_{\substack{r,s\in P_H\\ v,w\in P_L}} \widehat{V}(s/N)\eta_r(\sigma_va_{s+w}^*a_{-v}b_{-r}b_{r+v}-\gamma_vb_{r+v}^*b_{-r}^*a_{s+w}^*a_v\,\big]\,b_{-s}^*\,(\gamma_wb_w+\sigma_wb_{-w}^*)$$

and

$$\begin{aligned}
\Pi_{10} &= -\frac{\kappa}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \widehat{V}(r/N) b^*_{r+v}[b^*_{-r}, A]\big(\gamma_v b_v + \sigma_v b^*_{-v}\big) \\
\Pi_{11} &= -\frac{\kappa}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \widehat{V}(r/N)\gamma_v\, b^*_{r+v} b^*_{-r}\, [b_v, A] \\
\Pi_{12} &= -\frac{\kappa}{\sqrt{N}} \sum_{r\in P_H,\, v\in P_L} \widehat{V}(r/N) b^*_{r+v} b^*_{-r}\sigma_v\, [b^*_{-v}, A]
\end{aligned} \tag{4.210}$$

For all $j = 1,\dots,12$ (but not for $j = 0$) we have

$$\pm\big(\Pi_j + h.c.\big) \le \frac{C}{\sqrt{N}}\Big[(\mathcal{N}_+ + 1)(\mathcal{K} + 1) + (\mathcal{N}_+ + 1)^3\Big]. \tag{4.211}$$

Proof of Prop. 4.8.5. To show the proposition we write

$$e^{-A}\mathcal{H}_N\, e^{A} = \mathcal{H}_N + \int_0^1 ds\, e^{-sA}\, [\mathcal{H}_N, A] e^{sA} \tag{4.212}$$

From Lemma 4.4.3 we know that

$$[\mathcal{H}_N, A] = \Theta_0 + \Theta_0^* + \mathcal{E}^{(\mathcal{H})}_{N,1}$$

where

$$\pm\, \mathcal{E}^{(\mathcal{H})}_{N,1} \le C N^{-1/4}\Big[(\mathcal{N}_+ + 1)(\mathcal{K} + 1) + (\mathcal{N}_+ + 1)^3\Big]. \tag{4.213}$$

Hence, (4.212) implies that

$$\begin{aligned}
&e^{-A}\mathcal{H}_N\, e^{A} \\
&\quad = \mathcal{H}_N + \Theta_0 + \Theta_0^* \\
&\qquad + \int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{-s_2 A}\, \big[(\Theta_0 + \Theta_0^*), A\big] e^{s_2 A} + \int_0^1 ds\, e^{-sA}\, \mathcal{E}^{(\mathcal{H})}_{N,1}\, e^{sA}.
\end{aligned}$$

Using Lemma 4.8.6 and setting $\mathcal{E}^{(\mathcal{H})}_{N,2} = \sum_{j=1}^{12} \Pi_j + \text{h.c.}$ we finally obtain

$$\begin{aligned}
e^{-A}\mathcal{H}_N\, e^{A} = \mathcal{H}_N &+ \Theta_0 + \Theta_0^* + \Pi_0 \\
&+ \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3\, e^{-s_3 A}\, [2\Pi_0, A] e^{s_3 A} \\
&+ \int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{-s_2 A}\, \mathcal{E}^{(\mathcal{H})}_{N,2} e^{s_2 A} + \int_0^1 ds e^{-sA}\, \mathcal{E}^{(\mathcal{H})}_{N,1}\, e^{sA}.
\end{aligned}$$

Proposition 4.8.5 now follows combining (4.213) with the estimates (4.211) and with the observation that $\Pi_0 = -\Xi_0$, where Ξ_0 is defined in Lemma 4.8.4 and satisfies the bound (4.203). $\square$

Proof of Lemma 4.8.6. We write

$$[\Theta_0, A] = -\frac{1}{\sqrt{N}} \sum_{\substack{r\in P_H,\, v\in P_L \\ r+v\neq 0}} \kappa \widehat{V}(r/N)[b^*_{r+v}, A] b^*_{-r}\big(\gamma_v b_v + \sigma_v b^*_{-v}\big) + \sum_{j=10}^{12} \Pi_j$$

Using (4.196) and normal ordering the quartic terms (with the exception of the factor $(\gamma_v b_v + \sigma_v b^*_{-v})$) we obtain that

$$-\frac{1}{\sqrt{N}} \sum_{\substack{r\in P_H,\, v\in P_L \\ r+v\neq 0}} \kappa \widehat{V}(r/N)[b^*_{r+v}, A] b^*_{-r}\big(\gamma_v b_v + \sigma_v b^*_{-v}\big) = \sum_{j=0}^{9} \Pi_j$$

We now show (4.211). The bound for Π_1 follows from

$$\begin{aligned} |\langle \xi, \Pi_1 \xi\rangle| &\leq \frac{C}{N^2} \sum_{r\in P_H, v\in P_L} \frac{|\widehat{V}(r/N)|}{r^2} \Big[\|b_v(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + |\sigma_v|^2 \|(\mathcal{N}_+ + 1)\xi\|^2 \Big] \\ &\leq \frac{C}{N} \|(\mathcal{N}_+ + 1)\xi\|^2 \end{aligned}$$

To bound Π_j with j=2,3,4,7 one uses that $|P_L| \leq CN^{3/2}$ and $\sum_{r\in P_H} |\eta_r|^2 \leq CN^{-1/2}$. Hence

$$\begin{aligned} |\langle \xi, \Pi_2 \xi\rangle| &\leq \frac{k}{N} \sum_{\substack{r\in P_H \\ v\in P_L}} |\widehat{V}(r/N)||\eta_r||\sigma_v| \|b^*_{r+v}\xi\| \|(\gamma_{r+v} b_{-r-v} + \sigma_{r+v} b^*_{r+v})\xi\| \\ &\leq \frac{k}{N} \Big(\sum_{\substack{r\in P_H \\ v\in P_L}} |\eta_r|^2 |\sigma_v|^2 \|b^*_{r+v}\xi\|^2 \Big)^{1/2} \Big(\sum_{\substack{r\in P_H \\ v\in P_L}} \|(\gamma_{r+v} b_{-r-v} + \sigma_{r+v} b^*_{r+v})\xi\|^2 \Big)^{1/2} \\ &\leq \frac{C}{\sqrt{N}} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \end{aligned}$$

Similarly, with Cauchy-Schwarz we can bound Π_3 by

$$\begin{aligned} |\langle \xi, \Pi_3 \xi\rangle| \leq \frac{C}{N} &\Big(\sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{w\in P_L: \\ -w+r+v\in P_H}} |\eta_r|^2 \Big[\|b_v b_{w-r-v}\xi\|^2 + |\sigma_v|^2 \|b_{w-r-v}(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \Big] \Big)^{1/2} \\ &\times \Big(\sum_{\substack{r\in P_H \\ v\in P_L}} \sum_{\substack{w\in P_L: \\ -w+r+v\in P_H}} \Big[\|b_{-r} b_w \xi\|^2 + |\sigma_w|^2 \|b_{-r}(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \Big] \Big)^{1/2} \\ \leq \frac{C}{\sqrt{N}} &\|(\mathcal{N}_+ + 1)\xi\|^2 \end{aligned}$$

The terms Π_4 and Π_7 are bounded similarly. It is easy to check that Π_5 and Π_6 satisfy

(4.211). For example, we have

$$\begin{aligned}
|\langle \xi, \Pi_5 \xi\rangle| &\leq \frac{C}{N} \sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{w\in P_L\\ w+v\in P_H}} |\widehat{V}((v+w)/N)||\eta_r||\sigma_v| \, \|b_{w+v}(\mathcal{N}_+ +1)^{1/2}\xi\| \\
&\qquad \times \Big[\|b_{r+v}b_{-r}b_w(\mathcal{N}_+ +1)^{-1/2}\xi\| + |\sigma_w| \|b_{r+v}b_{-r}b^*_{-w}(\mathcal{N}_+ +1)^{-1/2}\xi\| \Big] \\
&\leq \frac{C}{N} \Big(\sum_{\substack{r\in P_H\\ v\in P_L}} \sum_{\substack{w\in P_L\\ w+v\in P_H}} |\eta_r|^2 |\sigma_v|^2 \|b_{w+v}(\mathcal{N}_+ +1)^{1/2}\xi\|^2 \Big)^{1/2} \\
&\qquad \times \Big(\sum_{\substack{r\in P_H,\\ v\in P_L}} \sum_{\substack{w\in P_L\\ w+v\in P_H}} \Big[\|b_{r+v}b_{-r}b_w(\mathcal{N}_+ +1)^{-1/2}\xi\|^2 \\
&\qquad\qquad\qquad\qquad + |\sigma_w| \|b_{r+v}b_{-r}b^*_{-w}(\mathcal{N}_+ +1)^{-1/2}\xi\|^2 \Big] \Big)^{1/2} \\
&\leq \frac{C}{N} \, \|(\mathcal{N}_+ +1)\xi\|^2
\end{aligned}$$

(In the term containing the creation operator b^*_w, we first sum over $\widetilde{v} = v + r$ and over r. This produces a factor $(\mathcal{N}_+ + 1)$ which can be moved through b^*_w. At this point, we can estimate b^*_w by an additional factor $(\mathcal{N}_+ + 1)^{1/2}$; with this procedure, we do not have to compute the commutator between b^*_w and the other annihilation operators). The bound for Π_6 is similar. As for Π_8, we decompose it as

$$\begin{aligned}
\Pi_8 = \frac{1}{N^2} &\sum_{\substack{r,s\in P_H\\ v,w\in P_L}} \kappa \widehat{V}(s/N)\eta_r(\gamma_v b^*_v + \sigma_v b_{-v}) b_{-r} b^*_{-s} a^*_{s+w} a_{r+v} \, (\gamma_w b_w + \sigma_w b^*_{-w}) \\
&+ \frac{1}{N^2} \sum_{\substack{r\in P_H\\ v,w\in P_L}} \kappa \widehat{V}((r+v)/N)\eta_r(\gamma_v b^*_v + \sigma_v b_{-v}) b_{-r} b^*_{-r-v+w} \, (\gamma_w b_w + \sigma_w b^*_{-w}) \\
&+ \frac{1}{N^2} \sum_{\substack{r,s\in P_H\\ v,w\in P_L}} \kappa \widehat{V}(s/N)\eta_r(\gamma_v b^*_v + \sigma_v b_{-v}) a^*_{s+w} a_{-r} b_{r+v} b^*_{-s} \, (\gamma_w b_w + \sigma_w b^*_{-w}) \\
= \Pi_8^{(1)} &+ \Pi_8^{(2)} + \Pi_8^{(3)}
\end{aligned}$$

The term $\Pi_8^{(1)}$ can be bounded commuting first the operator b_{-r} to the right, analogously to the estimates (4.207) and (4.208) for the term $\Xi_{10}^{(1)}$ in the proof of Lemma 4.8.4. Also the terms $\Pi_8^{(2)}$ (which is similar to $\Xi_{10}^{(3)}$ in (4.206)) and $\Pi_8^{(3)}$ can be treated similarly. We conclude that

$$\pm \Pi_8 \leq C N^{-1/2} (\mathcal{N}_+ + 1)^3 \,.$$

The operator Π_9 can be controlled as Π_8.

Finally, to bound the terms Π_{10}, Π_{11} and Π_{12} in (4.210), we can expand them using (4.196). The contributions arising from Π_{10} are similar to the terms $\Pi_1, \dots, \Pi_9$ and can be estimated analogously. On the other hand, the terms arising from Π_{11} and Π_{12} are similar to those arising from Ξ_{13} and Ξ_{14} in the proof of Lemma 4.8.4, and can be handled proceeding as we did to show (4.209), making use of the kinetic energy operator. $\square$

4.8.4 Proof of Theorem 4.3.3

Combining the results of Prop. 4.8.1, Prop. 4.8.3 and Prop. 4.8.5 we conclude that

$$\begin{aligned}
\mathcal{J}_N &= e^{-A}\,\mathcal{G}_N e^{A} \\
&= C_{\mathcal{G}_N} + \mathcal{Q}_{\mathcal{G}_N} + \mathcal{H}_N \\
&\quad + \frac{1}{N}\sum_{r\in P_H, v\in P_L} \kappa\big(\widehat{V}(r/N) + \widehat{V}((r+v)/N)\big)\eta_r \\
&\qquad\qquad \times\Big[\sigma_v^2 + (\gamma_v^2+\sigma_v^2)\, b_v^* b_v + \gamma_v\sigma_v\,\big(b_v b_{-v} + b_v^* b_{-v}^*\big)\Big] \\
&\quad + \mathcal{C}_N - \frac{1}{\sqrt{N}}\sum_{r\in P_H, v\in P_L} \kappa\widehat{V}(r/N)\Big[b_{r+v}^* b_{-r}^*\big(\gamma_v b_v + \sigma_v b_{-v}^*\big) + \text{h.c.}\Big] \\
&\quad + \widetilde{\mathcal{E}}_{\mathcal{J}_N}\,,
\end{aligned} \tag{4.214}$$

with an error operator $\widetilde{\mathcal{E}}_{\mathcal{J}_N}$ satisfying

$$\pm\,\widetilde{\mathcal{E}}_{\mathcal{J}_N} \le CN^{-1/4}\,\Big[(\mathcal{H}_N+1)(\mathcal{N}_+ +1) + (\mathcal{N}_+ +1)^3\Big]$$

We show now that the sum of the cubic terms on the fifth line of (4.214) also contributes to the error term. In fact we have

$$\begin{aligned}
&\mathcal{C}_N - \frac{1}{\sqrt{N}}\sum_{r\in P_H,\, v\in P_L} \kappa\widehat{V}(r/N)\,\Big[b_{r+v}^* b_{-r}^*\big(\gamma_v b_v + \sigma_v b_{-v}^*\big) + \text{h.c.}\Big] \\
&\qquad = \frac{1}{\sqrt{N}}\sum_{\substack{v\in P_H,\, r\in\Lambda_+^* \\ r+v\neq 0}} \kappa\widehat{V}(r/N)\,\Big[b_{r+v}^* b_{-r}^*\big(\gamma_v b_v + \sigma_v b_{-v}^*\big) + \text{h.c.}\Big] \\
&\qquad\quad + \frac{1}{\sqrt{N}}\sum_{\substack{v, r\in P_L \\ r+v\neq 0}} \kappa\widehat{V}(r/N)\,\Big[b_{r+v}^* b_{-r}^*\big(\gamma_v b_v + \sigma_v b_{-v}^*\big) + \text{h.c.}\Big] \\
&\qquad = \mathrm{Z}_1 + \mathrm{Z}_2
\end{aligned}$$

To bound Z_1 we use that $|v|^{-1} \leq N^{-1/2}$ for $v \in P_H$ and $\sum_{v\in P_H} |\sigma_v|^2 \leq CN^{-1/2}$. We find

$$\begin{aligned}
|\langle \xi, \mathrm{Z}_1 \xi\rangle| &\leq \frac{C}{\sqrt{N}} \sum_{\substack{v\in P_H,\, r\in\Lambda_+^*\\ r+v\neq 0}} |\widehat{V}(r/N)| \|b_{r+v} b_{-r}\xi\| \Big[\|b_v \xi\| + |\sigma_v| \|(\mathcal{N}_+ +1)^{1/2}\xi\|\Big] \\
&\leq \frac{C}{\sqrt{N}} \Big(\sum_{\substack{v\in P_H,\, r\in\Lambda_+^*\\ r+v\neq 0}} \frac{r^2}{v^2} \|b_{r+v} b_{-r}\xi\|^2\Big)^{1/2} \Big(\sum_{\substack{v\in P_H,\, r\in\Lambda_+^*\\ r+v\neq 0}} \frac{|\widehat{V}(r/N)|^2}{r^2} v^2 \|b_v\xi\|^2\Big)^{1/2} \\
&\quad + \frac{C}{\sqrt{N}} \Big(\sum_{\substack{v\in P_H,\, r\in\Lambda_+^*\\ r+v\neq 0}} r^2 \|b_{r+v} b_{-r}\xi\|^2\Big)^{1/2} \Big(\sum_{\substack{v\in P_H,\, r\in\Lambda_+^*\\ r+v\neq 0}} \frac{|\widehat{V}(r/N)|^2}{r^2} |\sigma_v|^2\Big)^{1/2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \|(\mathcal{N}_+ +1)^{1/2}\xi\| \\
&\leq \frac{C}{N^{1/4}} \|(\mathcal{K}+1)^{1/2} (\mathcal{N}_+ +1)^{1/2}\xi\|^2
\end{aligned}$$

The term Z_2 is bounded using Cauchy-Schwarz and the estimate $\sum_{r\in P_L} |r|^{-2} \leq CN^{1/2}$. We obtain

$$\begin{aligned}
|\langle \xi, \mathrm{Z}_2\xi\rangle| &\leq \frac{C}{\sqrt{N}} \Big(\sum_{\substack{r,v\in P_L\\ r+v\neq 0}} r^2 \|b_{r+v} b_{-r}\xi\|^2\Big)^{1/2} \Big(\sum_{\substack{r,v\in P_L,\\ r+v\neq 0}} \frac{1}{r^2} \Big[\|b_v\xi\|^2 + |\sigma_v|^2 \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2\Big] \Big)^{1/2} \\
&\leq \frac{C}{N^{1/4}} \|(\mathcal{N}_+ +1)^{1/2} (\mathcal{K}+1)^{1/2}\xi\|^2 .
\end{aligned}$$

Similarly, we can show that, in the term on the third and fourth line in (4.214), the restriction $r \in P_H, v \in P_L$ can be removed producing only a negligible error. We conclude that

$$\begin{aligned}
\mathcal{J}_N = C_{\mathcal{G}_N} &+ \mathcal{Q}_{\mathcal{G}_N} + \mathcal{H}_N \\
&+ \frac{1}{N} \sum_{p,q\in\Lambda_+^*} \kappa \widehat{V}((p+q)/N) \eta_q \Big[\sigma_p^2 + (\gamma_p^2 + \sigma_p^2)\, b_p^* b_p + \gamma_p \sigma_p \left(b_p b_{-p} + b_p^* b_{-p}^*\right)\Big] \\
&+ \frac{1}{N} \sum_{p,q\in\Lambda_+^*} \kappa \widehat{V}(q/N) \eta_q \Big[\sigma_p^2 + (\gamma_p^2 + \sigma_p^2)\, b_p^* b_p + \gamma_p \sigma_p \left(b_p b_{-p} + b_p^* b_{-p}^*\right)\Big] \\
&+ \bar{\mathcal{E}}_{\mathcal{J}_N}
\end{aligned}$$

with

$$\pm\, \bar{\mathcal{E}}_{\mathcal{J}_N} \leq \frac{C}{N^{1/4}} \Big[(\mathcal{H}_N + 1)(\mathcal{N}_+ +1) + (\mathcal{N}_+ +1)^3 \Big].$$

Theorem 4.3.3 now follows from the observation that

$$C_{\mathcal{G}_N} + \frac{1}{N} \sum_{p,q\in\Lambda_+^*} \kappa \widehat{V}((p+q)/N) \eta_q \sigma_p^2 + \frac{1}{N} \sum_{p,q\in\Lambda_+^*} \kappa \widehat{V}(q/N) \eta_q \sigma_p^2 = C_{\mathcal{J}_N} + \mathcal{O}(N^{-1})$$

and that

$$\mathcal{Q}_N + \mathcal{K} + \frac{1}{N} \sum_{p,q \in \Lambda_+^*} \kappa \widehat{V}((p+q)/N) \eta_q \Big[(\gamma_p^2 + \sigma_p^2)\, b_p^* b_p + \gamma_p \sigma_p \left(b_p b_{-p} + b_p^* b_{-p}^* \right) \Big]$$
$$+ \frac{1}{N} \sum_{p,q \in \Lambda_+^*} \kappa \widehat{V}(q/N) \eta_q \Big[(\gamma_p^2 + \sigma_p^2)\, b_p^* b_p + \gamma_p \sigma_p \left(b_p b_{-p} + b_p^* b_{-p}^* \right) \Big] = \widetilde{\mathcal{Q}}_N + \widetilde{\mathcal{E}}_{\mathcal{Q}_N}, \tag{4.215}$$

with

$$\pm\, \widetilde{\mathcal{E}}_{\mathcal{Q}_N} \leq \frac{C}{N} (\mathcal{N}_+ + 1)\,. \tag{4.216}$$

Here we used the fact that the contribution to $\mathcal{Q}_N$ arising from the last term in (4.63) and (4.64) cancels with the last sum on the l.h.s. of (4.215) (it is easy to check that the remainder corresponding to the momentum $q = 0$ satisfies (4.216)).

4.A Condensate Depletion

The goal of this short appendix is to prove the formula (4.9) for the number of orthogonal excitations of the condensate, in the ground state of (4.1).

We start with the observation that

$$\begin{aligned}\langle U_N \psi_N, \mathcal{N}_+ U_N \psi_N \rangle = &\Big\langle \Big[U_N \psi_N - e^{i\omega} e^{B(\eta)} e^{A} e^{B(\tau)} \Omega \Big], \mathcal{N}_+ U_N \psi_N \Big\rangle \\ &+ \Big\langle e^{i\omega} e^{B(\eta)} e^{A} e^{B(\tau)} \Omega, \mathcal{N}_+ \Big[U_N \psi_N - e^{i\omega} e^{B(\eta)} e^{A} e^{B(\tau)} \Omega \Big] \Big\rangle \\ &+ \Big\langle e^{B(\eta)} e^{A} e^{B(\tau)} \Omega, \mathcal{N}_+ e^{B(\eta)} e^{A} e^{B(\tau)} \Omega \Big\rangle \end{aligned} \tag{4.217}$$

From (4.139), Lemma 4.2.1 and Prop. 4.4.1 we conclude that

$$\Big| \langle U_N \psi_N, \mathcal{N}_+ U_N \psi_N \rangle - \Big\langle e^{B(\eta)} e^{A} e^{B(\tau)} \Omega, \mathcal{N}_+ e^{B(\eta)} e^{A} e^{B(\tau)} \Omega \Big\rangle \Big| \leq C N^{-1/8}$$

Proceeding as in Section 4.7.2 and recalling the notation $\gamma_p = \cosh \eta_p$, $\sigma_p = \sinh \eta_p$, we find

$$e^{-B(\eta)} \mathcal{N}_+ e^{B(\eta)} = \sum_{p \in \Lambda_+^*} \big[\, (\gamma_p^2 + \sigma_p^2) b_p^* b_p + \gamma_p \sigma_p (b_p^* b_{-p}^* + b_p b_{-p}) + \sigma_p^2 \,\big] + \widetilde{\mathcal{E}}_1$$

where $\pm\, \widetilde{\mathcal{E}}_1 \leq C N^{-1} (\mathcal{N}_+ + 1)^2$. By Lemma 4.8.2 and Proposition 4.4.2 we have

$$e^{-A} e^{-B(\eta)} \mathcal{N}_+ e^{B(\eta)} e^{A} = \sum_{p \in \Lambda_+^*} \big[\, (\gamma_p^2 + \sigma_p^2) b_p^* b_p + \gamma_p \sigma_p (b_p^* b_{-p}^* + b_p b_{-p}) + \sigma_p^2 \,\big] + \widetilde{\mathcal{E}}_2$$

with $\pm\, \widetilde{\mathcal{E}}_2 \leq C N^{-1/2} (\mathcal{N}_+ + 1)^2$. Conjugating with the generalized Bogoliubov transformation $e^{B(\tau)}$ and taking the vacuum expectation, we obtain

$$\begin{aligned}&\Big\langle e^{B(\eta)} e^{A} e^{B(\tau)} \Omega, \mathcal{N}_+ e^{B(\eta)} e^{A} e^{B(\tau)} \Omega \Big\rangle \\ &\qquad = \sum_{p \in \Lambda_+^*} \big[\sigma_p^2 + (\sigma_p^2 + \gamma_p^2) \sinh^2 \tau_p + 2 \gamma_p \sigma_p \sinh(\tau_p) \cosh(\tau_p) \big] + \mathcal{O}(N^{-1/2})\end{aligned}$$

With (4.107), we find

$$2\sinh^2\tau_p = \frac{F_p}{\sqrt{F_p^2 - G_p^2}} - 1\,, \qquad 2\sinh\tau_p\cosh\tau_p = \frac{-G_p}{\sqrt{F_p^2 - G_p^2}}\,.$$

Using (4.72), we arrive at

$$\begin{aligned}
&\Big\langle e^{B(\eta)}e^{A}e^{B(\tau)}\Omega, \mathcal{N}_+ e^{B(\eta)}e^{A}e^{B(\tau)}\Omega\Big\rangle \\
&\qquad = \sum_{p\in\Lambda_+^*} \frac{p^2 + \kappa\big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N}\big)_p - \sqrt{p^4 + 2p^2\kappa\big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N}\big)_p}}{2\sqrt{p^4 + 2p^2\kappa\big(\widehat{V}(\cdot/N) * \widehat{f}_{\ell,N}\big)_p}} + \mathcal{O}(N^{-1/2})
\end{aligned}$$

with $\widehat{f}_{\ell,N}$ as in (4.46). Proceeding as in the proof of (4.127), we conclude that

$$\Big\langle e^{B(\eta)}e^{A}e^{B(\tau)}\Omega, \mathcal{N}_+ e^{B(\eta)}e^{A}e^{B(\tau)}\Omega\Big\rangle = \sum_{p\in\Lambda_+^*} \frac{p^2 + 8\pi\mathfrak{a}_0 - \sqrt{p^4 + 16\pi\mathfrak{a}_0 p^2}}{2\sqrt{p^4 + 16\pi\mathfrak{a}_0 p^2}} + \mathcal{O}(N^{-1/2})$$

Eq. (4.9) follows by combining (4.217) with the last equation, since

$$1 - \langle\varphi_0, \gamma_N^{(1)}\varphi_0\rangle = N^{-1}\langle U_N\psi_N, \mathcal{N}_+ U_N\psi_N\rangle$$

Chapter 5

Fluctuations of N-Particle Quantum Dynamics around the Nonlinear Schrödinger Equation

In this chapter, we provide the details for the proofs of Theorem 1.7.1 and Theorem 1.7.2 in which we provide an effective norm approximation of the full many-body evolution of a Bose gas interacting through a short-range potential of the form $N^{3\beta-1}V(N^{\beta}.)$, for $\beta \in (0;1)$. Our main result is proved in the paper [19].

The following manuscript is a slightly modified version of [19]. Section 5.1 is a slightly shortened and rephrased version of [15, Section 1], since we already introduced the Fock space setting in which we work and related standard results in Section 1.2. Apart from this, the following sections appear as in the original article [19].

5.1 Main Results

In this chapter, we are concerned with an effective norm approximation of the full many-body evolution of a system of N bosons in $\Lambda = \mathbb{R}^3$, described by the Schrödinger equation

$$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t} \tag{5.1}$$

where

$$\Psi_{N,t} \in L^2_s(\mathbb{R}^{3N})$$

is the wave function and H_N is the Hamilton operator of the system. We will restrict our attention to Hamilton operators of the form

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + \frac{1}{N} \sum_{1\leq i<j\leq N} V_N(x_i - x_j) \tag{5.2}$$

with N-dependent two-body interaction potential

$$V_N(x) = N^{3\beta} V(N^{\beta} x). \tag{5.3}$$

Here $\beta \geq 0$ is a fixed parameter and $V \geq 0$ is a smooth, radially symmetric and compactly supported function on $\mathbb{R}^3$.

For $\beta = 0$, (5.2) is a mean-field Hamiltonian, describing a system of particles experiencing a large number of weak collisions. For $\beta = 1$, on the other hand, (5.2) corresponds to the Gross-Pitaevskii regime, where collisions are rare but strong. Physically, the Gross-Pitaevskii regime is more relevant for the description of trapped Bose-Einstein condensates. The mean-field regime, on the other hand, is more accessible to mathematical analysis. In this paper, we will study the solution of the Schrödinger equation (5.1) for intermediate regimes with $0 < \beta < 1$.

From the point of view of physics, it is interesting to study the solution of (5.1) for initial data approximating ground states of trapped systems; this corresponds to experimental settings where the evolution of an initially trapped Bose gas at very low temperature is observed after switching off the external fields.

It is known since [66, 79] that the ground state of a system of trapped bosons interacting through a two-body potential like the one appearing on the r.h.s. of (5.2) exhibits complete Bose-Einstein condensation (BEC); the one-particle reduced density associated with the ground state wave function $\psi_N \in L^2_s(\mathbb{R}^{3N})$ converges, as $N \to \infty$, towards the orthogonal projection onto a one-particle orbital $\varphi_0 \in L^2(\mathbb{R}^3)$.

Hence, we will be interested in the solution of (5.1) for initial data exhibiting BEC. Despite its linearity, for large N ($N \simeq 10^3 - 10^4$ in typical experiments) it is impossible to solve the many-body Schrödinger equation (5.1), neither analytically nor numerically. It is important, therefore, to find good approximations of the solution of (5.1) that are valid in the limit $N \to \infty$. A first step in this direction was achieved in [37] for $\beta < 1/2$ and in [40, 39] for the Gross-Pitaevskii regime with $\beta = 1$ (the same ideas can also be extended to all $\beta \in (0, 1)$), where it was proven that, for every fixed time $t \in \mathbb{R}$, the solution $\psi_{N,t}$ of (5.1) still exhibits BEC and that its one-particle reduced density converges to the orthogonal projection onto φ_t, given by the solution of the cubic nonlinear Schrödinger equation

$$i\partial_t \varphi_t = -\Delta \varphi_t + \sigma |\varphi_t|^2 \varphi_t \tag{5.4}$$

with the initial data $\varphi_{t=0} = \varphi$ and with coupling constant $\sigma = \int V(x)dx$ for $\beta < 1$ and $\sigma = 8\pi \mathfrak{a}_0$ for $\beta = 1$ (where $\mathfrak{a}_0$ denotes the scattering length of the unscaled potential V). The results of [37, 40, 39] have been revisited and improved further in [85, 11, 27, 20]. In the simpler case $\beta = 0$, i.e. in the mean-field regime, the convergence of the one-particle reduced density towards the orthogonal projection onto the solution of the nonlinear Hartree equation

$$i\partial_t \varphi_t = -\Delta \varphi_t + (V * |\varphi_t|^2)\varphi_t \tag{5.5}$$

has been proved in several situations; see, e.g., [98, 9, 41, 2, 35, 5, 43, 42, 57, 55, 6, 26, 4].

In this chapter, we are interested in the norm approximation to the many-body evolution, which is more precise than the convergence of the one-particle reduced density. It requires not only to follow the dynamics of the condensate, but also to take into account the evolution of its excitations.

As explained in Section 1.7, it is convient to switch to a Fock space representation in order to describe excitations and their dynamics In such a setting, it is instructive to study the time-evolution of coherent initial data, having the form

$$W(\sqrt{N}\varphi)\Omega = e^{-N/2}\left\{1, \varphi, \frac{\varphi^{\otimes 2}}{\sqrt{2!}}, \dots\right\} \tag{5.6}$$

for $\varphi \in L^2(\mathbb{R}^3)$ with $\|\varphi\| = 1$. Here $\Omega = \{1, 0, 0, \dots\}$ is the Fock space vacuum and, for any $f \in L^2(\mathbb{R}^3)$, $W(f) = \exp(a^*(f) - a(f))$ is a Weyl operator. The normalization of φ guarantees that

$$\langle W(\sqrt{N}\varphi)\Omega, \mathcal{N} W(\sqrt{N}\varphi)\Omega\rangle = N.$$

The time-evolution of initial coherent states of the form (5.6), generated by the natural extension of the Hamiltonian (5.2) to the Fock space $\mathcal{F}$

$$\mathcal{H}_N = \int dx a_x^*(-\Delta_x)a_x + \frac{1}{2N}\int \mathrm{d}x\,\mathrm{d}y\, V_N(x-y)a_x^* a_y^* a_y a_x =: \mathcal{K} + \mathcal{V}_N \tag{5.7}$$

has been studied for $\beta = 0$ in [52, 44], where it was proven that

$$\left\| e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi)\Omega - W(\sqrt{N}\varphi_t)\,\mathcal{U}^{\mathrm{f}}_{2,\mathrm{mf}}(t;0)\Omega \right\| \to 0 \tag{5.8}$$

as $N \to \infty$. Here φ_t denotes the solution of the Hartree equation (5.5) and $\mathcal{U}^{f}_{2,\mathrm{mf}}(t;s)$ is a unitary dynamics on $\mathcal{F}$ with a time-dependent generator that is quadratic in creation and annihilation operators[1]. This implies that $\mathcal{U}^{\mathrm{f}}_{2,\mathrm{mf}}(t;s)$ acts on creation and annihilation operators as a time-dependent Bogoliubov transformation $\Theta_{\mathrm{mf}}(t;s) : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ having the form

$$\Theta_{\mathrm{mf}}(t;s) = \begin{pmatrix} U_{\mathrm{mf}}(t;s) & \overline{V_{\mathrm{mf}}(t;s)} \\ V_{\mathrm{mf}}(t;s) & \overline{U_{\mathrm{mf}}(t;s)} \end{pmatrix}. \tag{5.9}$$

In other words, for any $f \in L^2(\mathbb{R}^3)$ and all $t, s \in \mathbb{R}$, we find

$$\mathcal{U}^{\mathrm{f}}_{2,\mathrm{mf}}(t;s)^* a(f)\,\mathcal{U}^{\mathrm{f}}_{2,\mathrm{mf}}(t;s) = a(U_{\mathrm{mf}}(t;s)f) + a^*(V_{\mathrm{mf}}(t;s)\bar{f}). \tag{5.10}$$

The time-dependent Bogoliubov transformation Θ_{mf} can be determined solving the partial differential equation

$$i\partial_t \Theta_{\mathrm{mf}}(t;s) = \mathcal{A}_{\mathrm{mf}}(t)\Theta_{\mathrm{mf}}(t;s) \tag{5.11}$$

with initial condition $\Theta_{\mathrm{mf}}(s;s) = 1$ and with generator

$$\mathcal{A}_{\mathrm{mf}}(t) = \begin{pmatrix} D(t) & -\overline{B(t)} \\ B(t) & -\overline{D(t)} \end{pmatrix}$$

where

$$\begin{aligned} D(t)f &= -\Delta f + (V * |\varphi_t|^2)f + (V * \overline{\varphi}_t f)\varphi_t \\ B(t)f &= (V * \overline{\varphi}_t f)\overline{\varphi}_t. \end{aligned}$$

[1] In the notation for $\mathcal{U}^{\mathrm{f}}_{2,\mathrm{mf}}$, the subscript mf and the superscript f refer to the fact that (5.8) holds in the mean-field regime with $\beta = 0$ for Fock space initial data

Thus, (5.8) allows us to describe the very complex many-body dynamics generated on $\mathcal{F}$ by the Hamiltonian (5.7) by solving the equation (5.5) for the condensate wave function and the equation (5.11) for the Bogoliubov transformation $\Theta_{\mathrm{mf}}(t;s)$ describing the evolution of fluctuations around the condensate.

The ideas of [52, 44] have been further developed in [95] and they have been used to prove a central limit theorem in [10, 22]. In [47, 48], norm approximations for the many-body dynamics in Fock space has been derived using different approaches.

To obtain a norm approximation for the mean-field time-evolution of N-particle initial data exhibiting BEC in a state with wave function $\varphi \in L^2(\mathbb{R}^3)$, it is very convenient to use the unitary map introduced in [64], mapping $L^2_s(\mathbb{R}^{3N})$ into the truncated Fock space

$$\mathcal{F}_{\perp\varphi}^{\leq N} = \bigoplus_{j=0}^{N} L^2_{\perp\varphi}(\mathbb{R}^3)^{\otimes_s N} \tag{5.12}$$

constructed over the orthogonal complement $L^2_{\perp\varphi}(\mathbb{R}^3)$ of the one-dimensional space spanned by the condensate wave function φ. Recall the definition of the map $U_N(\varphi) = U_\varphi : L^2_s(\mathbb{R}^{3N}) \to \mathcal{F}_{\perp\varphi}^{\leq N}$ from Section 1.2. The actions of U_φ on creation and annihilation operators follow the simple rules:

$$U_\varphi a^*(\varphi)a(\varphi)U_\varphi^* = N - \mathcal{N}, \tag{5.13}$$

$$U_\varphi a^*(f)a(\varphi)U_\varphi^* = a^*(f)\sqrt{N - \mathcal{N}}, \tag{5.14}$$

$$U_\varphi a^*(\varphi)a(g)U_\varphi^* = \sqrt{N - \mathcal{N}}a(g), \tag{5.15}$$

$$U_\varphi a^*(f)a(g)U_\varphi^* = a^*(f)a(g) \tag{5.16}$$

for all $f, g \in L^2_{\perp\varphi}(\mathbb{R}^3)$. Heuristically, U_φ factors out the condensate described by the wave function φ and it allows us to focus on its orthogonal excitations.

The unitary map U_φ was used in [63] to obtain a norm approximation for the many-body evolution in the mean-field regime with $\beta = 0$ (see [75] for a similar result). For N-particle initial data of the form $\psi_N = U_\varphi^* \xi_N$ with $\xi_N \in \mathcal{F}_{\perp\varphi}^{\leq N}$ having a finite expectation for the number of particles and for the kinetic energy operator, it was proven there that the solution of the many-body Schrödinger equation (5.1) is such that

$$\|U_{\varphi_t}\psi_{N,t} - \mathcal{U}_{2,\mathrm{mf}}(t;0)\xi_N\| \to 0 \tag{5.17}$$

as $N \to \infty$, where, similarly to (5.8), φ_t is the solution of (5.5) and $\mathcal{U}_{2,\mathrm{mf}}(t;s)$ is a unitary evolution on the Fock space, with a time-dependent generator quadratic in creation and annihilation operators (in fact $\mathcal{U}_{2,\mathrm{mf}}$ is very similar to the unitary evolution $\mathcal{U}^{\mathrm{f}}_{2,\mathrm{mf}}$ in (5.8), emerging in the mean field limit for coherent initial data on the Fock space). Eq. (5.17) is the analogous of (5.8) for N-particle initial data exhibiting BEC; it provides a norm-approximation of the many-body evolution in the mean-field regime in terms of the Hartree equation (5.5) and of a time-dependent Bogoliubov transformation very similar to (5.9).

The convergence (5.17) has been extended to intermediate regimes with $\beta < 1/3$ in [76] and with $\beta < 1/2$ in [77]. Before that, a norm approximation similar to (5.8) for the evolution of coherent initial data on the Fock space has been obtained with $\beta < 1/3$ in [49] and with $\beta < 1/2$ in [58]. A heuristic argument from [58] also shows that (5.8) or (5.17) cannot hold true for $\beta > 1/2$.

In regimes with $\beta > 1/2$ the short scale correlation structure developed by the solution of the many-body Schrödinger equation cannot be appropriately described by a time-dependent Bogoliubov transformation satisfying an equation of the form (5.11). To take into account correlations, it is useful to consider the ground state of the Neumann problem

$$\left[-\Delta + \frac{1}{2N} V_N\right] f_N = \lambda_N f_N \tag{5.18}$$

on the ball $|x| \leq \ell$, for a fixed $\ell > 0$. We fix $f_N(x) = 1$, for $|x| = \ell$, and we extend f_N to $\mathbb{R}^3$ requiring that $f_N(x) = 1$ for all $|x| \geq \ell$. Because of the scaling of the potential V_N, the scattering process takes place in the region $|x| \ll 1$; for this reason, the precise choice of ℓ is not very important, as long as ℓ is of order one (nevertheless, λ_N and f_N depend on ℓ, a dependence that is kept implicit in our notation). It is also useful to define $\omega_N = 1 - f_N$. For N sufficiently large, we have (see [16, Lemma 2.1])

$$\lambda_N = \frac{3b_0}{8\pi N \ell^3} + O(N^{\beta-2})$$

and, for all $x \in \mathbb{R}^3$,

$$0 \leq \omega_N(x) \leq \frac{C}{N(|x| + N^{-\beta})}, \qquad |\nabla \omega_N(x)| \leq \frac{C}{N(|x| + N^{-\beta})^2} \tag{5.19}$$

for a constant C, independent of N.

The solution of (5.18) can be used, first of all, to give a better approximation of the evolution of the condensate wave function, replacing the solution of the limiting nonlinear Schrödinger equation (5.4) with the solution of the modified, N-dependent, Hartree equation

$$i\partial \varphi_{N,t} = -\Delta \varphi_{N,t} + (V_N f_N * |\varphi_{N,t}|^2)\varphi_{N,t} \tag{5.20}$$

with initial data $\varphi_{N,0} = \varphi_0$ describing the condensate at time $t = 0$. Standard arguments in the analysis of dispersive partial differential equations imply that (5.20) is globally well-posed and that it propagates regularity; in particular, if $\varphi_0 \in H^4(\mathbb{R}^3)$, then [16, Appendix B]

$$\|\varphi_{N,t}\|_{H^1} \leq C, \quad \|\varphi_{N,t}\|_{H^4} \leq Ce^{Ct}, \quad \|\partial_t \varphi_{N,t}\|_{H^2} \leq Ce^{Ct}, \quad \forall t > 0. \tag{5.21}$$

Furthermore, (5.18) can be used to describe correlations among particles. To this end, let

$$T_{N,t} = \exp\left(\frac{1}{2}\int dxdy\; [k_{N,t}(x,y) a_x a_y - \text{h.c.}]\right) \tag{5.22}$$

with the integral kernel

$$k_{N,t}(x;y) = (Q_{N,t} \otimes Q_{N,t}) \left[-N\omega_N(x-y)\varphi^2_{N,t}((x+y)/2)\right] \tag{5.23}$$

where $Q_{N,t} = 1 - |\varphi_{N,t}\rangle\langle\varphi_{N,t}|$ is the orthogonal projection onto the orthogonal complement of the solution of the modified Hartree equation (5.20). It is important to observe that (5.23) is the integral kernel of a Hilbert-Schmidt operator. Abusing notation and denoting by $k_{N,t}$ both the Hilbert-Schmidt operator and its integral kernel, we easily find (using (5.19) and (5.21))

$$\begin{aligned} \|k_{N,t}\|_{\mathrm{HS}} &= \|k_{N,t}\|_2 \leq C \\ \|\nabla k_{N,t}\|_{\mathrm{HS}} &= \|k_{N,t}\nabla\|_{\mathrm{HS}} = \|\nabla_1 k_{N,t}\|_2 = \|\nabla_2 k_{N,t}\|_2 \leq C N^{\beta/2}. \end{aligned} \tag{5.24}$$

These bounds reflect the idea that, through $T_{N,t}$, we only produce a bounded number of excitations, causing however a large change in the energy.

Notice that the action of the Bogoliubov transformation (5.22) on creation and annihilation operators is explicit. For any $f \in L^2_{\perp\varphi_{N,t}}(\mathbb{R}^3)$, we find

$$\begin{aligned} T_{N,t} a(f) T^*_{N,t} &= a(\cosh_{k_{N,t}}(f)) + a^*(\sinh_{k_{N,t}}(\bar{f})) \\ T_{N,t} a^*(f) T^*_{N,t} &= a^*(\cosh_{k_{N,t}}(f)) + a(\sinh_{k_{N,t}}(\bar{f})) \end{aligned}$$

where $\cosh_{k_{N,t}}$ and $\sinh_{k_{N,t}}$ are the linear operators defined by the absolutely convergent series

$$\cosh_{k_{N,t}} = \sum_{n\geq 0} \frac{1}{(2n)!} \left(k_{N,t}\overline{k}_{N,t}\right)^n, \qquad \sinh_{k_{N,t}} = \sum_{n\geq 0} \frac{1}{(2n+1)!} (k_{N,t}\overline{k}_{N,t})^n k_{N,t}\,.$$

Using the Bogoliubov transformation $T_{N,t}$ to implement correlations, one can construct norm approximations for the many-body evolution that are valid also in regimes with $\beta > 1/2$. For Fock space initial data, it was recently proven in [16] that, for every $0 < \beta < 1$ and for every N large enough, there exists a unitary evolution $\mathcal{U}^{\beta}_{2,N}$ with a time-dependent generator quadratic in creation and annihilation operators, such that

$$\left\| e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T^*_{N,0}\Omega - W(\sqrt{N}\varphi_{N,t}) T^*_{N,t} \mathcal{U}^{\mathrm{f}}_{2,N}(t;0)\Omega \right\| \to 0$$

as $N \to \infty$ (to be more precise, in [16], the kernel $k_{N,t}$ was chosen slightly different, without the orthogonal projection $(Q_{N,t} \otimes Q_{N,t})$). In other words, for initial data of the form $W(\sqrt{N}\varphi)T_{N,0}\Omega$, describing an approximate coherent state, modified by the Bogoliubov transformation $T_{N,0}$ to take into account correlations, the full many-body time-evolution can be approximated in terms of the modified N-dependent Hartree equation (5.20) (describing the dynamics of the condensate), of the Bogoliubov transformation (5.22) (generating the correlation structure) and of the quadratic evolution $\mathcal{U}^{\mathrm{f}}_{2,N}$ (which, similarly to (5.10), also acts as a time-dependent Bogoliubov transformation). Using a related approach, a similar result has been established in [50] for $\beta < 2/3$.

Our aim in the present chapter is to obtain a norm-approximation for the many-body evolution of N-particle initial data exhibiting BEC for the whole range of parameters $0 < \beta < 1$. To reach this goal, we will combine ideas from [63] and [76, 77] with ideas from [16], in particular, with the idea of using Bogoliubov transformations of the form (5.22) to implement correlations. To state our main result, we define the unitary dynamics $\mathcal{U}_{2,N}(t;s)$ as the two-parameter unitary group on the Fock space $\mathcal{F}$ satisfying

$$i\partial_t \mathcal{U}_{2,N}(t;s) = \mathcal{G}_{2,N,t}\mathcal{U}_{2,N}(t;s), \qquad \mathcal{U}_{2,N}(s;s) = 1_{\mathcal{F}} \tag{5.25}$$

with the time-dependent quadratic generator $\mathcal{G}_{2,N,t}$ given by

$$\mathcal{G}_{2,N,t} = \eta_N(t) + (i\partial_t T_{N,t})T^*_{N,t} + \mathcal{G}^{\mathcal{V}}_{2,N,t} + \mathcal{G}^{\mathcal{K}}_{2,N,t} + \mathcal{G}^{\lambda_N}_{2,N,t} \tag{5.26}$$

with the phase $\eta_N(t)$ defined by

$$\begin{aligned}\eta_N(t) = &\frac{N+1}{2}\big\langle \varphi_{N,t}, [V_N(1-2f_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\big\rangle - \mu_N(t) \\ &+ \int dx\, \big(V_N * |\varphi_{N,t}|^2\big)(x)\|\mathrm{sh}_x\|^2 + \int dx\, \langle \nabla_x \mathrm{sh}_x, \nabla_x \mathrm{sh}_x\rangle \\ &+ \int dxdy\, K_{1,N,t}(x;y)\langle \mathrm{sh}_x, \mathrm{sh}_y\rangle + \mathrm{Re}\int dxdy\, K_{2,N,t}(x;y)\, \langle \mathrm{sh}_x, \mathrm{ch}_y\rangle \\ &+ \frac{1}{2N}\int dxdy\, V_N(x-y)\big|\big\langle \mathrm{sh}_x - \varphi_{N,t}(x)\mathrm{sh}(\varphi_{N,t}), \mathrm{ch}_y - \varphi_{N,t}(y)\mathrm{ch}(\varphi_{N,t})\big\rangle\big|^2\end{aligned} \tag{5.27}$$

with $\mu_N(t) = \big\langle \varphi_{N,t}, [(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\big\rangle$ and where the operators $\mathcal{G}^{\mathcal{V}}_{2,N,t}$, $\mathcal{G}^{\lambda_N}_{2,N,t}$ and $\mathcal{G}^{\mathcal{K}}_{2,N,t}$ are given by

$$\begin{aligned}\mathcal{G}^{\mathcal{V}}_{2,N,t} = &\int dx\, \big(V_N * |\varphi_{N,t}|^2\big)(x)\big[a^*(\mathrm{ch}_x)a(\mathrm{ch}_x) + a^*(\mathrm{ch}_x)a^*(\mathrm{sh}_x) \\ &\qquad + a(\mathrm{ch}_x)a(\mathrm{sh}_x) + a^*(\mathrm{sh}_x)a(\mathrm{sh}_x)\big] \\ &+ \int dxdy\, K_{1,N,t}(x;y)\big[a^*(\mathrm{ch}_x)a(\mathrm{ch}_y) + a^*(\mathrm{ch}_x)a^*(\mathrm{sh}_y) \\ &\qquad + a(\mathrm{ch}_y)a(\mathrm{sh}_x) + a^*(\mathrm{sh}_y)a(\mathrm{sh}_x)\big] \\ &+ \frac{1}{2}\int dxdy\, K_{2,N,t}(x;y)\big[a^*_x a^*(\mathrm{p}_y) + a^*_x a(\mathrm{sh}_y) + a^*(\mathrm{p}_x)a^*(\mathrm{p}_y) + a^*(\mathrm{p}_x)a(\mathrm{sh}_y) \\ &\qquad + a^*_y a^*(\mathrm{p}_x) + a^*_y a(\mathrm{sh}_x) + a^*(\mathrm{p}_y)a(\mathrm{sh}_x) + a(\mathrm{sh}_x)a(\mathrm{sh}_y) + \mathrm{h.c.}\big] \\ &+ \frac{1}{2}\Big[\langle \varphi_{N,t}, V_N * |\varphi_{N,t}|^2\varphi_{N,t}\rangle a^*(\varphi_{N,t})a^*(\varphi_{N,t}) \\ &\qquad - 2a^*(\varphi_{N,t})a^*\big([V_N * |\varphi_{N,t}|^2]\varphi_{N,t}\big) + \mathrm{h.c.}\Big], \\ \mathcal{G}^{\lambda_N}_{2,N,t} = &N\lambda_N\int dxdy\, f_N(x-y)\chi(|x-y|\leq \ell)\varphi^2_{N,t}((x+y)/2)a^*_x a^*_y + \mathrm{h.c.}\end{aligned} \tag{5.28}$$

and

$$
\begin{aligned}
\mathcal{G}^{\mathcal{K}}_{2,N,t} = & \int dx\; \big[a_x^*(-\Delta_x)a_x + a_x^* a(-\Delta_x \mathrm{p}_x) + a_x^* a^*(-\Delta_x \mathrm{v}_x) + a_x^* a^*(-\Delta_x \mathrm{r}_x) \\
& \qquad + a^*(-\Delta_x \mathrm{p}_x) a(\mathrm{ch}_x) + a^*(-\Delta_x \mathrm{p}_x) a^*(\mathrm{sh}_x) + a(-\Delta_x \mathrm{r}_x) a_x \\
& \qquad + a(-\Delta_x \mathrm{v}_x) a_x + a(\mathrm{sh}_x) a(-\Delta_x \mathrm{p}_x) + a^*(-\Delta_x \mathrm{r}_x) a(\mathrm{k}_x) \\
& \qquad + a^*(-\Delta_x \mathrm{r}_x) a(\mathrm{r}_x) + a^*(\mathrm{k}_x) a(-\Delta_x \mathrm{r}_x) + a^*\big(\nabla_x \mathrm{k}_x\big) a\big(\nabla_x \mathrm{k}_x\big)\big] \\
& + \frac{1}{2}\int dxdy\; N\omega_N(x-y)\big[\varphi_{N,t}((x+y)/2)\Delta\varphi_{N,t}((x+y)/2) \\
& \qquad\qquad + \nabla\varphi_{N,t}((x+y)/2)\cdot\nabla\varphi_{N,t}((x+y)/2)\big] a_x^* a_y^* + \text{h.c.}
\end{aligned}
\tag{5.29}
$$

Here, we use the notation $j_x(\cdot) := j(\cdot; x)$ for any $j \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, the decompositions $\sinh_{k_{N,t}} = \mathrm{sh} = \mathrm{k} + \mathrm{r}$ and $\cosh_{k_{N,t}} = \mathrm{ch} = 1 + \mathrm{p}$ as well as

$$
k_{N,t}(x;y) = -N\omega_N(x-y)\varphi^2_{N,t}((x+y)/2) + \mathrm{v}(x;y); \quad \forall x, y \in \mathbb{R}^3
$$

Finally, $Q_{N,t} = 1 - |\varphi_{N,t}\rangle\langle\varphi_{N,t}|$ denotes the orthogonal projection onto $\{\varphi_{N,t}\}^\perp$.

We are now ready to state our first main result, providing a norm-approximation for the many-body evolution of N-particle initial data exhibiting BEC. To this end, let us first collect some conditions that will be required throughout the paper.

Hypothesis A: We assume that $0 < \beta < 1$. We suppose, moreover, the interaction potential $V : \mathbb{R}^3 \to \mathbb{R}$ to be smooth, radially symmetric, compactly supported and pointwise non-negative. Furthermore, we choose f_N to be the solution of the Neumann problem (5.18) on the ball $|x| \le \ell$, for a sufficiently small[2] (but fixed, independent of N) parameter $\ell > 0$. Finally, we let $\varphi_{N,t}$ be the solution of the N-dependent nonlinear Hartree equation (5.20) with initial data $\varphi_0 \in H^4(\mathbb{R}^3)$.

Theorem 5.1.1. *Assume that Hypothesis A holds true. Let $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ with $\|\xi_N\| = 1$ and*

$$
\langle \xi_N, (\mathcal{K} + \mathcal{N})\xi_N\rangle \le C. \tag{5.30}
$$

Let $\Psi_{N,t}$ be the solution of the Schrödinger equation (5.1) with initial data

$$
\Psi_{N,0} = U^*_{\varphi_0} 1^{\le N} T^*_{N,0}\xi_N \tag{5.31}
$$

and let $\mathcal{U}_{2,N}(t;s)$ be the unitary dynamics on $\mathcal{F}$ defined in (5.25). Then, for all $\alpha < \min(\beta/2, (1-\beta)/2)$, there exists a constant $C > 0$ such that

$$
\big\| U_{\varphi_{N,t}}\Psi_{N,t} - T^*_{N,t}\mathcal{U}_{2,N}(t;0)\,\xi_N \big\|^2 \le C N^{-\alpha}\exp(C\exp(C|t|)) \tag{5.32}
$$

for all N sufficiently large and all $t \in \mathbb{R}$.

[2] The smallness of ℓ is used because it implies that the kernel $k_{N,t}$ introduced in (5.23) has a small Hilbert-Schmidt norm; this in turn implies that conjugation with the Bogoliubov transformation $T_{N,t}$ produces only small changes in the number of particles operator; see Proposition 5.3.3.

Since the quadratic evolution $\mathcal{U}_{2,N}(t;s)$ depends on N, it is natural to ask what happens as $N \to \infty$. Proceeding similarly to [16], we observe that $k_{N,t}$ can be approximated, for large N, by the limiting kernel

$$k_t(x;y) = (Q_t \otimes Q_t)\left[-\omega_\infty(x-y)\varphi_t^2((x+y)/2)\right] \tag{5.33}$$

where φ_t is the solution of the nonlinear Schrödinger equation (5.4), $Q_t = 1 - |\varphi_t\rangle\langle\varphi_t|$ is the projection onto the orthogonal complement of φ_t and where ω_∞ is given by

$$\omega_\infty(x) := \begin{cases} \frac{b_0}{8\pi}\left[\frac{1}{|x|} - \frac{3}{2\ell} + \frac{|x|^2}{2\ell^3}\right] & \text{for } |x| \le \ell, \\ 0 & \text{for } |x| > \ell \end{cases} \tag{5.34}$$

With k_t, we can define a new Bogoliubov transformation

$$T_t = \exp\left[\frac{1}{2}\int dxdy k_t(x;y)a_x a_y - \text{h.c.}\right] \tag{5.35}$$

Replacing $\cosh_{k_{N,t}}$, $\sinh_{k_{N,t}}$, $\mathrm{p}_{k_{N,t}}$ and $\mathrm{r}_{k_{N,t}}$ by their counterparts $\cosh_{k_t}$, $\sinh_{k_t}$, p_{k_t} and r_{k_t}, replacing $\varphi_{N,t}$ by φ_t, the convolution $V_N * (\cdot)$ by $b_0\delta * (\cdot)$, the eigenvalue $N\lambda_N$ by its first order approximation $3b_0/(8\pi\ell^3)$, $N\omega_N$ by ω_∞ and, finally, replacing $f_N = 1-\omega_N$ by $f_\infty = 1$ in the operators $\mathcal{G}^{\mathcal{V}}_{2,N,t}, \mathcal{G}^{\lambda_N}_{2,N,t}, \mathcal{G}^{\mathcal{K}}_{2,N,t}$ in (5.28) and (5.29), we can define limiting operators $\mathcal{G}^{\mathcal{V}}_{2,t}, \mathcal{G}^{\lambda}_{2,t}, \mathcal{G}^{\mathcal{K}}_{2,t}$ and we can use them to define the limiting generator

$$\mathcal{G}_{2,t} = (i\partial_t T_t)T_t^* + \mathcal{G}^{\mathcal{V}}_{2,t} + \mathcal{G}^{\mathcal{K}}_{2,t} + \mathcal{G}^{\lambda}_{2,t} \tag{5.36}$$

and the corresponding limiting fluctuation dynamics $\mathcal{U}_2$ by

$$i\partial_t \mathcal{U}_2(t;s) = \mathcal{G}_{2,t}\mathcal{U}_2(t;s) \quad \mathcal{U}_2(s;s) = 1_{\mathcal{F}} \tag{5.37}$$

We are now ready to state our second main result.

Theorem 5.1.2. *Assume that Hypothesis A holds true. Let $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ with $\|\xi_N\| = 1$ and (5.30). Let $\Psi_{N,t}$ be the solution of the Schrödinger equation (5.1) with initial data (5.31) and let $\mathcal{U}_2(t;0)$ be the unitary dynamics on $\mathcal{F}$ defined in (5.37). Then, for all $\alpha < \min(\beta/2, (1-\beta)/2)$, there exists a constant $C > 0$ such that*

$$\left\|U_{\varphi_{N,t}}\Psi_{N,t} - e^{-i\int_0^t d\tau\, \eta_N(\tau)}\, T^*_{N,t}\mathcal{U}_2(t;0)\,\xi_N\right\|^2 \le CN^{-\alpha}\exp(C\exp(C|t|)) \tag{5.38}$$

for all N sufficiently large and all $t \in \mathbb{R}$.

Theorem 5.1.1 and Theorem 5.1.2 apply to the study of the time-evolution of initial data of the form

$$\psi_{N,0} = U^*_{\varphi_0}1^{\le N}T^*_{N,0}\xi_N \tag{5.39}$$

for a $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ satisfying the bound

$$\langle \xi_N, [\mathcal{K} + \mathcal{N}]\,\xi_N\rangle \le C \tag{5.40}$$

uniformly in N. It is natural to ask under which assumptions on $\psi_{N,0}$ is it possible to find $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ such that (5.39) and (5.40) hold true. The answer is given in our last main theorem.

Theorem 5.1.3. *Assume Hypothesis A holds true. Let $\Psi_{N,0} \in L^2_s(\mathbb{R}^{3N})$ with reduced one-particle density matrix $\gamma_{N,0}$ such that*

$$\operatorname{tr} |\gamma_{N,0} - |\varphi_0\rangle\langle\varphi_0|| \leq CN^{-1} \tag{5.41}$$

and

$$\left| \frac{1}{N} \langle \Psi_{N,0}, H_N \Psi_{N,0} \rangle - \left[\|\nabla\varphi_0\|^2 + \frac{1}{2} \langle \varphi_0, (V_N f_n * |\varphi_0|^2)\varphi_0 \rangle \right] \right| \leq CN^{-1} \tag{5.42}$$

Let $\Psi_{N,t}$ be the solution of the Schrödinger equation (5.1) with initial data $\psi_{N,0}$ and let $\mathcal{U}_2(t;0)$ be the unitary dynamics on $\mathcal{F}$ defined in (5.37). Then, for all $\alpha < \min(\beta/2, (1-\beta)/2)$, there exists a constant $C > 0$ such that

$$\begin{aligned} \big\| T_{N,t} U_{\varphi_{N,t}} \Psi_{N,t} - e^{-i\int_0^t d\tau\, \eta_N(\tau)}\, \mathcal{U}_2(t;0)\, T_{N,0}\, U_{\varphi_{N,0}} \Psi_{N,0} \big\|^2 & \\ \leq CN^{-\alpha} \exp(C \exp(C|t|)) & \end{aligned} \tag{5.43}$$

for all N sufficiently large and all $t \in \mathbb{R}$.

Remarks:

1) Recall that, although this is not reflected in our notation, the family of Bogoliubov transformations $T_{N,t}$ and the quadratic evolutions $\mathcal{U}_{2,N}(t;0)$ in Theorem 5.1.1 and $\mathcal{U}_2(t;0)$ in Theorem 5.1.2 and in Theorem 5.1.3 depend on the choice of the length scale $\ell > 0$ in (5.18). This parameter is chosen small enough, but fixed.

2) The bounds (5.32), (5.38) and (5.43) give norm approximations of the full many-body dynamics of initial data exhibiting BEC in terms of Fock space dynamics $\mathcal{U}_{2,N}(t;0)$ or $\mathcal{U}_2(t;0)$ with quadratic generators, of the family of time-dependent Bogoliubov transformation $T_{N,t}$ and of the solution $\varphi_{N,t}$ of the modified Hartree equation.

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5.2 Outline of the proof

In this section we explain the overall strategy of the proof. As in Theorem 5.1.1, we denote by $\Psi_{N,t}$ the solution of the N-particle Schrödinger equation (5.1) with the initial data $\Psi_{N,0} = U^*_{\varphi_{N,0}} 1^{\leq N} T^*_{N,0} \xi_N$, where $\xi_N \in \mathcal{F}^{\leq N}_{\perp\varphi}$ is such that

$$\langle \xi_N, (\mathcal{N} + \mathcal{K})\xi_N \rangle \leq C$$

uniformly in N. Furthermore, we denote by $\varphi_{N,t}$ the solution of the modified, N-dependent, nonlinear Hartree equation (5.20), with initial data $\varphi_0 \in H^4(\mathbb{R}^3)$.

5.2.1 Fluctuation evolution

First of all, we apply the map $U_{\varphi_{N,t}}$, defined in (**??**), to $\Psi_{N,t}$. This allows us to remove the condensate described at time t by $\varphi_{N,t}$ and to focus on the orthogonal fluctuations. We set

$$\Phi_{N,t} = U_{\varphi_{N,t}} \Psi_{N,t}, \tag{5.44}$$

and we observe that $\Phi_{N,t} \in \mathcal{F}^{\leq N}_{\perp \varphi_{N,t}}$ satisfies the equation

$$i\partial_t \Phi_{N,t} = \mathcal{L}_{N,t} \Phi_{N,t}, \tag{5.45}$$

with the initial data $\Phi_{N,0} = 1^{\leq N} T^*_{N,0} \xi_N$ and the generator

$$\mathcal{L}_{N,t} = (i\partial_t U_{\varphi_{N,t}}) U^*_{N,t} + U_{\varphi_{N,t}} H_N U^*_{\varphi_{N,t}}. \tag{5.46}$$

Using (5.13) and computing the first term on the r.h.s. of (5.46) as in [63], we obtain

$$\begin{aligned}
\mathcal{L}_{N,t} &= \frac{N+1}{2} \left\langle \varphi_{N,t}, [V_N(1-2f_N) * |\varphi_{N,t}|^2]\varphi_{N,t} \right\rangle - \mu_N(t) \\
&+ \frac{1}{2} \left\langle \varphi_{N,t}, [V_N * |\varphi_{N,t}|^2]\varphi_{N,t} \right\rangle \frac{\mathcal{N}(\mathcal{N}+1)}{N} \\
&+ \Big[\sqrt{N} \Big[a^*(Q_{N,t}[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}) - a^*(Q_{N,t}[V_N * |\varphi_{N,t}|^2]\varphi_{N,t}) \frac{\mathcal{N}}{N} \Big] \sqrt{\frac{N-\mathcal{N}}{N}} \\
&\qquad\qquad + \text{h.c.} \Big] \\
&+ \mathrm{d}\Gamma\Big(-\Delta + (V_N f_N) * |\varphi_{N,t}|^2 + K_{1,N,t} - \mu_{N,t} \Big) \\
&+ \mathrm{d}\Gamma\Big(Q_{N,t}(V_N\omega_N * |\varphi_{N,t}|^2) Q_{N,t} \Big) - \mathrm{d}\Gamma\Big(Q_{N,t}(V_N * |\varphi_{N,t}|^2) Q_{N,t} + K_{1,N,t} \Big) \frac{\mathcal{N}}{N} \\
&+ \Big[\frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, K_{2,N,t}(x,y) a^*_x a^*_y \frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N} + \text{h.c.} \Big] \\
&+ \Big[\frac{1}{\sqrt{N}} \int dx dy dx' dy' \ \ (Q_{N,t} \otimes Q_{N,t} V_N Q_{N,t} \otimes 1)(x,y;x',y') \varphi_{N,t}(y') a^*_x a^*_y a_{x'} \sqrt{\frac{N-\mathcal{N}}{N}} \\
&\qquad\qquad + \text{h.c.} \Big] \\
&+ \frac{1}{2N} \int dx dy dx' dy' \ \ (Q_{N,t} \otimes Q_{N,t} V_N Q_{N,t} \otimes Q_{N,t})(x,y;x',y') a^*_x a^*_y a_{x'} a_{y'}
\end{aligned} \tag{5.47}$$

with

$$\mu_N(t) := \left\langle \varphi_{N,t}, [(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t} \right\rangle .$$

5.2.2 Modified fluctuation evolution

Next, we have to remove the singular correlation structure from $\Phi_{N,t}$. Since $\Psi_{N,t} = U^*_{\varphi_{N,t}} \Phi_{N,t}$ and since $U^*_{\varphi_{N,t}}$ just adds products of solutions of the nonlinear equation

(5.20), it is clear that all correlations developed by $\Psi_{N,t}$ must be contained in $\Phi_{N,t}$. As a consequence, at least for $\beta > 1/2$, the time evolution of $\Phi_{N,t}$ cannot be generated by a quadratic Hamiltonian, not even approximately in the limit of large N. To remove correlations from $\Phi_{N,t}$ we would like to follow the idea of [16] and apply the Bogoliubov transformation $T_{N,t}$ defined in (5.22). Unfortunately, $T_{N,t}$ does not preserve the number of particles, and therefore it does not leave the truncated Fock space $\mathcal{F}^{\leq N}_{\perp\varphi_{N,t}}$ invariant. Since $T_{N,t}$ only creates few particles (the bound (5.24) implies that $T_{N,t}\mathcal{N}T^*_{N,t} \leq C\mathcal{N}$), this should not be a serious obstacle. To circumvent it, it seems natural to give up the restriction on the number of particles and consider $\Phi_{N,t}$ as a vector in the untruncated Fock space $\mathcal{F}_{\perp\varphi_{N,t}}$. The drawback of this approach is the fact that the generator $\mathcal{L}_{N,t}$ computed in (5.47) is defined only on sectors with at most N particles. So, we proceed as follows; first we approximate $\Phi_{N,t}$ by a new, modified, fluctuation vector $\widetilde{\Phi}_{N,t}$, whose dynamics is governed by a modified generator $\widetilde{\mathcal{L}}_{N,t}$ which, on the one hand, is close to $\mathcal{L}_{N,t}$ when acting on vectors with a small number of particles and, on the other hand, is well-defined on the full untruncated Fock space $\mathcal{F}_{\perp\varphi_{N,t}}$. To define $\widetilde{\mathcal{L}}_{N,t}$ we proceed as follows. Starting from the expression on the r.h.s. of (5.47), we replace first of all the factor $\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}$ by $N-\mathcal{N}$; the error is small, since

$$|\sqrt{(N-x)(N-x-1)} - (N-x)| \leq 1$$

for all $x \in \mathbb{N}$.

Secondly, we replace $\sqrt{N-\mathcal{N}}$ by $\sqrt{N}G_b(\mathcal{N}/N)$ where

$$G_b(t) := \sum_{n=0}^{b} \frac{(2n)!}{(n!)^2 4^n (1-2n)} t^n. \tag{5.48}$$

Indeed, the polynomial $G_b(t)$ is the Taylor series for $\sqrt{1-t}$ around $t=0$; it satisfies

$$|\sqrt{1-t} - G_b(t)| \leq Ct^{b+1}, \quad \forall t \in [0,1]. \tag{5.49}$$

for a constant $C > 0$ depending on b. Here $b \in \mathbb{N}$ is a large, fixed number, that will be specified later.

Finally, we add a term of the form $C_b e^{C_b|t|}\mathcal{N}(\mathcal{N}/N)^{2b}$ with a sufficiently large constant C_b that will also be specified later. Since the generators $\mathcal{L}_N$ and $\widetilde{\mathcal{L}}_N$ will act on states with small number of particles, we expect this term to have a negligible effect on the dynamics (on the other hand, it allows us to better control the energy). With these changes, we obtain the modified generator

$$
\begin{aligned}
&\widetilde{\mathcal{L}}_{N,t} = \frac{N+1}{2}\left\langle \varphi_{N,t}, [V_N(1-2f_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\right\rangle - \mu_N(t) \\
&+ \frac{1}{2}\left\langle \varphi_{N,t}, [V_N * |\varphi_{N,t}|^2]\varphi_{N,t}\right\rangle \frac{\mathcal{N}(\mathcal{N}+1)}{N} \\
&+ \Big[\sqrt{N}\Big[a^*(Q_{N,t}[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}) - a^*(Q_{N,t}[V_N * |\varphi_{N,t}|^2]\varphi_{N,t})\frac{\mathcal{N}}{N}\Big]G_b(\mathcal{N}/N) \\
&\qquad\qquad\qquad + \text{h.c.}\Big] \\
&+ \mathrm{d}\Gamma\Big(-\Delta + (V_N f_N) * |\varphi_{N,t}|^2 + K_{1,N,t} - \mu_{N,t}\Big) \\
&+ \mathrm{d}\Gamma\Big(Q_{N,t}(V_N\omega_N * |\varphi_{N,t}|^2)Q_{N,t}\Big) - \mathrm{d}\Gamma\Big(Q_{N,t}(V_N * |\varphi_{N,t}|^2)Q_{N,t} + K_{1,N,t}\Big)\frac{\mathcal{N}}{N} \\
&+ \left[\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, K_{2,N,t}(x,y)a_x^*a_y^*\frac{N-\mathcal{N}}{N} + \text{h.c.}\right] \\
&+ \left[\frac{1}{\sqrt{N}}\int dxdydx'dy' \;\; (Q_{N,t}\otimes Q_{N,t}V_N Q_{N,t}\otimes 1)(x,y;x',y')\varphi_{N,t}(y')a_x^*a_y^*a_{x'}G_b(\mathcal{N}/N)\right. \\
&\qquad\qquad\qquad \left. + \text{h.c.}\right] \\
&+ \frac{1}{2N}\int dxdydx'dy' \;\; (Q_{N,t}\otimes Q_{N,t}V_N Q_{N,t}\otimes Q_{N,t})(x,y;x',y')a_x^*a_y^*a_{x'}a_{y'} \\
&+ C_b e^{C_b|t|}\mathcal{N}(\mathcal{N}/N)^{2b}.
\end{aligned} \tag{5.50}
$$

Using this modified generator, we define the modified fluctuation dynamics $\widetilde{\Phi}_{N,t}$ as the solution of the Schrödinger equation

$$
i\partial_t\widetilde{\Phi}_{N,t} = \widetilde{\mathcal{L}}_{N,t}\widetilde{\Phi}_{N,t}, \tag{5.51}
$$

with the initial data $\widetilde{\Phi}_{N,0} = T^*_{N,0}\xi_N$. Observe that $\widetilde{\Phi}_{N,t} \in \mathcal{F}_{\perp\varphi_{N,t}}$. Indeed, arguing as in [63, Lemma 9], we have

$$
\begin{aligned}
\frac{d}{dt}\|a(\varphi_{N,t})\widetilde{\Phi}_{N,t}\|^2 =& i\left\langle\widetilde{\Phi}_{N,t}, \left[\widetilde{\mathcal{L}}_{N,t}, a^*(\varphi_{N,t})a(\varphi_{N,t})\right]\widetilde{\Phi}_{N,t}\right\rangle \\
&+ 2\mathrm{Im}\left\langle\widetilde{\Phi}_{N,t}, a^*(i\partial_t\varphi_{N,t})a(\varphi_{N,t})\widetilde{\Phi}_{N,t}\right\rangle = 0,
\end{aligned} \tag{5.52}
$$

because, using that $[a^*(\varphi_{N,t})a(\varphi_{N,t}), \mathcal{N}] = 0$, we find

$$
\begin{aligned}
\left[\widetilde{\mathcal{L}}_{N,t}, a^*(\varphi_{N,t})a(\varphi_{N,t})\right] &= \left[\mathrm{d}\Gamma\big(-\Delta + (V_N f_N) * |\varphi_{N,t}|^2\big), a^*(\varphi_{N,t})a(\varphi_{N,t})\right] \\
&= a^*\big([-\Delta + (V_N f_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\big)a(\varphi_{N,t}) - \text{h.c.} \\
&= a^*(i\partial_t\varphi_{N,t})a(\varphi_{N,t}) - \text{h.c.}
\end{aligned}
$$

Notice moreover that we find it more convenient to choose the initial data for the modified dynamics slightly different from the initial data for the original fluctuation dynamics (we do not include the cutoff to $\mathcal{N} \leq N$ in the definition of $\widetilde{\Phi}_{N,0}$). Nevertheless, it is possible to prove that the two dynamics remain close; this is the content of the next lemma, which is the first step in the proof of Theorem 5.1.1.

Lemma 5.2.1. *Assume Hypothesis A holds true. Let $\Phi_{N,t}$ be as defined in* (5.45) *and $\widetilde{\Phi}_{N,t}$ as in* (5.51). *Here, we assume that the parameters $b \in \mathbb{N}$ and $C_b > 0$ in (5.50) are large enough, and that $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ is such that $\|\xi_N\| \leq 1$ and*

$$\langle \xi_N, \big[\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}\big]\xi_N\rangle \leq C \tag{5.53}$$

uniformly in N. Then, for all $\alpha < (1-\beta)/2$, there exists a constant $C > 0$ such that

$$\|\Phi_{N,t} - \widetilde{\Phi}_{N,t}\|^2 \leq CN^{-\alpha} \exp(C \exp(C|t|))$$

for all $t \in \mathbb{R}$.

5.2.3 Bogoliubov transformation

Next, we apply the Bogoliubov transformation (5.22) to the modified fluctuation evolution $\widetilde{\Phi}_{N,t}$ defined in (5.51). We set

$$\xi_{N,t} = T_{N,t}\widetilde{\Phi}_{N,t} \tag{5.54}$$

Then $\xi_{N,t} \in \mathcal{F}_{\perp\varphi_{N,t}}$ (with no restriction on the number of particles) and it solves the Schrödinger equation

$$i\partial_t \xi_{N,t} = \mathcal{G}_{N,t}\xi_{N,t}, \tag{5.55}$$

with the initial data $\xi_{N,0} = \xi_N$ and the generator

$$\mathcal{G}_{N,t} = (i\partial_t T_{N,t})T^*_{N,t} + T_{N,t}\widetilde{\mathcal{L}}_{N,t}T^*_{N,t}. \tag{5.56}$$

As explained above, the application of the Bogoliubov transformation $T_{N,t}$ takes care of correlations and makes it possible for us to approximate the evolution (5.55) with the unitary evolution $\mathcal{U}_{2,N}$, having the quadratic generator (5.26). This is the content of the next lemma.

Lemma 5.2.2. *Assume Hypothesis A holds true. Let $\xi_{N,t}$ be defined as in (5.54) and $\xi_{2,N,t} = \mathcal{U}_{2,N}(t;0)\xi_N$ with the unitary evolution $\mathcal{U}_{2,N}$ defined in (5.25). Here, we assume that the parameters $b \in \mathbb{N}$ and $C_b > 0$ in (5.50) are large enough, and that $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ is such that $\|\xi_N\| \leq 1$ and (5.53) holds true. Then there exists $C > 0$ such that*

$$\|\xi_{N,t} - \xi_{2,N,t}\|^2 \leq CN^{-\alpha} \exp(C \exp(C|t|)),$$

for all $t \in \mathbb{R}$, with $\alpha = \min(\beta/2, (1-\beta)/2)$.

Theorem 5.1.1 is a consequence of Lemma 5.2.1 and Lemma 5.2.2, up to the remark that the assumption on the sequence $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ appearing in Theorem 5.1.1 is weaker than the assumption (5.53) appearing in both lemmas. So, to conclude the proof of Theorem 5.1.1, we need an additional localization argument, which will be explained in Section 5.5.

To prove Theorem 5.1.2 we will then compare $\xi_{2,N,t}$ with $\xi_{2,t} = \mathcal{U}_2(t;0)\xi_N$, where $\mathcal{U}_2$ is the limiting evolution defined in (5.37), by controlling the difference between the two generators.

Finally, Theorem 5.1.3 will follow from Theorem 5.1.2, by proving that, under the assumptions (5.41) and (5.42), it is possible to write $\psi_{N,0} = U^*_{\varphi_0}\mathbf{1}^{\leq N} T^*_{N,0}\xi_N$ with a sequence $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ satisfying the condition (5.30).

The rest of the paper is organized as follows. In Section 5.3 we show Lemma 5.2.2. In Section 5.4, we prove Lemma 5.2.1 making use of some energy estimates. Finally, in Section 5.5, we conclude the proof of our three main theorems.

5.3 Analysis of Bogoliubov transformed dynamics

In this section, we prove Lemma 5.2.2. To this end, we need to study the properties of the generator $\mathcal{G}_{N,t}$ defined in (5.56).

Proposition 5.3.1. *Assume that Hypothesis A holds true. Then, there exists a constant $C > 0$ and, for every fixed $b \in \mathbb{N}$, a constant $K_b > 0$ such that the generator $\mathcal{G}_{N,t}$ in (5.56) can be written as*

$$\mathcal{G}_{N,t} = \mathcal{G}_{2,N,t} + \mathcal{V}_N + C_b e^{C_b|t|}\mathcal{N}(\mathcal{N}/N)^{2b} + \mathcal{E}_{N,t} \tag{5.57}$$

with the quadratic generator $\mathcal{G}_{2,N,t}$, defined as in (5.26)*, satisfying the estimates*

$$\begin{aligned} \pm(\mathcal{G}_{2,N,t} - \eta_N(t) - \mathcal{K}) &\leq Ce^{C|t|}(\mathcal{N}+1) \\ \pm\big[\mathcal{G}_{2,N,t}, i\mathcal{N}\big] &\leq Ce^{C|t|}(\mathcal{N}+1) \\ \pm\partial_t\big(\mathcal{G}_{2,N,t} - \eta_N(t)\big) &\leq Ce^{C|t|}(\mathcal{N}+1) \end{aligned} \tag{5.58}$$

and the error operator $\mathcal{E}_{N,t}$ such that, with $\alpha = \min(\beta/2, (1-\beta)/2)$,

$$\begin{aligned} \pm\mathcal{E}_{N,t} \leq\; &\delta\mathcal{V}_N + N^{-\beta/2}\mathcal{K} + Ce^{C|t|}\max(N^{-\alpha},\delta^{-1})(\mathcal{N}+1) \\ &+ K_b e^{Ct}\max(\delta,\delta^{-1})\frac{(\mathcal{N}+1)^2}{N} \\ &+ \Big[K_b\delta^{-1}e^{C|t|} + \frac{1}{2}C_b e^{C_b|t|}\Big](\mathcal{N}+1)(\mathcal{N}/N)^{2b}, \\ \pm i[\mathcal{N},\mathcal{E}_{N,t}] \leq\; &\delta\mathcal{V}_N + N^{-\beta/2}\mathcal{K} + Ce^{C|t|}\max(N^{-\alpha},\delta^{-1})(\mathcal{N}+1) \\ &+ K_b e^{C|t|}\max(\delta,\delta^{-1})\frac{(\mathcal{N}+1)^2}{N} + K_b e^{C|t|}(\mathcal{N}+1)(\mathcal{N}/N)^{2b}, \\ \pm\partial_t\mathcal{E}_{N,t} \leq\; &\delta\mathcal{V}_N + N^{-\beta/2}\mathcal{K} + Ce^{Ct}\max(N^{-\alpha},\delta^{-1})(\mathcal{N}+1) \\ &+ K_b e^{C|t|}\max(\delta,\delta^{-1})\frac{(\mathcal{N}+1)^2}{N} + K_b e^{C|t|}(\mathcal{N}+1)(\mathcal{N}/N)^{2b} \end{aligned} \tag{5.59}$$

for all $\delta > 0$, for all $t \in \mathbb{R}\backslash\{0\}$ and for all choices of the constant C_b in the definition of $\mathcal{G}_{N,t}$ (recall that $b \in \mathbb{N}$ and C_b enter $\mathcal{G}_{N,t}$ through the definition of $\widetilde{\mathcal{L}}_{N,t}$ in (5.50)).

As a simple corollary of Proposition 5.3.1, we can show that the expectation of the energy and the expectation and certain moments of the number of particles operator are approximately preserved along the evolution generated by $\mathcal{G}_{N,t}$; this bound will play an important role in the rest of our analysis (in particular, in the proof of Lemma 5.4.2 below).

Corollary 5.3.2. *Assume Hypothesis A holds true. Let $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ with $\|\xi_N\| \leq 1$ and such that*

$$\left\langle \xi_N, \left[\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}\right] \xi_N \right\rangle \leq C \tag{5.60}$$

uniformly in N (where $b \in \mathbb{N}$ is the parameter entering the definition of $\mathcal{G}_{N,t}$ through (5.50)). Let $\xi_{N,t}$ be the solution of (5.55) and $\xi_{2,N,t} = \mathcal{U}_{2,N}(t;0)\xi_N$ with the quadratic dynamics $\mathcal{U}_{2,N}$ defined in (5.25). Then, for every $b \in \mathbb{N}$ and for sufficiently large $C_b > 0$, there exists a constant $C > 0$ such that

$$\begin{aligned}
\left\langle \xi_{2,N,t}, \left[\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}\right] \xi_{2,N,t} \right\rangle &\leq C \exp(C \exp(C|t|)) \\
\langle \xi_{N,t}, \left[\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}\right] \xi_{N,t} \rangle &\leq C \exp(C \exp(C|t|))
\end{aligned}$$

for all $t \in \mathbb{R}$.

Proof. From (5.58) and (5.59) with $\delta = 1/2$ we find that, if $C_b > 0$ is large enough,

$$\begin{aligned}
\mathcal{G}_{N,t} &\geq \eta_N(t) + \frac{1}{2}\mathcal{H}_N - Ce^{C|t|}(\mathcal{N}+1) + \frac{1}{4}C_b e^{C_b|t|}\mathcal{N}(\mathcal{N}/N)^{2b} \\
\mathcal{G}_{N,t} &\leq \eta_N(t) + 2\mathcal{H}_N + Ce^{C|t|}(\mathcal{N}+1) + 2C_b e^{C_b|t|}\mathcal{N}(\mathcal{N}/N)^{2b}
\end{aligned} \tag{5.61}$$

and also

$$\begin{aligned}
i[\mathcal{G}_{N,t}, \mathcal{N}] &\leq Ce^{C|t|}(\mathcal{N}+1) + \mathcal{H}_N + K_b e^{C|t|}\mathcal{N}(\mathcal{N}/N)^{2b} \\
&\leq Ce^{C|t|}(\mathcal{G}_{N,t} - \eta_N(t)) + Ce^{C|t|}(\mathcal{N}+1) \\
\partial_t(\mathcal{G}_{N,t} - \eta_N(t)) &\leq Ce^{C|t|}(\mathcal{N}+1) + \mathcal{H}_N + K_b e^{C|t|}\mathcal{N}(\mathcal{N}/N)^{2b} \\
&\leq Ce^{C|t|}(\mathcal{G}_{N,t} - \eta_N(t)) + Ce^{C|t|}(\mathcal{N}+1)
\end{aligned} \tag{5.62}$$

We have, for any $t > 0$,

$$\begin{aligned}
&\partial_t \langle \xi_{N,t}, (\mathcal{G}_{N,t} - \eta_N(t) + Ce^{Ct}\mathcal{N})\xi_{N,t}\rangle \\
&\qquad = Ce^{Ct}\langle \xi_{N,t}, i[\mathcal{G}_{N,t}, \mathcal{N}]\xi_{N,t}\rangle + \left\langle \xi_{N,t}, \left(\partial_t\left(\mathcal{G}_{N,t} - \eta_N(t)\right) + C^2 e^{Ct}\mathcal{N}\right)\xi_{N,t}\right\rangle.
\end{aligned}$$

Thus, from (5.62),

$$\begin{aligned}
&\partial_t \langle \xi_{N,t}, (\mathcal{G}_{N,t} - \eta_N(t) + Ce^{Ct}\mathcal{N})\xi_{N,t}\rangle \\
&\qquad \leq \widetilde{C}\exp(\widetilde{C}t)\langle \xi_{N,t}, (\mathcal{G}_{N,t} - \eta_N(t) + Ce^{Ct}(\mathcal{N}+1))\xi_{N,t}\rangle
\end{aligned}$$

for a sufficiently large constant $\widetilde{C} > 0$. Grönwall's lemma yields

$$\begin{aligned}&\langle \xi_{N,t}, (\mathcal{G}_{N,t} - \eta_N(t) + Ce^{Ct}\mathcal{N})\xi_{N,t}\rangle \\ &\qquad \leq \widetilde{C} \exp(\widetilde{C} \exp(\widetilde{C}t))\langle \xi_N, (\mathcal{G}_{N,0} - \eta_N(t) + C(\mathcal{N} + 1))\xi_N\rangle.\end{aligned}$$

From (5.61), we conclude that, for a sufficiently large constant $C > 0$,

$$\begin{aligned}&\langle \xi_{N,t}, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b})\xi_{N,t}\rangle \\ &\qquad \leq C \exp(C \exp(Ct))\langle \xi_N, (\mathcal{H}_N + \mathcal{N} + 1 + \mathcal{N}(\mathcal{N}/N)^{2b})\xi_N\rangle.\end{aligned}$$

The case $t < 0$ can be treated analogously. To obtain the estimates for $\xi_{2,N,t}$ we follow exactly the same strategy, with generator $\mathcal{G}_{N,t}$ replaced by $\mathcal{G}_{2,N,t}$. □

An important ingredient in the proof of Proposition 5.3.1 is the following result, whose proof can be found, for example, in [20]; it controls the growth of moments of the number of particles operator under the action of the Bogoliubov transformation $T_{N,t}$.

Proposition 5.3.3. *Assume Hypothesis A holds true and let $T_{N,t}$ denote the Bogoliubov transformation defined in (5.22). Then, for every fixed $k \in \mathbb{N}$ and $\delta > 0$, there exists $C > 0$ such that*

$$\pm (T_{N,t}\mathcal{N}^k T_{N,t}^* - \mathcal{N}^k) \leq \delta \mathcal{N}^k + C. \tag{5.63}$$

Remark that (5.63) requires smallness of the parameter $\ell > 0$ in (5.18) (an assumption that is included in Hypothesis A). With no assumption on the size of $\ell > 0$, (5.63) remains true, but only for $\delta > 0$ large enough.

To show Proposition 5.3.1, we are going to consider first a simplified version of the generator $\mathcal{G}_{N,t}$, given by

$$\mathcal{G}^{\mathrm{c}}_{N,t} = (i\partial_t T_{N,t})T_{N,t}^* + T_{N,t}\mathcal{L}^{\mathrm{c}}_{N,t}T_{N,t}^*. \tag{5.64}$$

with $\mathcal{L}^{\mathrm{c}}_{N,t}$ given by

$$\begin{aligned}\mathcal{L}^{\mathrm{c}}_{N,t} =& \frac{N+1}{2}\left\langle \varphi_{N,t}, [V_N(1-2f_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\right\rangle - \mu_N(t) \\ &+ \left[\sqrt{N} a^*(Q_{N,t}[V_N\omega_N * |\varphi_{N,t}|^2]\varphi_{N,t}) + \mathrm{h.c.}\right] \\ &+ \mathrm{d}\Gamma\Big(-\Delta + (V_N f_N) * |\varphi_{N,t}|^2 + K_{1,N,t} - \mu_{N,t}\Big) \\ &+ \left[\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, K_{2,N,t}(x,y) a_x^* a_y^* + \mathrm{h.c.}\right] \\ &+ \Bigg[\frac{1}{\sqrt{N}}\int dxdydx'dy'\, (Q_{N,t}\otimes Q_{N,t}V_N Q_{N,t}\otimes 1)(x,y;x',y')\varphi_{N,t}(y') a_x^* a_y^* a_{x'} \\ &\qquad\qquad + \mathrm{h.c.}\Bigg] \\ &+ \frac{1}{2N}\int dxdydx'dy'\, (Q_{N,t}\otimes Q_{N,t}V_N Q_{N,t}\otimes Q_{N,t})(x,y;x',y') a_x^* a_y^* a_{x'} a_{y'}.\end{aligned} \tag{5.65}$$

The reason for considering first the generator $\mathcal{G}^{\rm c}_{N,t}$ is the fact that this is essentially the operator generating the fluctuation dynamics studied in [16] for approximately coherent initial data. The only difference is the fact that, here, we always project onto the orthogonal complement of $\varphi_{N,t}$. The presence of the projection Q_t, however, does not substantially affect the analysis of [16]. With only small and local modifications of the proof of [16, Theorem 3.1], we obtain the following proposition.

Proposition 5.3.4. *Assume Hypothesis A holds true. Let $\mathcal{G}^{\rm c}_{N,t}$ be as defined in (5.64). Then, we have*

$$\mathcal{G}^{c}_{N,t} = \mathcal{G}_{2,N,t} + \mathcal{V}_N + \mathcal{E}^{c}_{N,t} \tag{5.66}$$

where the quadratic generator $\mathcal{G}_{2,N,t}$ is defined in (5.26) *and satisfies the estimates (5.58) and where there exists a constant $C > 0$ such that the error operator $\mathcal{E}^{\rm c}_{N,t}$ satisfies*

$$\begin{aligned}
\pm \mathcal{E}^{c}_{N,t} &\leq \delta \mathcal{V}_N + N^{-\beta/2}\mathcal{K} + Ce^{C|t|}\max(N^{-\alpha},\delta^{-1})(\mathcal{N}+1) \\
&\quad + Ce^{C|t|}\max(\delta,\delta^{-1})(\mathcal{N}+1)^2/N, \\
\pm i[\mathcal{N},\mathcal{E}^{c}_{N,t}] &\leq \delta \mathcal{V}_N + N^{-\beta/2}\mathcal{K} + Ce^{C|t|}\max(N^{-\alpha},\delta^{-1})(\mathcal{N}+1) \\
&\quad + Ce^{C|t|}\max(\delta,\delta^{-1})(\mathcal{N}+1)^2/N, \\
\pm \partial_t \mathcal{E}^{c}_{N,t} &\leq \delta \mathcal{V}_N + N^{-\beta/2}\mathcal{K} + Ce^{C|t|}\max(N^{-\alpha},\delta^{-1})(\mathcal{N}+1) \\
&\quad + Ce^{C|t|}\max(\delta,\delta^{-1})(\mathcal{N}+1)^2/N
\end{aligned} \tag{5.67}$$

for all $\delta > 0$ and $t \in \mathbb{R}$.

Observe that, in [16, Theorem 3.1], the operators $\mathcal{K}^2$ and $\mathcal{N}^2$ (the square of the kinetic energy and of the number of particles operators) are also used to control the error operator $\mathcal{E}^{\rm c}_{N,t}$ (see, in particular, [16, Eq. (3.3)]). In (5.67), these operators do not appear; instead, we make use of the potential energy $\mathcal{V}_N$ (which will be later bounded, on sectors with small number of particles, by the kinetic energy operator; see (5.73)).

Using Proposition 5.3.4, we can proceed with the proof of Proposition 5.3.1, where we only have to control the contributions to $\mathcal{G}_{N,t}$ arising from the difference $\widetilde{\mathcal{L}}_{N,t} - \mathcal{L}^{\rm c}_{N,t}$.

Proof of Proposition 5.3.1. From the definitions (5.56) and (5.64) we have

$$\mathcal{E}_{N,t} = T_{N,t}\left(\widetilde{\mathcal{L}}_{N,t} - \mathcal{L}^{\rm c}_{N,t}\right)T^*_{N,t} - C_b e^{C_b|t|}\mathcal{N}(\mathcal{N}/N)^{2b} + \mathcal{E}^{\rm c}_{N,t} \tag{5.68}$$

We already know from Proposition 5.3.4 that $\mathcal{E}^{\rm c}_{N,t}$ satisfies the desired bounds. So, we focus on the first two terms on the r.h.s. of (5.68). Comparing (5.50) with (5.65), we conclude that

$$T_{N,t}\left(\widetilde{\mathcal{L}}_{N,t} - \mathcal{L}^{\rm c}_{N,t}\right)T^*_{N,t} - C_b e^{C_b|t|}\mathcal{N}(\mathcal{N}/N)^{2b} = \sum_{j=1}^{7} B_j$$

with

$$
\begin{aligned}
B_1 &= \frac{1}{2}\left\langle \varphi_{N,t}, [V_N * |\varphi_{N,t}|^2]\varphi_{N,t}\right\rangle T_{N,t}\frac{\mathcal{N}(\mathcal{N}+1)}{N}T^*_{N,t} \\
B_2 &= T_{N,t}(\mathcal{L}^{(1)}_{N,t} + \mathcal{L}^{(3)}_{N,t})\,(G_b(\mathcal{N}/N)-1)\,T^*_{N,t} + \text{h.c.} \\
B_3 &= -\,T_{N,t}a^*(Q_{N,t}[V_N * |\varphi_{N,t}|^2]\varphi_{N,t})\frac{\mathcal{N}}{\sqrt{N}}G_b(\mathcal{N}/N)T^*_{N,t} + \text{h.c.} \\
B_4 &= T_{N,t}\mathrm{d}\Gamma\Big(Q_{N,t}(V_N\omega_N * |\varphi_{N,t}|^2)Q_{N,t}\Big)T^*_{N,t} \\
B_5 &= -\,T_{N,t}\mathrm{d}\Gamma\Big(Q_{N,t}(V_N * |\varphi_{N,t}|^2)Q_{N,t} + K_{1,N,t}\Big)\frac{\mathcal{N}}{N}T^*_{N,t} \\
B_6 &= -\,\frac{1}{2}\,T_{N,t}\int dxdy\; K_{2,N,t}(x,y)a^*_x a^*_y\frac{\mathcal{N}}{N}T^*_{N,t} + \text{h.c.} \\
B_7 &= C_b e^{C_b|t|}\left(T_{N,t}\mathcal{N}(\mathcal{N}/N)^{2b}T^*_{N,t} - \mathcal{N}(\mathcal{N}/N)^{2b}\right)
\end{aligned}
$$

where we introduced the notation

$$
\begin{aligned}
\mathcal{L}^{(1)}_{N,t} &= \sqrt{N}a^*(Q_{N,t}[V_N\omega_N * |\varphi_{N,t}|^2]\varphi_{N,t}) + \text{h.c.} \\
\mathcal{L}^{(3)}_{N,t} &= \frac{1}{\sqrt{N}}\int dxdydx'dy'\,(Q_{N,t}\otimes Q_{N,t}V_N Q_{N,t}\otimes 1)(x,y;x',y')\varphi_{N,t}(y')a^*_x a^*_y a_{x'} + \text{h.c.}
\end{aligned}
$$

Next, we control the operators $B_1,\dots,B_7$, one after the other.

Bound for B_1: From Proposition 5.3.3 and (5.21), we find immediately

$$
0 \le B_1 \le C(\mathcal{N}+1)^2/N\,.
$$

Bound for B_2: To bound the expectation of B_2, we write

$$
B_2 = \Big[T_{N,t}\mathcal{L}^{(1)}_{N,t}T^*_{N,t} + T_{N,t}\mathcal{L}^{(3)}_{N,t})T^*_{N,t}\Big]\,T_{N,t}\,(G_b(\mathcal{N}/N)-1)\,T^*_{N,t} \tag{5.69}
$$

The operator in the parenthesis can be computed as in [16, Section 3]. The most singular contribution is the cubic term

$$
\frac{1}{\sqrt{N}}\int dxdy\,V_N(x-y)a^*_x a^*_y a_x\varphi_{N,t}(y)
$$

Inserted in (5.69), it produces an operator, let us denote it by $\widetilde{B}_2$, whose expectation can be bounded by

$$
\begin{aligned}
|\langle\xi,\widetilde{B}_2\xi\rangle| &= \Big|\frac{1}{\sqrt{N}}\int dxdyV_N(x-y)\varphi_{N,t}(y)\langle\xi, a^*_x a^*_y a_x T_{N,t}\,(G_b(\mathcal{N}/N)-1)\,T^*_{N,t}\xi\rangle\Big| \\
&\le \frac{1}{\sqrt{N}}\int dxdyV_N(x-y)|\varphi_{N,t}(y)|\|a_x a_y\xi\|\|a_x T_{N,t}\,(G_b(\mathcal{N}/N)-1)\,T^*_{N,t}\xi\| \\
&\le \frac{\delta}{2N}\int dxdyV_N(x-y)\|a_x a_y\xi\|^2 \\
&\quad + C\delta^{-1}e^{C|t|}\int dxdyV_N(x-y)\|a_x T_{N,t}\,(G_b(\mathcal{N}/N)-1)\,T^*_{N,t}\xi\|^2 \\
&\le \delta\langle\xi,\mathcal{V}_N\xi\rangle + K_b\delta^{-1}e^{C|t|}\left\langle\xi,(\mathcal{N}+1)\left[((\mathcal{N}+1)/N)^2 + ((\mathcal{N}+1)/N)^{2b}\right]\xi\right\rangle
\end{aligned}
$$

for any $\delta > 0$ and for an appropriate constant K_b depending on the choice of b. Here we used Proposition 5.3.3. Other terms contributing to B_2 can be bounded in a similar fashion. We conclude that

$$\pm B_2 \leq \delta \mathcal{V}_N + K_b \delta^{-1} e^{C|t|} (\mathcal{N}+1) \left[((\mathcal{N}+1)/N)^2 + ((\mathcal{N}+1)/N)^{2b} \right]$$

Bound for B_3: Let us now deal with B_3. Since $\|Q_{N,t}[V_N * |\varphi_{N,t}|^2]\varphi_{N,t}\| \leq C \exp(C|t|)$, we obtain, with Cauchy-Schwarz,

$$\pm B_3 \leq K_b \delta e^{C|t|} \frac{(\mathcal{N}+1)^2}{N} + K_b e^{C|t|} \delta^{-1} \mathcal{N} + K_b e^{C|t|} \delta^{-1} (\mathcal{N}+1)(\mathcal{N}/N)^{2b}$$

for every $\delta > 0$ and for an appropriate constant $K_b > 0$ depending on $b \in \mathbb{N}$.
Bound for B_4: From (5.19), we have

$$\|Q_{N,t}(V_N \omega_N * |\varphi_{N,t}|^2) Q_{N,t}\|_\infty \leq C N^{\beta-1} e^{C|t|}$$

Hence, with Proposition 5.3.3, we find

$$\pm B_4 \leq C N^{\beta-1} \leq C N^{\beta-1} (\mathcal{N}+1)$$

Bound for B_5: Similarly, since $\|K_{1,N,t}\| = \|Q_{N,t} \widetilde{K}_{1,N,t} Q_{N,t}\| \leq \|\widetilde{K}_{1,N,t}\|$,

$$\begin{aligned} \|\widetilde{K}_{1,N,t}\| &= \sup_{\|f\|_{L^2}=1} \left| \int \overline{f(x)} \varphi_{N,t}(x) V_N(x-y) \overline{\varphi_{N,t}(y)} f(y) \, \mathrm{d}x \, \mathrm{d}y \right| \\ &\leq \sup_{\|f\|_{L^2}=1} \|\varphi_{N,t}\|_{L^\infty}^2 \int \frac{|f(x)|^2 + |f(y)|^2}{2} V_N(x-y) \, \mathrm{d}x \, \mathrm{d}y \leq \; C e^{C|t|} \end{aligned}$$

and $\|Q_{N,t}(V_N * |\varphi_{N,t}|^2) Q_{N,t}\|_\infty \leq C \exp(C|t|)$, we obtain with Proposition 5.3.3 that

$$\pm B_5 \leq C e^{C|t|} \frac{(\mathcal{N}+1)^2}{N}.$$

Bound for B_6: Proceeding as in [16, Prop. 3.5] we find

$$\begin{aligned} B_6 = & -\frac{1}{2N} \int dxdy \; K_{2,N,t}(x;y) \langle \mathrm{sh}_x, \mathrm{ch}_y \rangle T_{N,t} \mathcal{N} T_{N,t}^* \\ & + \frac{1}{2N} \int dxdy \; V_N(x-y) \varphi_{N,t}(x) \varphi_{N,t}(y) a_x^* a_y^* T_{N,t} \mathcal{N} T_{N,t}^* \\ & + \frac{1}{N} \, \mathcal{E}_{6,N} T_{N,t}^* \mathcal{N} T_{N,t} + \text{h.c.} \end{aligned} \tag{5.70}$$

where the operator $\mathcal{E}_{6,N}$ is such that

$$\mathcal{E}_{6,N}^2 \leq C e^{C|t|} (\mathcal{N}+1)^2. \tag{5.71}$$

Since

$$\left| \int dxdy \; K_{2,N,t}(x;y) \langle \mathrm{sh}_x, \mathrm{ch}_y \rangle \right| \leq C e^{C|t|} \, ,$$

the expectation of the first term on the r.h.s. of (5.70) is bounded, with Proposition 5.3.3, by

$$\Big|\frac{1}{2N}\int dxdy\; K_{2,N,t}(x;y)\langle \mathrm{sh}_x, \mathrm{ch}_y\rangle \,\langle \xi, T_{N,t}\mathcal{N}T^*_{N,t}\xi\rangle\Big| \leq CN^{-1}\langle \xi, (\mathcal{N}+1)\xi\rangle$$

The expectation of the second term on the r.h.s. of (5.70) can be controlled by

$$\begin{aligned}
&\Big|\frac{1}{2N}\int dxdy\; V_N(x-y)\varphi_{N,t}(x)\varphi_{N,t}(y)\,\langle \xi, a^*_x a^*_y T_{N,t}\mathcal{N}T^*_{N,t}\xi\rangle\Big| \\
&\qquad \leq \frac{1}{2N}\int dxdy\; V_N(x-y)|\varphi_{N,t}(x)|\,|\varphi_{N,t}(y)|\|a_x a_y\xi\|\,\|T_{N,t}\mathcal{N}T^*_{N,t}\xi\| \\
&\qquad \leq \frac{1}{2N}\int dxdy\; V_N(x-y)\left[\delta\|a_x a_y\xi\|^2 + \delta^{-1}|\varphi_{N,t}(x)|^2\,|\varphi_{N,t}(y)|^2\|\mathcal{N}T^*_{N,t}\xi\|\right] \\
&\qquad \leq \delta\langle \xi, \mathcal{V}_N\xi\rangle + C\delta^{-1}N^{-1}e^{C|t|}\langle \xi, (\mathcal{N}+1)^2\xi\rangle
\end{aligned}$$

where we used once again Proposition 5.3.3. As for the last term on the r.h.s. of (5.70), it can be estimated using (5.71) and Proposition 5.3.3. We conclude that

$$\pm B_6 \leq \delta\mathcal{V}_N + C\delta^{-1}e^{C|t|}\frac{(\mathcal{N}+1)^2}{N}$$

for any $\delta > 0$.
Bound for B_7: with Proposition 5.3.3 we find

$$\pm B_7 \leq \frac{1}{2}C_b e^{C_b|t|}(\mathcal{N}+1)((\mathcal{N}+1)/N))^{2b}$$

if $\ell > 0$ in (5.18) is chosen sufficiently small.

Combining all these bounds with the bounds (5.67) for the error term $\mathcal{E}^c_{N,t}$, we obtain the first estimate in (5.59) for the error term $\mathcal{E}_{N,t}$.

The bound for the commutator $i[\mathcal{N}, \mathcal{E}_{N,t}]$ follows from the observation that the commutator with $\mathcal{N}$ of every monomial A in creation and annihilation operators appearing in $\mathcal{E}_{N,t}$ is given by λA, where $\lambda \in \{0, \pm 1, \pm 2, \pm 3\}$. Hence, $[i\mathcal{N}, \mathcal{E}_{N,t}]$ can be bounded exactly like we did for $\mathcal{E}_{N,t}$.

Similarly, the bound for the time-derivative $\partial_t\mathcal{E}_{N,t}$ is established by noticing that the time derivative of every monomial A contributing to $\mathcal{E}_{N,t}$ is the sum of finitely many terms having again the same form of A, just with one factor $\varphi_{N,t}$ replaced by the time derivative $\partial_t\varphi_{N,t}$ (the generator $\mathcal{G}_{N,t}$ only depends on time through the solution $\varphi_{N,t}$ of the nonlinear Hartree equation (5.20)). Therefore, to bound $\partial_t\mathcal{E}_{N,t}$ we proceed exactly as we did for $\mathcal{E}_{N,t}$, with the only difference that, sometimes, we have to use the bound for $\partial_t\varphi_{N,t}$ in (5.21) rather than the corresponding bound for $\varphi_{N,t}$. □

With Proposition 5.3.1, we are now ready to prove Lemma 5.2.2.

Proof of Lemma 5.2.2. Let $\alpha = \min(\beta, 1-\beta)/2$ and $M = N^\alpha$. We have

$$\|\widetilde{\xi}_{N,t} - \xi_{2,N,t}\|^2 = 2\left[1 - \mathrm{Re}\,\langle \xi_{N,t}, \xi_{2,N,t}\rangle\right]$$

and we decompose, with $M/2 \leq m \leq M$,

$$\begin{aligned}\langle \xi_{N,t}, \xi_{2,N,t}\rangle &= \langle \xi_{N,t}, 1^{\leq m}\xi_{2,N,t}\rangle + \langle \xi_{N,t}, 1^{>m}\xi_{2,N,t}\rangle \\ &= \frac{2}{M}\sum_{m=M/2+1}^{M} \left[\langle \xi_{N,t}, 1^{\leq m}\xi_{2,N,t}\rangle + \langle \xi_{N,t}, 1^{>m}\xi_{2,N,t}\rangle\right].\end{aligned}$$

where we used the notation $1^{\leq m} = 1(\mathcal{N} \leq m)$ and $1^{>m} = 1 - 1^{\leq m}$.
Many-particle sectors. From Cauchy-Schwarz and the bounds in Corollary 5.3.2, we find

$$\begin{aligned}|\langle \xi_{N,t}, 1^{>m}\xi_{2,N,t}\rangle| &\leq \|1^{>m}\xi_{N,t}\|.\|1^{>m}\xi_{2,N,t}\| \\ &\leq \langle \xi_{N,t}, (\mathcal{N}/m)\xi_{N,t}\rangle^{1/2}\langle \xi_{2,N,t}, (\mathcal{N}/m)\xi_{2,N,t}\rangle^{1/2} \\ &\leq CM^{-1}\exp(C\exp(C|t|)).\end{aligned}$$

for a constant $C > 0$ depending on b. Averaging over $m \in [M/2+1, M]$, we conclude that

$$\Big|\frac{2}{M}\sum_{m=M/2+1}^{M}\langle \xi_{N,t}, 1^{>m}\xi_{2,N,t}\rangle\Big| \leq CN^{-\alpha}\exp(C\exp(C|t|)). \tag{5.72}$$

Few-particle sectors. From the Schrödinger equations for $\xi_{N,t}$ and $\xi_{2,N,t}$, we find

$$\operatorname{Re}\frac{d}{dt}\langle \xi_{N,t}, 1^{\leq m}\xi_{2,N,t}\rangle = \operatorname{Im}\Big\langle \xi_{N,t}, \Big[(\mathcal{G}_{N,t} - \mathcal{G}_{2,N,t})1^{\leq m} + [\mathcal{G}_{2,N,t}, 1^{\leq m}]\Big]\xi_{2,N,t}\Big\rangle.$$

Using Proposition 5.3.1, in particular (5.59) with $\delta = N^{\alpha}$, we obtain

$$\begin{aligned}&\pm(\mathcal{G}_{N,t} - \mathcal{G}_{2,N,t}) \\ &\quad\leq \Big[N^{\alpha}\mathcal{V}_N + N^{\alpha}(\mathcal{N}+1)^2/N + (\mathcal{N}+1)(\mathcal{N}/N)^{2b} + N^{-\alpha}(\mathcal{K}+\mathcal{N}+1)\Big]C\exp(Ct)\end{aligned}$$

for a constant $C > 0$ depending on b. We choose $b \in \mathbb{N}$ large enough so that $2b(\alpha-1) < -\alpha$ (i.e. $b > \alpha/(2(1-\alpha))$). Then, using the simple operator estimate

$$0 \leq \mathcal{V}_N \leq CN^{\beta-1}\mathcal{K}\mathcal{N} \tag{5.73}$$

which follows by quantization of the two-body estimate $V_N(x-y) \leq CN^{\beta}(-\Delta_x - \Delta_y)$, projecting to the sector with $\mathcal{N} \leq m+2$ (where $m \leq N^{\alpha}$), and using also the inequality $2\alpha - 1 < -\alpha$ (since, by definition, $\alpha < 1/4$) we find

$$\pm 1^{\leq m+2}(\mathcal{G}_{N,t} - \mathcal{G}_{2,N,t})1^{\leq m+2} \leq CN^{-\alpha}(\mathcal{K}+\mathcal{N}+1)\exp(C|t|). \tag{5.74}$$

Since $\mathcal{G}_{N,t} - \mathcal{G}_{2,N,t}$ contains terms with at most two creation operators, we have the obvious identity

$$(\mathcal{G}_{N,t} - \mathcal{G}_{2,N,t})1^{\leq m} = 1^{\leq m+2}(\mathcal{G}_{N,t} - \mathcal{G}_{2,N,t})1^{\leq m+2}1^{\leq m}.$$

From (5.74) we find, by Cauchy-Schwarz,

$$\begin{aligned}&|\langle \xi_{N,t}, (\mathcal{G}_{N,t} - \mathcal{G}_{2,N,t}) 1^{\leq m} \xi_{2,N,t}\rangle| \\ &= |\langle \xi_{N,t}, 1^{\leq m+2} (\mathcal{G}_{N,t} - \mathcal{G}_{2,N,t}) 1^{\leq m+2} 1^{\leq m} \xi_{2,N,t}\rangle| \\ &\leq CN^{-\alpha} \exp(C|t|) \langle \xi_{N,t}, (\mathcal{K} + \mathcal{N} + 1)\xi_{N,t}\rangle^{1/2} \langle 1^{\leq m} \xi_{2,N,t}, (\mathcal{K} + \mathcal{N} + 1) \xi_{2,N,t}\rangle^{1/2}.\end{aligned} \tag{5.75}$$

Inserting the energy estimates in Corollary 5.3.2, we find that

$$|\langle \xi_{N,t}, (\mathcal{G}_{N,t} - \mathcal{G}_{2,N,t}) 1^{\leq m} \xi_{2,N,t}\rangle| \leq CN^{-\alpha} \exp(\exp(C|t|)).$$

In (5.75), we used the fact that, if D is a self-adjoint and F a non-negative operator on a Hilbert space $\mathfrak{h}$ with $\pm D \leq F$ then, for every $\phi, \psi \in \mathfrak{h}$, we have (using the fact that $D + F \geq 0$)

$$\begin{aligned}|\langle \phi, D\psi\rangle| &\leq |\langle \phi, (D+F)\psi\rangle| + |\langle \phi, F\psi\rangle| \\ &\leq \kappa \langle \phi, (D+F)\phi\rangle + \kappa^{-1} \langle \psi, (D+F)\psi\rangle + \kappa \langle \phi, F\phi\rangle + \kappa^{-1} \langle \psi, F\psi\rangle \\ &\leq 3\kappa \langle \phi, F\phi\rangle + 3\kappa^{-1} \langle \psi, F\psi\rangle\end{aligned}$$

for every $\kappa > 0$. With $\kappa = \langle \psi, F\psi\rangle^{1/2} \langle \phi, F\phi\rangle^{-1/2}$, we find

$$|\langle \phi, D\psi\rangle| \leq 6 \langle \phi, F\phi\rangle^{1/2} \langle \psi, F\psi\rangle^{1/2}$$

Next, we turn to the commutator $[\mathcal{G}_{2,N,t}, 1^{\leq m}]$. We observe that

$$[\mathcal{G}_{2,N,t}, 1^{\leq m}] = 1^{>m} \mathcal{G}_{2,N,t} 1^{\leq m} - 1^{\leq m} \mathcal{G}_{2,N,t} 1^{>m}. \tag{5.76}$$

Consider the first term on the r.h.s. of (5.76). Only terms in $\mathcal{G}_{2,N,t}$ with two creation operators give a non-vanishing contribution; hence,

$$\begin{aligned}&\langle \xi_1, 1^{>m} \mathcal{G}_{2,N,t} 1^{\leq m} \xi_2\rangle \\ &\qquad = \left\langle \xi_1, \left[\chi(\mathcal{N} = m+2) \mathcal{G}_{2,N,t} \chi(\mathcal{N} = m) + \chi(\mathcal{N} = m+1) \mathcal{G}_{2,N,t} \chi(\mathcal{N} = m)\right] \xi_2 \right\rangle\end{aligned}$$

Estimating terms in $\mathcal{G}_{2,N,t}$ with two creation operators similarly as in Proposition 5.3.4, we obtain

$$\begin{aligned}&\Big| \frac{2}{M} \sum_{m=M/2+1}^{M} \langle \xi_{N,t}, i[\mathcal{G}_{2,N,t}, 1^{\leq m}] \xi_{2,N,t}\rangle \Big| \\ &\leq CM^{-1} \exp(C|t|) \langle \xi_{N,t}, (\mathcal{N}+1)\xi_{N,t}\rangle^{1/2} \langle \xi_{2,N,t}, (\mathcal{N}+1)\xi_{2,N,t}\rangle^{1/2} \\ &\leq CN^{-\alpha} \exp(C \exp(C|t|)).\end{aligned}$$

where we used Corollary 5.3.2 and the choice $M = N^{\alpha}$. In summary, we have proved that

$$\Big| \frac{2}{M} \sum_{m=M/2+1}^{M} \frac{d}{dt} \langle \xi_{N,t}, 1^{\leq m} \xi_{2,N,t}\rangle \Big| \leq CN^{-\alpha} \exp(C \exp(C|t|)).$$

Consequently,

$$\begin{aligned}
&\mathrm{Re}\,\frac{2}{M}\sum_{m=M/2+1}^{M}\langle\xi_{N,t},1^{\leq m}\xi_{2,N,t}\rangle\\
&\geq \mathrm{Re}\,\frac{2}{M}\sum_{m=M/2+1}^{M}\langle\xi_{N,0},1^{\leq m}\xi_{2,N,0}\rangle - CN^{-\alpha}\exp(C\exp(C|t|)).
\end{aligned}$$

With the assumption (5.53) on the initial datum $\xi_{N,0}=\xi_{2,N,0}=\xi_N$, we find

$$\begin{aligned}
\langle\xi_{N,0},1^{\leq m}\xi_{2,N,0}\rangle &= \|1^{\leq m}\xi_N\|^2 = 1-\|1^{>m}\xi_N\|^2\\
&\geq 1-\langle\xi_N,(\mathcal{N}/m)\xi_N\rangle \geq 1-CM^{-1} = 1-CN^{-\alpha}.
\end{aligned}$$

Thus

$$\mathrm{Re}\,\frac{2}{M}\sum_{m=M/2+1}^{M}\langle\xi_{N,t},1^{\leq m}\xi_{2,N,t}\rangle \geq 1-CN^{-\alpha}\exp(C\exp(C|t|)).$$

Combining the latter bound with (5.72), we arrive at

$$\mathrm{Re}\,\langle\xi_{N,t},\xi_{2,N,t}\rangle \geq 1-CN^{-\alpha}\exp(C\exp(C|t|)).$$

We conclude that

$$\|\xi_{N,t}-\xi_{2,N,t}\|^2 \leq 2(1-\mathrm{Re}\,\langle\xi_{N,t},\xi_{2,N,t}\rangle) \leq CN^{-\alpha}\exp(C\exp(C|t|)).$$

□

The localization argument used in the above proof is similar to that in [77, **?**]. The main idea is to employ the operator inequality (5.73) in the sector of few particles. This argument will be used again below.

5.4 Approximation of fluctuation dynamics

In this section, we show Lemma 5.2.1. To this end, we will make use of the following energy estimates.

Lemma 5.4.1. *Assume Hypothesis A holds true. Let $\xi_N \in \mathcal{F}_\perp$ with $\|\xi_N\|\leq 1$ and*

$$\langle\xi_N,(\mathcal{H}_N+\mathcal{N}+\mathcal{N}^2/N)\xi_N\rangle \leq C, \tag{5.77}$$

uniformly in N. Let $\Phi_{N,t}$ be as defined in (5.45). *Then there exists a constant $C>0$ such that*

$$\langle\Phi_{N,t},(\mathcal{H}_N+\mathcal{N})\Phi_{N,t}\rangle \leq CN^{\beta}\exp(C\exp(C|t|)) \tag{5.78}$$

for all $t\in\mathbb{R}$.

Proof. We recall that $\Phi_{N,t}$ solves the Schrödinger equation (5.45) with the generator (5.46) that can be decomposed into

$$\mathcal{L}_{N,t} = C_{N,t} + \mathcal{H}_{N,t} + \mathcal{R}_{N,t}$$

with the constant part

$$C_{N,t} = \frac{N+1}{2} \left\langle \varphi_{N,t}, [V_N(1-2f_N) * |\varphi_{N,t}|^2]\varphi_{N,t} \right\rangle - \mu_{N,t}, \tag{5.79}$$

the projected Hamilton operator

$$\begin{aligned}\mathcal{H}_{N,t} &= d\Gamma(-\Delta) \\ &\quad + \frac{1}{2N} \int dxdydx'dy' \; [(Q_{N,t} \otimes Q_{N,t})V_N(Q_{N,t} \otimes Q_{N,t})]\,(x,y;x',y')a_x^* a_y^* a_{x'} a_{y'}\end{aligned}$$

and the rest

$$\mathcal{R}_{N,t} = \sum_{i=1}^{7} \mathcal{R}^i_{N,t}$$

where

$$\begin{aligned}
\mathcal{R}^1_{N,t} &= \frac{1}{2} \left\langle \varphi_{N,t}, [V_N * |\varphi_{N,t}|^2]\varphi_{N,t} \right\rangle \frac{\mathcal{N}(\mathcal{N}+1)}{N} \\
\mathcal{R}^2_{N,t} &= \Big[a^*(Q_{N,t}[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}) - a^*(Q_{N,t}[V_N * |\varphi_{N,t}|^2]\varphi_{N,t})\frac{\mathcal{N}}{N} \Big] \sqrt{N-\mathcal{N}} \\
&\quad + \text{h.c.} \\
\mathcal{R}^3_{N,t} &= \mathrm{d}\Gamma\Big((V_N f_N) * |\varphi_{N,t}|^2 + K_{1,N,t} - \mu_{N,t} \Big) + \mathrm{d}\Gamma\Big(Q_{N,t}(V_N\omega_N * |\varphi_{N,t}|^2)Q_{N,t} \Big) \\
\mathcal{R}^4_{N,t} &= -\mathrm{d}\Gamma\Big(Q_{N,t}(V_N * |\varphi_{N,t}|^2)Q_{N,t} + K_{1,N,t} \Big)\frac{\mathcal{N}}{N} \\
\mathcal{R}^5_{N,t} &= \frac{1}{2} \int \mathrm{d}x\,\mathrm{d}y\, K_{2,N,t}(x,y)a_x^* a_y^* + \text{h.c.} \\
\mathcal{R}^6_{N,t} &= \frac{1}{2} \int \mathrm{d}x\,\mathrm{d}y\, K_{2,N,t}(x,y)a_x^* a_y^* \left(\frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N} - 1 \right) + \text{h.c.} \\
\mathcal{R}^7_{N,t} &= \frac{1}{\sqrt{N}} \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y'\, (Q_{N,t} \otimes Q_{N,t}V_N Q_{N,t} \otimes 1)(x,y;x',y') \\
&\qquad\qquad \times a_x^* a_y^* a_{x'} \varphi_{N,t}(y') \sqrt{\frac{N-\mathcal{N}}{N}} + \text{h.c.}
\end{aligned} \tag{5.80}$$

The proof of Lemma 5.4.1 is divided into three steps. In the first step, we bound the rest operator $\mathcal{R}_{N,t}$, its commutator with $\mathcal{N}$ and its time derivative, through the number of particles operator $\mathcal{N}$ and the Hamiltonian $\mathcal{H}_N$. In the second step we use these bounds and, with Grönwall's Lemma, we control the expectation on the r.h.s. of (5.78) in terms of its initial value at time $t = 0$. Finally, in the third step, we control the expectation of $\mathcal{H}_N$ and $\mathcal{N}$ in the initial state $\Phi_{N,0} = T_{N,0}\xi_N$ through the expectation of the same operators in the state ξ_N, making use of the assumption (5.77).

Step 1. We claim that, for all $\delta > 0$ there exists $C > 0$ with

$$\begin{aligned}\pm\mathcal{R}_{N,t} &\leq \delta\mathcal{V}_N + Ce^{C|t|}(\mathcal{N} + N^\beta)\\ \pm i[\mathcal{R}_{N,t}, \mathcal{N}] &\leq \delta\mathcal{V}_N + C_\varepsilon e^{Ct}(\mathcal{N} + N^\beta)\\ \pm\partial_t\mathcal{R}_{N,t} &\leq \delta\mathcal{V}_N + C_\varepsilon e^{Ct}(\mathcal{N} + N^\beta).\end{aligned} \tag{5.81}$$

as operator inequality on $\mathcal{F}^{\leq N}_{\perp\varphi_{N,t}}$. We will focus on the proof of the bound for $\mathcal{R}_{N,t}$. The other two estimates in (5.81) can be shown similarly, since the commutator $i[\mathcal{R}_{N,t}, \mathcal{N}]$ and the derivative $\partial_t\mathcal{R}_{N,t}$ contain the same terms appearing in $\mathcal{R}_{N,t}$, multiplied by a constant in $\{0, \pm1, \pm2\}$ in the first case and with a factor $\varphi_{N,t}$ replaced by its derivative $\partial_t\varphi_{N,t}$ in the second case. We follow here [77, Theorem 3], where more details can be found.

Step 1.1: Since

$$\left\langle\varphi_{N,t}, [V_N * |\varphi_{N,t}|^2]\varphi_{N,t}\right\rangle \leq \|V_N\|_{L^1}\|\varphi_{N,t}\|^4_{L^4} \leq C$$

and $\mathcal{N}/N \leq 1$ on the truncated Fock space $\mathcal{F}^{\leq N}_{\perp\varphi_{N,t}}$, we have

$$0 \leq \mathcal{R}^1_{N,t} \leq C\mathcal{N}.$$

Step 1.2: We divide $\mathcal{R}^2_{N,t} = \mathcal{R}^{2,1}_{N,t} + \mathcal{R}^{2,2}_{N,t}$ with

$$\begin{aligned}\mathcal{R}^{2,1}_{N,t} &= \Big[a^*(Q_{N,t}[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t})\Big]\sqrt{N - \mathcal{N}} + \text{h.c.},\\ \mathcal{R}^{2,2}_{N,t} &= \sqrt{N}\Big[a^*(Q_{N,t}[V_N * |\varphi_{N,t}|^2]\varphi_{N,t})\frac{\mathcal{N}}{N}\Big]\sqrt{\frac{N - \mathcal{N}}{N}} + \text{h.c.}\end{aligned} \tag{5.82}$$

Using the Cauchy–Schwarz inequality, we find, for arbitrary $\xi \in \mathcal{F}^{\leq N}_{\perp\varphi_{N,t}}$,

$$\left|\langle\xi, \mathcal{R}^{2,1}_{N,t}\xi\rangle\right| \leq \sqrt{N}\left\|Q_{N,t}[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\right\|_2\|\mathcal{N}^{1/2}\xi\|\|\xi\|$$

Since, with (5.19),

$$\left\|Q_{N,t}[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\right\|_{L^2} \leq \left\|[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\right\|_{L^2} \leq CN^{\beta-1}e^{C|t|}$$

we conclude that

$$\pm\mathcal{R}^{2,1}_{N,t} \leq Ce^{C|t|}\left(N^{2\beta-1} + \mathcal{N}\right) \leq Ce^{C|t|}(N^\beta + \mathcal{N}).$$

As for the second term in (5.82), using

$$\left\| [V_N * |\varphi_{N,t}|^2]\varphi_{N,t} \right\|_{L^2} \le C\,,$$

the Cauchy–Schwarz inequality and the fact that, on $\mathcal{F}^{\le N}_{\perp\varphi_{N,t}}$, $\mathcal{N}/N \le 1$, we find hat

$$\pm\mathcal{R}^{2,2}_{N,t} \le Ce^{C|t|}\mathcal{N}.$$

Step 1.3: Recall that for an operator B on $L^2(\mathbb{R}^3)$ we have $\pm d\Gamma(B) \le \|B\|\mathcal{N}$. Since

$$\begin{aligned} \|(V_N f_N) * |\varphi_{N,t}|^2\|_{L^\infty} &\le \|\varphi_{N,t}\|^2_{L^\infty}\|V_N f_N\|_{L^1} \le Ce^{Ct} \\ |\mu_{N,t}| &= \big| \left\langle \varphi_{N,t}, [(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t} \right\rangle \big| \\ &\le CN^{\beta-1}e^{C|t|} \\ \|Q_{N,t}(V_N\omega_N) * |\varphi_{N,t}|^2 Q_{N,t}\| &\le \|(V_N\omega_N) * |\varphi_{N,t}|^2\|_{L^\infty} \\ &\le CN^{\beta-1}e^{C|t|} \end{aligned}$$

and

$$\begin{aligned} \|K_{1,N,t}\| &= \|Q_{N,t}\widetilde{K}_{1,N,t}Q_{N,t}\| \le \|\widetilde{K}_{1,N,t}\| \\ &= \sup_{\|f\|_{L^2}=1} \left| \int \overline{f(x)}\varphi_{N,t}(x)V_N(x-y)\overline{\varphi_{N,t}(y)}f(y)\,\mathrm{d}x\,\mathrm{d}y \right| \\ &\le \frac{\|\varphi_{N,t}\|^2_{L^\infty}}{2} \sup_{\|f\|_{L^2}=1} \int (|f(x)|^2 + |f(y)|^2)V_N(x-y)dxdy \le Ce^{C|t|} \end{aligned} \tag{5.83}$$

we conclude that

$$\pm\mathcal{R}^{3}_{N,t} \le Ce^{C|t|}\mathcal{N}.$$

Step 1.4: Proceeding similarly to Step 3 and using the fact that $d\Gamma(B)$ commutes with $\mathcal{N}$, we find

$$\pm\mathcal{R}^{4}_{N,t} \le Ce^{C|t|}\mathcal{N}.$$

Step 1.5: To bound the term $\mathcal{R}^5_{N,t}$ we observe that, for any $\delta > 0$,

$$\begin{aligned} &\delta \mathrm{d}\Gamma(1-\Delta) \pm \Big[\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, K_{2,N,t}(x,y)a^*_x a^*_y + \mathrm{h.c.}\Big] \\ &\qquad\qquad \ge -\frac{1}{2\delta}\left\|(1-\Delta)^{-1/2}K^*_{2,N,t}\right\|^2_{\mathrm{HS}} \ge -\frac{1}{2\delta}\left\|(1-\Delta)^{-1/2}\widetilde{K}^*_{2,N,t}\right\|^2_{\mathrm{HS}} \end{aligned}$$

from [78, Lemma 9]. Since $\widetilde{K}_{2,N,t}(x;y) = V_N(x-y)\varphi_{N,t}(x)\varphi_{N,t}(y)$, we find

$$\begin{aligned}
&\left\|(1-\Delta)^{-1/2}\widetilde{K}^*_{2,N,t}\right\|^2_{\mathrm{HS}} \\
&\qquad = \operatorname{tr}\, \widetilde{K}_{2,N,t}(1-\Delta)^{-1}\widetilde{K}^*_{2,N,t} \\
&\qquad = C\int dxdydz\, V_N(x-y)\,\frac{e^{-|y-z|}}{|y-z|}\,V_N(z-x)\,|\varphi_{N,t}(x)|^2\varphi_{N,t}(y)\varphi_{N,t}(z) \\
&\qquad \le \|\varphi_{N,t}\|^2_\infty\|\varphi_{N,t}\|^2_2\int V_N(z)\left[V_N * \frac{1}{|.|}\right](z)dz \\
&\qquad \le Ce^{C|t|}\int \frac{|\widehat{V}_N(p)|^2}{p^2}dp = Ce^{C|t|}\int \frac{|\widehat{V}(p/N^\beta)|^2}{p^2}dp \le CN^\beta e^{C|t|}\int \frac{|\widehat{V}(p)|^2}{p^2}dp \\
&\qquad \le CN^\beta e^{C|t|}
\end{aligned}$$

We obtain that, for any $\delta > 0$,

$$\pm\mathcal{R}^5_{N,t} = \pm\Big[\frac{1}{2}\int dxdy\, K_{2,N,t}(x,y)a^*_x a^*_y + \text{h.c.}\Big] \le \delta d\Gamma(1-\Delta) + C\delta^{-1}N^\beta e^{C|t|}$$

Step 1.6: To bound $\mathcal{R}^6_{N,t}$, we observe that, by Cauchy-Schwarz, we have

$$\begin{aligned}
|\langle\xi,\mathcal{R}^6_{N,t}\xi\rangle| \le{}& C\int dxdy\, V_N(x-y)|\varphi_{N,t}(x)||\varphi_{N,t}(y)|\|a_x a_y\xi\| \\
&\times\left\|\left(\frac{\sqrt{N-\mathcal{N})(N-\mathcal{N}-1)}}{N}-1\right)\xi\right\| \\
\le{}& \frac{C}{\sqrt{N}}\int dxdy\, V_N(x-y)|\varphi_{N,t}(x)||\varphi_{N,t}(y)|\|a_x a_y\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\| \\
\le{}& \delta\langle\xi,\mathcal{V}_N\xi\rangle + C\delta^{-1}\|(\mathcal{N}+1)^{1/2}\xi\|^2
\end{aligned}$$

which implies that

$$\pm\mathcal{R}^6_{N,t} \le \delta\mathcal{V}_N + C\delta^{-1}(\mathcal{N}+1)$$

Step 1.7: For $\xi \in \mathcal{F}^{\le N}_{\perp\varphi_{N,t}}$, we have, using Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|\langle\xi,\mathcal{R}^7_{N,t}\xi\rangle\right| &\le \frac{1}{\sqrt{N}}\int V_N(x-y)|\varphi_{N,t}(y)|\,\|a_x a_y\xi\|\,\left\|a_x\sqrt{\frac{N-\mathcal{N}}{N}}\xi\right\|dxdy \\
&\le \delta\langle\xi,\mathcal{V}_N\xi\rangle + C\|\varphi_{N,t}\|^2_\infty\langle\xi,\mathcal{N}\xi\rangle
\end{aligned}$$

and therefore

$$\pm\mathcal{R}^7_{N,t} \le \delta\mathcal{V}_N + C\delta^{-1}e^{C|t|}\mathcal{N}$$

Combining the results of Step 1.1 - Step 1.7, we obtain (5.81).

Step 2. There exists a constant $C > 0$ such that

$$\langle \Phi_{N,t}, (\mathcal{H}_N + \mathcal{N})\Phi_{N,t}\rangle \leq C \exp(C \exp(C|t|))\langle \Phi_{N,0}, (\mathcal{H}_N + \mathcal{N} + N^\beta)\Phi_{N,0}\rangle \tag{5.84}$$

for all $t \in \mathbb{R}$.

We focus on $t > 0$ (the case $t < 0$ can be handled similarly). We have

$$\begin{aligned}
&\partial_t \big\langle \Phi_{N,t}, \big(\mathcal{L}_{N,t} - C_{N,t} + Ce^{Ct}(\mathcal{N} + N^\beta)\big)\Phi_{N,t}\big\rangle \\
&\quad = Ce^{Ct}\langle \Phi_{N,t}, i[\mathcal{R}_{N,t}, \mathcal{N}]\Phi_{N,t}\rangle + \langle \Phi_{N,t}, (\partial_t \mathcal{R}_{N,t} + C^2 e^{Ct}(\mathcal{N} + N^\beta))\Phi_{N,t}\rangle.
\end{aligned}$$

The second and third bound in (5.81) imply that there exists a constant $\widetilde{C}$ such that

$$\begin{aligned}
&\partial_t \langle \Phi_{N,t}, (\mathcal{L}_{N,t} - C_{N,t} + Ce^{Ct}(\mathcal{N} + N^\beta))\Phi_{N,t}\rangle \\
&\quad \leq \widetilde{C} e^{\widetilde{C}t}\langle \Phi_{N,t}, (\mathcal{L}_{N,t} - C_{N,t} + Ce^{Ct}(\mathcal{N} + N^\beta))\Phi_{N,t}\rangle.
\end{aligned}$$

Grönwall's Lemma gives

$$\begin{aligned}
&\langle \Phi_{N,t}, (\mathcal{L}_{N,t} - C_{N,t} + Ce^{Ct}(\mathcal{N} + N^\beta))\Phi_{N,t}\rangle \\
&\quad \leq \widetilde{C} \exp(\widetilde{C} \exp(\widetilde{C}t))\langle \Phi_{N,0}, (\mathcal{L}_{N,0} - C_{N,0} + \mathcal{N} + N^\beta)\Phi_{N,0}\rangle
\end{aligned}$$

The first inequality in (5.81) implies (5.84).

Step 3. To finish the proof we need to show that, with the assumption

$$\langle \xi_N, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}^2/N)\xi_N\rangle \leq C\,, \tag{5.85}$$

we have

$$\langle \Phi_{N,0}, (\mathcal{H}_N + \mathcal{N})\Phi_{N,0}\rangle \leq CN^\beta. \tag{5.86}$$

To reach this goal, we observe, first of all, that

$$\begin{aligned}
\langle \Phi_{N,0}, (\mathcal{H}_N + \mathcal{N})\Phi_{N,0}\rangle &= \langle 1^{\leq N} T^*_{N,0}\xi_N, (\mathcal{H}_N + \mathcal{N})1^{\leq N} T^*_{N,0}\xi_N\rangle \\
&\leq \langle \xi_N, T_{N,0}\mathcal{H}_N T^*_{N,0}\xi_N\rangle + C
\end{aligned} \tag{5.87}$$

by Proposition 5.3.3 and (5.85). To bound the remaining expectation on the r.h.s. of (5.87), we compute (see [16, Section 3, in particular Prop. 3.3 and Prop. 3.11])

$$\begin{aligned}
&T_{N,0}\mathcal{H}_N T^*_{N,0} \\
&\quad = \mathcal{H}_N + \|\nabla_2 \mathrm{sinh}_{k_{N,0}}\|^2 + N\int dxdy\, \big[\Delta\omega_N(x-y)\varphi_0^2((x+y)/2)\, a^*_x a^*_y + \text{h.c.}\big] \\
&\qquad + \frac{1}{2N}\int dxdy\, V_N(x-y)|\langle \mathrm{sh}_x - \varphi_0(x)\mathrm{sh}_{k_{N,0}}(\varphi_0), \mathrm{ch}_y - \varphi_0(y)\mathrm{ch}_{k_{N,0}}(\varphi_0)\rangle|^2 \\
&\qquad + \frac{1}{2}\int dxdy\, V_N(x-y)\big[-\omega_N(x-y)\varphi_0^2((x+y)/2)a^*_x a^*_y + \text{h.c.}\big] + \delta_N.
\end{aligned} \tag{5.88}$$

where we used the notation sh_x to indicate the function $\mathrm{sh}_x(z) = \mathrm{sinh}_{k_{N,0}}(x;z)$ and similarly for ch_x (in this case, a distribution) and where the operator δ_N is such that

$$\pm\delta_N \leq \mathcal{H}_N + C(\mathcal{N} + \mathcal{N}^2/N + 1)$$

(in fact, the constant in front of $\mathcal{H}_N$ could be chosen arbitrarily small, but we are not going to use this fact here). With (5.24), we find

$$\|\nabla_2 \mathrm{sinh}_{k_{N,0}}\|^2 \leq CN^\beta$$

Furthermore, integrating by parts, using (5.19), the assumption $\varphi_0 \in H^4(\mathbb{R}^3)$ and (5.85), we obtain

$$\begin{aligned}
&\Big| N \int dxdy \; \Delta\omega_N(x-y)\varphi_0^2((x+y)/2)\langle \xi_N, a_x^* a_y^* \xi_N\rangle \Big| \\
&\qquad \leq \int dx \, \|a_x \xi_N\| \, \|a^*(N\nabla\omega_N(x-\cdot)\nabla_x \varphi_0^2((x+\cdot)/2))\xi_N\| \\
&\qquad\quad + \int dx \, \|\nabla_x a_x \xi_N\| \, \|a^*(N\nabla\omega_N(x-\cdot)\varphi_0^2((x+\cdot)/2))\xi_N\| \\
&\qquad \leq \|(\mathcal{N}+1)^{1/2}\xi_N\| \int dx \, \|a_x \xi_N\| \|N\nabla\omega_N(x-\cdot)\nabla_x\varphi_0^2((x+\cdot)/2))\|_2 \\
&\qquad\quad + \|(\mathcal{N}+1)^{1/2}\xi_N\| \int dx \, \|\nabla_x a_x\| \|N\nabla\omega_N(x-\cdot)\varphi_0^2((x+\cdot)/2))\|_2 \\
&\qquad \leq CN^\beta \|(\mathcal{N}+\mathcal{K}+1)^{1/2}\xi_N\|^2 \leq CN^\beta .
\end{aligned}$$

Let us now consider the fourth term on the r.h.s. of (5.88). The most singular contribution is bounded by

$$\begin{aligned}
&\frac{1}{2N}\int dxdy \, V_N(x-y) |\langle \mathrm{sh}_x, \mathrm{ch}_y\rangle|^2 \\
&\quad \leq \frac{1}{2N}\int dxdy \, V_N(x-y)|\mathrm{sh}_{k_{N,0}}(x;y)|^2 \\
&\quad\quad + \frac{1}{2N}\int dxdy \, V_N(x-y)\Big|\int dz \, \mathrm{sh}_{k_{N,0}}(x;z)\mathrm{p}(y;z)\Big|^2 \\
&\quad \leq N^{\beta-1}\big(\|\nabla_1 \mathrm{sh}_{k_{N,0}}\|^2 + \|\nabla_2 \mathrm{sh}_{k_{N,0}}\|^2\big) + \frac{1}{2N}\int dxdy \, V_N(x-y)\|\mathrm{sh}_x\|^2\|\mathrm{p}_y\|^2 \\
&\quad \leq CN^{2\beta-1}
\end{aligned}$$

where we used Cauchy-Schwarz and the operator inequality

$$V_N(x-y) \leq CN^\beta(-\Delta_x - \Delta_y) \, .$$

Finally, let us consider the fifth term on the r.h.s. of (5.88). Using Cauchy-Schwarz, (5.19) and (5.21), we find

$$\begin{aligned}
&\left| \int dxdy \; V_N(x-y)\, \omega_N(x-y) \varphi_0^2((x+y)/2) \langle \xi, a_x^* a_y^* \xi \rangle \right| \\
&\quad \leq C \langle \xi, \mathcal{V}_N \xi \rangle + C \int dxdy \; V_N(x-y) N |\omega_N(x-y)|^2 \, |\varphi_0((x+y)/2)|^4 \\
&\quad \leq C\delta \langle \xi, \mathcal{V}_N \xi \rangle + C N^{2\beta-1}
\end{aligned}$$

From (5.88), we conclude with (5.85) that

$$\langle \xi, T_{N,0} \mathcal{H}_N T_{N,0}^* \xi \rangle \leq C N^\beta$$

Together with (5.87), this implies (5.86). □

A bound similar to the one in Lemma 5.4.1 also holds for the modified evolution $\widetilde{\Phi}_{N,t}$ introduced in (5.51).

Lemma 5.4.2. *Assume Hypothesis A holds true. Let $\xi_N \in \mathcal{F}_{\perp \varphi_0}$ with $\|\xi_N\| \leq 1$ and*

$$\langle \xi_N, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}) \xi_N \rangle \leq C,$$

uniformly in N. Let $\widetilde{\Phi}_{N,t}$ be as defined in (5.51). *We assume here that the parameter $C_b > 0$ in (5.50) is large enough. Then there exists a constant $C > 0$ such that*

$$\langle \widetilde{\Phi}_{N,t}, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}) \widetilde{\Phi}_{N,t} \rangle \leq C N^\beta \exp(C \exp(C|t|)). \tag{5.89}$$

for all $t \in \mathbb{R}$.

Proof. Consider the Bogoliubov transformed dynamics $\xi_{N,t} = T_{N,t} \widetilde{\Phi}_{N,t}$ as defined in (5.54). Then

$$\begin{aligned}
\langle \widetilde{\Phi}_{N,t}, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}) \widetilde{\Phi}_{N,t} \rangle &= \langle \xi_{N,t}, T_{N,t} (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}) T_{N,t}^* \xi_{N,t} \rangle \\
&\leq C N^\beta \langle \xi_{N,t}, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}) \xi_{N,t} \rangle
\end{aligned}$$

where we proceeded exactly as in Step 3 in the proof of Lemma 5.4.1 to bound the expectation of $T_{N,t} \mathcal{H}_N T_{N,t}^*$ and we applied Proposition 5.3.3 to bound the other terms. Now we apply Corollary 5.3.2 to conclude that, if $\ell > 0$ is small enough in (5.18) and if $C_b > 0$ is large enough in (5.50), there exists a constant $C > 0$ such that

$$\langle \widetilde{\Phi}_{N,t}, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b}) \widetilde{\Phi}_{N,t} \rangle \leq C N^\beta \exp(C \exp(C|t|))$$

for all $t \in \mathbb{R}$. □

Remark that Corollary 5.3.2 and Proposition 5.3.3 actually imply the stronger (compared with (5.89)) estimate $\langle \widetilde{\Phi}_{N,t}, \mathcal{N} \widetilde{\Phi}_{N,t} \rangle \leq C \exp(C \exp(C|t|))$ for the expectation of $\mathcal{N}$.

Using Lemma 5.4.1 and Lemma 5.4.2 we are now ready to prove Lemma 5.2.1.

Proof of Lemma 5.2.1. Note that

$$\|\Phi_{N,t} - \widetilde{\Phi}_{N,t}\|^2 = 2\big(1 - \mathrm{Re}\,\langle \Phi_{N,t}, \widetilde{\Phi}_{N,t}\rangle\big).$$

With the notation $1^{\leq m} = 1(\mathcal{N} \leq m)$ and $1^{>m} = 1 - 1^{\leq m}$, we can decompose

$$\langle \Phi_{N,t}, \widetilde{\Phi}_{N,t}\rangle = \langle \Phi_{N,t}, 1^{\leq m}\widetilde{\Phi}_{N,t}\rangle + \langle \Phi_{N,t}, 1^{>m}\widetilde{\Phi}_{N,t}\rangle. \tag{5.90}$$

Instead of fixing m, we take the average over $m \in [M/2+1, M]$ with an even number $1 \ll M \ll N$. This gives

$$\langle \Phi_{N,t}, \widetilde{\Phi}_{N,t}\rangle = \frac{2}{M}\sum_{m=M/2+1}^{M}\Big(\langle \Phi_{N,t}, 1^{\leq m}\widetilde{\Phi}_{N,t}\rangle + \langle \Phi_{N,t}, 1^{>m}\widetilde{\Phi}_{N,t}\rangle\Big). \tag{5.91}$$

We are going to choose $M = N^{1-\varepsilon}$ with $\varepsilon > 0$ a sufficiently small that will be specified later. Next, we estimate the two terms on the r.h.s. of (5.91).

Many-particle sectors. With $1^{>m} \leq \mathcal{N}/m$ and Lemma 5.4.2, we have

$$\begin{aligned}|\langle \Phi_{N,t}, 1^{>m}\widetilde{\Phi}_{N,t}\rangle| &\leq \|\Phi_{N,t}\|\|1^{>m}\widetilde{\Phi}_{N,t}\|\\ &\leq \langle \widetilde{\Phi}_{N,t}, (\mathcal{N}/m)\widetilde{\Phi}_{N,t}\rangle^{1/2} \leq C\sqrt{\frac{N^\beta}{M}}\exp(C\exp(C|t|)).\end{aligned}$$

Thus

$$\frac{2}{M}\sum_{m=M/2+1}^{M}|\langle \Phi_{N,t}, 1^{>m}\widetilde{\Phi}_{N,t}\rangle| \leq C\sqrt{\frac{N^\beta}{M}}\exp(C\exp(C|t|)). \tag{5.92}$$

Few-particle sectors. From the Schrödinger equations (5.45) and (5.51) for $\Phi_{N,t}$ and $\widetilde{\Phi}_{N,t}$, we obtain

$$\frac{d}{dt}\mathrm{Re}\,\langle \Phi_{N,t}, 1^{\leq m}\widetilde{\Phi}_{N,t}\rangle = \mathrm{Im}\,\langle \Phi_{N,t}, (\mathcal{L}_{N,t}1^{\leq m} - 1^{\leq m}\widetilde{\mathcal{L}}_{N,t})\widetilde{\Phi}_{N,t}\rangle$$

We can write

$$\mathcal{L}_{N,t}1^{\leq m} - 1^{\leq m}\widetilde{\mathcal{L}}_{N,t} = (\mathcal{L}_{N,t} - \widetilde{\mathcal{L}}_{N,t})1^{\leq m} + [\widetilde{\mathcal{L}}_{N,t}, 1^{\leq m}].$$

Bound for $(\mathcal{L}_{N,t} - \widetilde{\mathcal{L}}_{N,t})1^{\leq m}$. We have

$$\begin{aligned}(\mathcal{L}_{N,t} - \widetilde{\mathcal{L}}_{N,t})1^{\leq m} &= A_1\Big[\sqrt{1-\mathcal{N}/N} - G_b(\mathcal{N}/N)\Big]1^{\leq m} + \mathrm{h.c.}\\ &\quad + A_2\frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)} - (N-\mathcal{N})}{N}1^{\leq m} + \mathrm{h.c.}\\ &\quad - C_b e^{C_b t}\mathcal{N}(\mathcal{N}/N)^{2b}1^{\leq m}\end{aligned} \tag{5.93}$$

with the two operators

$$
\begin{aligned}
A_1 &= \sqrt{N}\big[a^*(Q_{N,t}[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}) - a^*(Q_{N,t}[V_N * |\varphi_{N,t}|^2]\varphi_{N,t})(\mathcal{N}/N)\big] \\
&\quad + \frac{1}{\sqrt{N}}\int dxdydx'dy'\,(Q_{N,t}\otimes Q_{N,t}V_NQ_{N,t}\otimes 1)(x,y;x',y')a_x^*a_y^*a_{x'}\varphi_{N,t}(y') \\
A_2 &= \frac{1}{2}\int dxdy\, K_{2,N,t}(x;y)a_x^*a_y^*.
\end{aligned}
\tag{5.94}
$$

To bound the r.h.s. of (5.93) we are going to use the following proposition.

Proposition 5.4.3. *Assume the interaction potential V to be smooth, spherically symmetric, compactly supported and non-negative. Then, for all vectors $\xi_1, \xi_2 \in \mathcal{F}_{\perp\varphi_{N,t}}$, we have the bounds*

$$|\langle\xi_1, A_1\xi_2\rangle| \leq C\exp(C|t|)\big\langle\xi_1, \big(N^{2\beta-1} + (\mathcal{N}/N)^2 + \mathcal{V}_N\big)\xi_1\big\rangle^{1/2}\langle\xi_2, (\mathcal{N}+1)\xi_2\rangle^{1/2}$$

and

$$|\langle\xi_1, A_2\xi_2\rangle| \leq C\sqrt{N}\exp(C|t|)\langle\xi_1, \mathcal{V}_N\xi_1\rangle^{1/2}\|\xi_2\|.$$

Proof. First we consider A_1. Using

$$a^*(g)a(g) \leq a(g)a^*(g) \leq (\mathcal{N}+1)\|g\|_{L^2}^2$$

and

$$
\begin{aligned}
\|Q_{N,t}[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\|_{L^2} &\leq \|[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t}\|_{L^2} \\
&\leq \|V_N\|_{L^1}\|\omega_N\|_{L^\infty}\|\varphi_{N,t}\|_{L^\infty}^2\|\varphi_{N,t}\|_{L^2} \\
&\leq CN^{\beta-1}\exp(C|t|),
\end{aligned}
$$

we have

$$
\begin{aligned}
&|\langle\xi_1, \sqrt{N}a^*(Q_{N,t}[(V_N\omega_N) * |\varphi_{N,t}|^2]\varphi_{N,t})\xi_2\rangle| \\
&\qquad \leq CN^{\beta-1/2}\exp(C|t|)\|\xi_1\|\langle\xi_2, (\mathcal{N}+1)\xi_2\rangle^{1/2}
\end{aligned}
$$

and

$$|\langle\xi_1, a^*(Q_{N,t}[V_N * |\varphi_{N,t}|^2]\varphi_{N,t})\frac{\mathcal{N}}{N}\xi_2\rangle| \leq C\exp(C|t|)\langle\xi_1, (\mathcal{N}/N)^2\xi_1\rangle^{1/2}\langle\xi_2, \mathcal{N}\xi_2\rangle^{1/2}.$$

Moreover[3]

$$
\begin{aligned}
&\Big|\Big\langle\xi_1, \frac{1}{\sqrt{N}}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y'\,(Q_{N,t}\otimes Q_{N,t}V_NQ_{N,t}\otimes 1)(x,y;x',y')a_x^*a_y^*a_{x'}\varphi_{N,t}(y')\xi_2\Big\rangle\Big| \\
&= \Big|\frac{1}{\sqrt{N}}\int \mathrm{d}x\,\mathrm{d}y\,V_N(x-y)\varphi_{N,t}(y)\langle a_xa_y\xi_1, a_x\xi_2\rangle\Big| \\
&\leq \frac{1}{\sqrt{N}}\int \mathrm{d}x\,\mathrm{d}y\,V_N(x-y)|\varphi_{N,t}(y)|\|a_xa_y\xi_1\|\|a_x\xi_2\| \\
&\leq \|\varphi_{N,t}\|_{L^\infty}\Big(\frac{1}{N}\int \mathrm{d}x\,\mathrm{d}y\,V_N(x-y)\|a_xa_y\xi_1\|^2\Big)^{1/2}\Big(\int \mathrm{d}x\,\mathrm{d}yV_N(x-y)\|a_x\xi_2\|^2\Big)^{1/2} \\
&\leq C\exp(C|t|)\langle\xi_1, \mathcal{V}_N\xi_1\rangle^{1/2}\langle\xi_2, \mathcal{N}\xi_2\rangle^{1/2}.
\end{aligned}
$$

[3]Note that the projection $Q_{N,t}$ has no effect in the excited Fock space $\mathcal{F}_{\perp\varphi_{N,t}}$

To prove the bound for A_2, we estimate

$$
\begin{aligned}
|\langle \xi_1, A_2 \xi_2\rangle| &= \Big| \int \mathrm{d}x\, \mathrm{d}y V_N(x-y) \varphi_{N,t}(x) \varphi_{N,t}(y) \langle a_x a_y \xi_1, \xi_2\rangle \Big| \\
&\leq \|\varphi_{N,t}\|_{L^\infty} \Big(\int \mathrm{d}x\, \mathrm{d}y\, V_N(x-y) \|a_x a_y \xi_1\|^2 \Big)^{1/2} \\
&\qquad \times \Big(\int \mathrm{d}x\, \mathrm{d}y V_N(x-y) |\varphi_{N,t}(x)|^2 \|\xi_2\|^2 \Big)^{1/2} \\
&\leq C\sqrt{N} \exp(C|t|) \langle \xi_1, \mathcal{V}_N \xi_1\rangle^{1/2} \|\xi_2\|.
\end{aligned}
$$

This ends the proof of the proposition. □

We control now the operators on the r.h.s. of (5.93). Obviously,

$$
\mathcal{N}(\mathcal{N}/N)^{2b} 1^{\leq m} \leq CM(M/N)^{2b}.
$$

and, therefore,

$$
|\langle \Phi_{N,t}, \mathcal{N}(\mathcal{N}/N)^{2b} 1^{\leq m} \widetilde{\Phi}_{N,t}\rangle| \leq CM(M/N)^{2b}.
$$

Using Proposition 5.4.3 with

$$
\xi_1 = \Phi_{N,t}, \quad \xi_2 = \big[\sqrt{1-\mathcal{N}/N} - G_b(\mathcal{N}/N)\big] 1^{\leq m}\, \widetilde{\Phi}_{N,t},
$$

combined with the simple bound

$$
|\sqrt{1-\mathcal{N}/N} - G_b(\mathcal{N}/N)| 1^{\leq m} \leq C(M/N)^{b+1}
$$

that follows from (5.49) and with the estimates in Lemma 5.4.1 and Lemma 5.4.2, we obtain

$$
\Big| \langle \Phi_{N,t}, A_1(\sqrt{1-\mathcal{N}/N} - G_b(\mathcal{N}/N)) 1^{\leq m} \widetilde{\Phi}_{N,t}\rangle \Big| \leq C(M/N)^{b+1} N^{\beta} \exp(C\exp(C|t|)).
$$

Using again Proposition 5.4.3 with

$$
\xi_1 = \Phi_{N,t}, \quad \xi_2 = \big[\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)} - N - \mathcal{N}\big] 1^{\leq m} \widetilde{\Phi}_{N,t}\,,
$$

the simple bound

$$
|\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)} - N - \mathcal{N}| \leq 1,
$$

and the bounds in Lemma 5.4.1 and Lemma 5.4.2, we also obtain

$$
\Big| \Big\langle \Phi_{N,t}, A_2 \frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)} - N - \mathcal{N}}{N} 1^{\leq m} \widetilde{\Phi}_{N,t} \Big\rangle \Big| \leq C N^{\frac{\beta-1}{2}} \exp(C\exp(C|t|)).
$$

The hermitian conjugated terms can be controlled analogously (Proposition 5.4.3 provides bounds for A_1^*, A_2^*, as well, switching ξ_1 and ξ_2). In summary, we have shown that

$$
\begin{aligned}
&\Big| \langle \Phi_{N,t}, (\mathcal{L}_{N,t} - \widetilde{\mathcal{L}}_{N,t}) 1^{\leq m} \widetilde{\Phi}_{N,t}\rangle \Big| \\
&\qquad \leq C\Big[N^{\beta-1} + M(M/N)^{2b} + (M/N)^{b+1} N^{\beta} \Big] \exp(C\exp(C|t|)).
\end{aligned}
$$

Bound for $[\widetilde{\mathcal{L}}_{N,t}, 1^{\leq m}]$. We can decompose

$$[\widetilde{\mathcal{L}}_{N,t}, 1^{\leq m}] = 1^{\leq m}\widetilde{\mathcal{L}}_{N,t}1^{>m} - 1^{>m}\widetilde{\mathcal{L}}_{N,t}1^{\leq m}. \tag{5.95}$$

Let us focus on $1^{>m}\widetilde{\mathcal{L}}_{N,t}1^{\leq m}$; the other term can be treated similarly. With the operators A_1, A_2 defined in (5.94), we have

$$\begin{aligned} 1^{>m}\widetilde{\mathcal{L}}_{N,t}1^{\leq m} &= 1^{>m}\Big(A_1 G_p(\mathcal{N}/N) + A_2\frac{N-\mathcal{N}}{N}\Big)1^{\leq m} \\ &= A_1 G_p(\mathcal{N}/N)1(\mathcal{N}=m) + A_2\frac{N-\mathcal{N}}{N}1(m-1\leq \mathcal{N}\leq m). \end{aligned} \tag{5.96}$$

Here we used the fact that A_1 creates exactly one particle while A_2 creates exactly two particles. All other terms in $\widetilde{\mathcal{L}}_{N,t}$ leave the number of particles invariant, and therefore do not contribute to (5.96). Thus

$$\begin{aligned} \sum_{m=M/2+1}^{M} 1^{>m}\widetilde{\mathcal{L}}_{N,t}1^{\leq m} &= A_1 G_p(\mathcal{N}/N)1(M/2<\mathcal{N}\leq M) \\ &\quad + A_2\frac{N-\mathcal{N}}{N}\Big[1(M/2<\mathcal{N}\leq M) + 1(M/2\leq \mathcal{N}<M)\Big]. \end{aligned}$$

Using Proposition 5.4.3 with

$$\xi_1 = \Phi_{N,t}, \quad \xi_2 = G_p(\mathcal{N}/N)1(M/2<\mathcal{N}\leq M)\widetilde{\Phi}_{N,t},$$

combined with the simple estimate (recall that we will choose $M \ll N$)

$$|G_p(\mathcal{N}/N)|1(M/2<\mathcal{N}\leq M)\leq C$$

and with the bounds in Lemma 5.4.1 and in Lemma 5.4.2, we obtain

$$\langle \Phi_{N,t}, A_1 G_p(\mathcal{N}/N)1(M/2<\mathcal{N}\leq M)\widetilde{\Phi}_{N,t}\rangle \leq CN^{\beta}\exp(C\exp(C|t|)).$$

Similarly, using again Proposition 5.4.3 and Lemma 5.4.2, we find

$$\begin{aligned} \langle \Phi_{N,t}, A_2(1-\mathcal{N}/N)\Big[1(M/2<\mathcal{N}\leq M) + 1(M/2\leq \mathcal{N}<M)\Big]\widetilde{\Phi}_{N,t}\rangle \\ \leq CN^{\frac{\beta+1}{2}}\exp(C\exp(C|t|)). \end{aligned}$$

Thus, we conclude that

$$\frac{2}{M}\Big|\sum_{m=M/2+1}^{M}\langle \Phi_{N,t}, [\widetilde{\mathcal{L}}_{N,t}, 1^{\leq m}]\widetilde{\Phi}_{N,t}\rangle\Big| \leq C\frac{N^{\frac{\beta+1}{2}}}{M}\exp(C\exp(C|t|)). \tag{5.97}$$

In summary, we have proved that

$$\begin{aligned} &\Big|\mathrm{Re}\,\frac{2}{M}\sum_{m=M/2+1}^{M}\frac{d}{dt}\langle \Phi_{N,t}, 1^{\leq m}\widetilde{\Phi}_{N,t}\rangle\Big| \\ &\qquad \leq C\Big[N^{\frac{\beta-1}{2}} + M\Big(\frac{M}{N}\Big)^{2b} + N^{\beta}\Big(\frac{M}{N}\Big)^{b+1} + \frac{N^{\frac{\beta+1}{2}}}{M}\Big]\exp(C\exp(C|t|)). \end{aligned}$$

Conclusion of the proof. For every $\alpha < (1-\beta)/2$, we can choose $M = N^{1-\varepsilon}$ with a sufficiently small $\varepsilon > 0$, and then b sufficiently large to obtain

$$\Big|\mathrm{Re}\,\frac{2}{M}\sum_{m=M/2+1}^{M}\frac{d}{dt}\langle\Phi_{N,t},1^{\leq m}\widetilde{\Phi}_{N,t}\rangle\Big| \leq CN^{-\alpha}\exp(C\exp(C|t|)).$$

Integrating over t, we find

$$\begin{aligned}\mathrm{Re}\,\frac{2}{M}&\sum_{m=M/2+1}^{M}\langle\Phi_{N,t},1^{\leq m}\widetilde{\Phi}_{N,t}\rangle\\ &\geq \mathrm{Re}\,\frac{2}{M}\sum_{m=M/2+1}^{M}\langle\Phi_{N,0},1^{\leq m}\widetilde{\Phi}_{N,0}\rangle - CN^{-\alpha}\exp(C\exp(C|t|)).\end{aligned}$$

On the other hand, using the assumption $\Phi_{N,0} = 1^{\leq N}T^*_{N,0}\xi_N$, $\widetilde{\Phi}_{N,0} = T^*_{N,0}\xi_N$ we have the lower bound

$$\begin{aligned}\langle\Phi_{N,0},1^{\leq m}\widetilde{\Phi}_{N,0}\rangle &= \|1^{\leq m}T^*_{N,0}\xi_N\|^2 = 1 - \|1^{>m}T^*_{N,0}\xi_N\|^2\\ &\geq 1 - \langle T^*_{N,0}\xi_N,(\mathcal{N}/m)T^*_{N,0}\xi_N\rangle\\ &\geq 1 - C\langle\xi_N,(\mathcal{N}/m)\xi_N\rangle \geq 1 - C/M.\end{aligned}$$

Here we have used Proposition 5.3.3 in the second last estimate and the assumption on ξ_N for the last inequality. Thus

$$\mathrm{Re}\,\frac{2}{M}\sum_{m=M/2+1}^{M}\langle\Phi_{N,t},1^{\leq m}\widetilde{\Phi}_{N,t}\rangle \geq 1 - CN^{-\alpha}\exp(C\exp(C|t|)) - CM^{-1}$$

Combining with (5.92) and using the choice $M = N^{1-\varepsilon}$ for a sufficiently small $\varepsilon > 0$, we obtain

$$\mathrm{Re}\,\langle\Phi_{N,t},\widetilde{\Phi}_{N,t}\rangle \geq 1 - CN^{-\alpha}\exp(C\exp(C|t|)).$$

Consequently,

$$\|\Phi_{N,t}-\widetilde{\Phi}_{N,t}\|^2 \leq 2(1-\mathrm{Re}\,\langle\Phi_{N,t},\widetilde{\Phi}_{N,t}\rangle) \leq CN^{-\alpha}\exp(C\exp(C|t|)).$$

□

5.5 Proof of main Results

Combining Lemma 5.2.1 and Lemma 5.2.2, we can prove our first main theorem.

Proof of Theorem 5.1.1. Fix $\alpha < \min(\beta/2, (1-\beta)/2)$. To begin with, let us choose a sequence $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ with $\|\xi_N\| \leq 1$ and with

$$\langle \xi_N, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b})\xi_N \rangle \leq C \tag{5.98}$$

uniformly in N. This assumption is stronger than the assumption (5.30) in the theorem; at the end, we will show how to relax it.

Assuming (5.98), we consider the many-body evolution

$$\Psi_{N,t} = e^{-itH_N} U^*_{\varphi_0} 1^{\leq N} T^*_{N,0} \xi_N$$

and we factor out the condensate, defining, as in (5.44), $\Phi_{N,t} = U_{\varphi_{N,t}} \Psi_{N,t}$. To prove Theorem 5.1.1, we have to compare $\Phi_{N,t}$ with the (Bogoliubov transformed) effective evolution $T^*_{N,t}\xi_{2,N,t} = T^*_{N,t}\mathcal{U}_{2,N}(t;0)\xi_N$. To this end, we recall the definition (5.51) of the modified fluctuation dynamics $\widetilde{\Phi}_{N,t}$, and we bound

$$\left\|\Phi_{N,t} - T^*_{N,t}\xi_{2,N,t}\right\| \leq \|\Phi_{N,t} - \widetilde{\Phi}_{N,t}\| + \|\widetilde{\Phi}_{N,t} - T^*_{N,t}\xi_{2,N,t}\| \leq \|\Phi_{N,t} - \widetilde{\Phi}_{N,t}\| + \|\xi_{N,t} - \xi_{2,N,t}\|$$

where, as in (5.54), we set $\xi_{N,t} = T_{N,t}\widetilde{\Phi}_{N,t}$ and we used the unitarity of $T_{N,t}$. Combining Lemma 5.2.1 and Lemma 5.2.2 (which can be used, because of the additional assumption (5.98)), we conclude that there exists a constant $C > 0$ such that

$$\left\|\Phi_{N,t} - T^*_{N,t}\xi_{2,N,t}\right\|_{\mathcal{F}} \leq CN^{-\alpha} \exp(C\exp(C|t|)) \tag{5.99}$$

for all $t \in \mathbb{R}$ and all N large enough. This proves Theorem 5.1.1 under the additional assumption (5.98).

Now, let us assume that the sequence $\xi_N \in \mathcal{F}_{\perp\varphi_0}$ is normalized $\|\xi_N\| = 1$, but, instead of (5.98), that it only satisfies the weaker bound

$$\langle \xi_N, (\mathcal{K} + \mathcal{N})\xi_N \rangle \leq C\,, \tag{5.100}$$

uniformly in N. We choose $M = N^{2\alpha}$ and we decompose

$$\xi_N = 1^{\leq M}\xi_N + 1^{>M}\xi_N$$

Then, using unitarity of the maps $U_{\varphi_{N,t}}$, $T_{N,t}$, $e^{iH_N t}$ and $\mathcal{U}_{2,N}(t;0)$, we obtain

$$\begin{aligned}\|\Phi_{N,t} - T^*_{N,t}\xi_{2,N,t}\| &= \|U_{\varphi_{N,t}} e^{-itH_N} U^*_{\varphi_0} 1^{\leq N} T^*_{N,0}\xi_N - T^*_{N,t}\mathcal{U}_{2,N}(t;0)\xi_N\| \\ &\leq \|U_{\varphi_{N,t}} e^{-itH_N} U^*_{\varphi_0} 1^{\leq N} T^*_{N,0} 1^{\leq M}\xi_N - T^*_{N,t}\mathcal{U}_{2,N}(t;0)1^{\leq M}\xi_N\| \\ &\quad + 2\|1^{>M}\xi_N\|\end{aligned} \tag{5.101}$$

On the one hand, using Markov's inequality and (5.100), we have

$$\|1^{>M}\xi_N\|^2 = \langle \xi_N, 1^{>M}\xi_N \rangle \leq M^{-1}\langle \xi_N, \mathcal{N}\xi_N \rangle \leq CN^{-2\alpha}$$

On the other hand, the sequence $\widetilde{\xi}_N = 1^{\leq M}\xi_N$ is such that $\|\widetilde{\xi}_N\| \leq \|\xi_N\| = 1$ and

$$\langle \widetilde{\xi}_N, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b})\widetilde{\xi}_N\rangle \leq \langle \xi_N, (\mathcal{K} + \mathcal{N} + 1)\xi_N\rangle \leq C \tag{5.102}$$

by (5.100). Here we used the bound $\mathcal{V}_N \leq CN^{\beta-1}(\mathcal{K}+1)(\mathcal{N}+1)$ for the potential energy, which implies, by the choice of $M = N^{2\alpha}$ and of $\alpha \leq (1-\beta)/2$, that $\mathcal{V}_N 1^{\leq M} \leq C(\mathcal{K}+1)$. Because of (5.102), we can apply the convergence (5.99), established under the additional assumption (5.98), to estimate the first term on the r.h.s. of (5.101). We obtain that (this time only under the assumption (5.100))

$$\|\Phi_{N,t} - T^*_{N,t}\xi_{2,N,t}\| \leq CN^{-\alpha}\exp(C\exp(C|t|))$$

This concludes the proof of Theorem 5.1.1. □

To show Theorem 5.1.2, we compare the difference between the generators of the quadratic evolutions $\mathcal{U}_{2,N}$ and $\mathcal{U}_2$ defined in (5.25) and, respectively, in (5.37).

Proposition 5.5.1. *Assume Hypothesis A holds true. Let $\mathcal{G}_{2,N,t}$ and $\mathcal{G}_{2,t}$ be as defined in (5.26) and in (5.36) (and $\eta_N(t)$ as in (5.27)). Then there exists $C > 0$ such that, with $\alpha = \min(\beta/2, (1-\beta)/2)$,*

$$\begin{aligned}|\langle \xi_1, (\mathcal{G}_{2,N,t} - \eta_N(t) - \mathcal{G}_{2,t})\xi_2\rangle| \\ \leq CN^{-\alpha}\exp(C\exp(C|t|))\|(\mathcal{K}+\mathcal{N}+1)^{1/2}\xi_1\|\|(\mathcal{N}+1)^{1/2}\xi_2\|\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathcal{F}_{\perp\varphi_{N,t}}$ and all $t \in \mathbb{R}$.

The proof of Proposition 5.5.1 can be found in [16, Lemmas 5.1, 5.2, 5.3, 5.4], up to very minor modifications.

Proof of Theorem 5.1.2. As in the proof of Theorem 5.1.1, we first assume that

$$\langle \xi_N, (\mathcal{H}_N + \mathcal{N} + \mathcal{N}(\mathcal{N}/N)^{2b})\xi_N\rangle \leq C \tag{5.103}$$

uniformly in N. With $\theta_N(t) := -\int_0^t d\tau\, \eta_N(\tau)$ we find

$$\frac{d}{dt}\left\|\xi_{2,N,t} - e^{i\theta_N(t)}\xi_{2,t}\right\|^2 = 2\,\mathrm{Im}\left\langle \xi_{2,N,t}, \left[\mathcal{G}_{2,N,t} - \eta_N(t) - \mathcal{G}_{2,t}\right]e^{i\theta_N(t)}\xi_{2,t}\right\rangle$$

Proposition 5.5.1 above implies that

$$\begin{aligned}&\frac{d}{dt}\left\|\xi_{2,N,t} - e^{i\theta_N(t)}\xi_{2,t}\right\|^2 \\ &\qquad \leq CN^{-\alpha}\exp(C\exp(C|t|))\langle \xi_{2,N,t}, (\mathcal{K}+\mathcal{N}+1)\xi_{2,N,t}\rangle^{1/2}\langle \xi_{2,t}, (\mathcal{N}+1)\xi_{2,t}\rangle^{1/2} \\ &\qquad \leq CN^{-\alpha}\exp(C\exp(C|t|))\end{aligned}$$

Here we used Corollary 5.3.2 (with the additional assumption (5.103)) and the analogous bound

$$\langle \xi_{2,t}, (\mathcal{N}+1)\xi_{2,t}\rangle \leq C\exp(C\exp(C|t|)) \tag{5.104}$$

for the limiting dynamics $\xi_{2,t}$. Eq. (5.104) can be proven similarly to the bound for $\xi_{2,N,t}$ in Corollary 5.3.2 (with estimates for the generator $\mathcal{G}_{2,t}$ analogous to (5.58)). Integrating in time, we conclude that

$$\left\|\xi_{2,N,t} - e^{i\theta_N(t)}\xi_{2,t}\right\|^2 \leq CN^{-\alpha}\exp(C\exp(C|t|))$$

for all $t \in \mathbb{R}$. Combining the last bound with Theorem 5.1.1, we obtain

$$\begin{aligned}\|U_{\varphi_{N,t}}\Psi_{N,t} - e^{-i\theta_N(t)}T^*_{N,t}\xi_{2,t}\| &\leq \|U_{\varphi_{N,t}}\Psi_{N,t} - T^*_{N,t}\xi_{2,N,t}\| + \|\xi_{2,N,t} - e^{-i\theta_N(t)}\xi_{2,t}\| \\ &\leq CN^{-\alpha/2}\exp(C\exp(C|t|))\end{aligned}$$

This proves Theorem 5.1.2 under the additional assumption (5.103). To relax this condition, we proceed exactly as in the proof of Theorem 5.1.1. We omit the details. □

Finally, Theorem 5.1.3 follows immediately combining Theorem 5.1.2 with the following proposition, which is a modification of the analysis in [20, Section 6].

Proposition 5.5.2. *Assume Hypothesis A holds true. Let $\psi_N \in L^2_s(\mathbb{R}^{3N})$ with reduced one-particle density γ_N such that*

$$a_N := \mathrm{tr}\;|\gamma_N - |\varphi_0\rangle\langle\varphi_0|| \leq CN^{-1} \tag{5.105}$$

and

$$b_N := \left|\frac{1}{N}\langle\psi_N, H_N\psi_N\rangle - \left[\|\nabla\varphi_0\|_2^2 + \frac{1}{2}\langle\varphi_0, [V_N f_N * |\varphi_0|^2]\varphi_0\rangle\right]\right| \leq CN^{-1} \tag{5.106}$$

Set $\xi_N = T_{N,0}U_{\varphi_0}\psi_N$ with the Bogoliubov transformation $T_{N,0}$ defined in (5.22). Then, we have $\psi_N = U^*_{\varphi_0}\mathbf{1}^{\leq N}T^*_{N,0}\xi_N$ and

$$\langle\xi_N, [\mathcal{K} + \mathcal{N}]\,\xi_N\rangle \leq C$$

uniformly in N.

Proof. First of all, we remark that, with Proposition 5.3.3 and (5.13),

$$\begin{aligned}\langle\xi_N, \mathcal{N}\xi_N\rangle &= \langle T_{N,0}U_{\varphi_0}\psi_N, \mathcal{N}T_{N,0}U_{\varphi_0}\psi_N\rangle \\ &\leq C\langle U_{\varphi_0}\psi_N, (\mathcal{N}+1)U_{\varphi_0}\psi_N\rangle \\ &= C\,[N - \langle\psi_N, a^*(\varphi_0)a(\varphi_0)\psi_N\rangle] + C \\ &= CN\,[1 - \langle\varphi_0, \gamma_N\varphi_0\rangle] + C \leq CNa_N + C.\end{aligned}$$

To bound $\langle\xi_N, \mathcal{K}\xi_N\rangle$, we use $\mathcal{K} \leq \mathcal{H}_N$ and the first bound in (5.61), which implies that

$$\begin{aligned}\langle\xi_N, \mathcal{H}_N\xi_N\rangle &\leq 2\langle\xi_N, (\mathcal{G}_{N,0} - \eta_N(0))\xi_N\rangle + C\langle\xi_N, (\mathcal{N}+1)\xi_N\rangle \\ &\leq 2\langle\xi_N, (\mathcal{G}_{N,0} - \eta_N(0))\xi_N\rangle + CNa_N + C.\end{aligned}$$

Hence, the proposition follows from (5.105) and (5.106) if we can show that

$$\langle \xi_N, \big[\mathcal{G}_{N,0} - \eta_N(0)\big]\xi_N\rangle \leq \frac{1}{4}\langle \xi_N, \mathcal{H}_N \xi_N\rangle + CN(a_N + b_N) + C. \tag{5.107}$$

To prove (5.107) we observe that, from the definition (5.56) of $\mathcal{G}_{N,0}$ and since $\xi_N = T_{N,0}U_{\varphi_0}\psi_N$,

$$\begin{aligned}\langle \xi_N, \big[\mathcal{G}_{N,0} - \eta_N(0)\big]\xi_N\rangle =& \langle U_{\varphi_0}\psi_N, \big[T^*_{N,0}(i\partial_t T_{N,t})_{|t=0} + \mathcal{L}_{N,0} - \eta_N(0)\big]U_{\varphi_0}\psi_N\rangle \\ &+ \langle U_{\varphi_0}\psi_N, \big[\widetilde{\mathcal{L}}_{N,0} - \mathcal{L}_{N,0}\big]U_{\varphi_0}\psi_N\rangle.\end{aligned} \tag{5.108}$$

From the proof of Lemma 6.2 and of Theorem 1.1 in [20, Section 6], we find

$$\langle U_{\varphi_0}\psi_N, \big[T^*_{N,0}(i\partial_t T_{N,t})_{|t=0} + \mathcal{L}_{N,0} - \eta_N(0)\big]U_{\varphi_0}\psi_N\rangle \leq CN(a_N + b_N) + C. \tag{5.109}$$

Therefore, it is enough to consider the second term on the r.h.s. of (5.108). From the definitions (5.50) of $\widetilde{\mathcal{L}}_{N,0}$ and (5.47) of $\mathcal{L}_{N,0}$, we have (see also (5.93))

$$\widetilde{\mathcal{L}}_{N,0} - \mathcal{L}_{N,0} = \sum_{j=1}^{4} D_j,$$

with the operators

$$\begin{aligned}
D_1 =& \sqrt{N}\Big[a^*\big(Q_{N,0}\big[(V_N\omega_N) * |\varphi_0|^2\big]\varphi_0\big) - a^*(Q_{N,0}[V_N * |\varphi_0|^2]\varphi_0)(\mathcal{N}/N)\Big] \\
&\times (G_b(\mathcal{N}/N) - \sqrt{1-\mathcal{N}/N}) + \text{h.c.} \\
D_2 =& \frac{1}{2}\int dxdy K_{2,N,0}(x;y)a^*_x a^*_y \frac{(N-\mathcal{N}) - \sqrt{(N-\mathcal{N})(N-1-\mathcal{N})}}{N} + \text{h.c.} \\
D_3 =& \frac{1}{\sqrt{N}}\int dxdy (Q_{N,0}\otimes Q_{N,0}V_N Q_{N,0}\otimes 1)(x,y;x',y')a^*_x a^*_y a_{x'}\varphi_0(y') \\
&\times (G_b(\mathcal{N}/N) - \sqrt{1-\mathcal{N}/N}) + \text{h.c.} \\
D_4 =& C_b\mathcal{N}(\mathcal{N}/N)^{2b}.
\end{aligned}$$

Using $|\sqrt{1-z} - G_b(z)| \leq Cz^{b+1}$ for all $z > 0$, we easily arrive at

$$\big|\langle U_{\varphi_0}\psi_N, D_1 U_{\varphi_0}\psi_N\rangle\big| \leq C\langle U_{\varphi_0}\psi_N, \mathcal{N}U_{\varphi_0}\psi_N\rangle \leq CNa_N + C. \tag{5.110}$$

Since, for $z \in (0,1)$,

$$\big|(1-z) - \sqrt{(1-z)(1-z-1/N)}\big| \leq C/N$$

we obtain that, for any $\delta > 0$ (recall that $Q_{N,0}$ has no effect on states in $\mathcal{F}^{\leq N}_{\perp\varphi_0}$),

$$\begin{aligned}
&|\langle U_{\varphi_0}\psi_N, D_2 U_{\varphi_0}\psi_N\rangle| \\
&\quad\leq \int dxdy\; N^{3\beta-1}V(N^\beta(x-y))\big(\delta^{-1}N|\varphi_0(x)|^2|\varphi_0(y)|^2 + \delta N^{-1}\|a_x a_y U_{\varphi_0}\psi_N\|^2\big) \\
&\quad\leq \delta N^{-1}\langle U_{\varphi_0}\psi_N, \mathcal{V}_N U_{\varphi_0}\psi_N\rangle + C.
\end{aligned}$$

As in Step 3 of the proof of Lemma 5.4.1, we can estimate

$$\begin{aligned}\delta N^{-1}\langle U_{\varphi_0}\psi_N, \mathcal{V}_N U_{\varphi_0}\psi_N\rangle &= \delta N^{-1}\langle \xi_N, T_{N,0}\mathcal{H}_N T^*_{N,0}\xi_N\rangle \\ &\leq \delta\langle \xi_N, \mathcal{H}_N\xi_N\rangle + CNa_N + C.\end{aligned} \tag{5.111}$$

Choosing, for example, $\delta = 1/8$, we conclude that

$$|\langle U_{\varphi_0}\psi_N, D_2 U_{\varphi_0}\psi_N\rangle| \leq \frac{1}{8}\langle \xi_N, \mathcal{H}_N\xi_N\rangle + CNa_N + C. \tag{5.112}$$

As for the expectation of D_3, we proceed similarly as in the proof of Proposition 5.3.1 (in particular, in the bound for the operator B_2). Using again the bound $|\sqrt{1-z} - G_b(z)| \leq Cz^{b+1}$ for all $z \in (0;1)$, we find that, for every $\delta > 0$ there exists $C > 0$ such that

$$\begin{aligned}&\big|\langle U_{\varphi_0}\psi_N, D_3 U_{\varphi_0}\psi_N\rangle\big| \\ &= \frac{1}{\sqrt{N}}\left|\int dxdy\, V_N(x-y)\varphi_0(y)\langle \xi_N, T_{N,0}\, a^*_x a^*_y a_x\big(G_b(\mathcal{N}/N) - \sqrt{1-\mathcal{N}/N}\big)T^*_{N,0}\xi_N\rangle\right| \\ &\leq \delta\langle \xi_N, \mathcal{V}_N\xi_N\rangle + C\langle \xi_N, (\mathcal{N}+1)\xi_N\rangle.\end{aligned}$$

Choosing $\delta = 1/8$, we obtain

$$\big|\langle U_{\varphi_0}\psi_N, D_3 U_{\varphi_0}\psi_N\rangle\big| \leq \frac{1}{8}\langle \xi_N, \mathcal{H}_N\xi_N\rangle + CNa_N + C. \tag{5.113}$$

Finally, since $U_{\varphi_0}\psi_N$ has at most N particles, we easily find that

$$0 \leq \langle U_{\varphi_0}\psi_N, D_4 U_{\varphi_0}\psi_N\rangle \leq CNa_N + C.$$

Combining the last bound with (5.110), (5.112) and (5.113), we conclude that

$$|\langle U_{\varphi_0}\psi_N, \big[\widetilde{\mathcal{L}}_{N,0} - \mathcal{L}_{N,0}\big]U_{\varphi_0}\psi_N\rangle| \leq \frac{1}{4}\langle \xi_N, \mathcal{H}_N\xi_N\rangle + CNa_N + C$$

Together with (5.109) and (5.108), we obtain (5.107). □

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